# Chapter 4
# **Force System Resultants**

# Chapter Objectives

- Concept of moment of a force in two and three dimensions.
- Method for finding the moment of a force about a specified axis.
- Define the moment of a couple.
- Determine the resultants of non-concurrent force systems.
- Reduce a simple distributed loading to a resultant force having a specified location

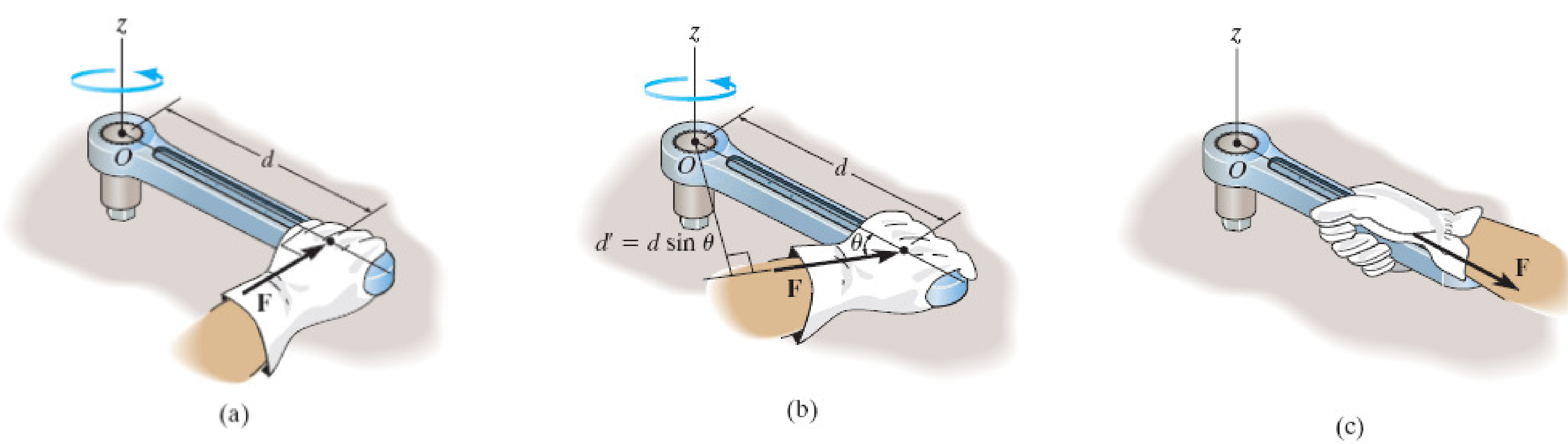
# Chapter Outline

1. Moment of a Force – Scalar Formation
2. Cross Product
3. Moment of Force – Vector Formulation
4. Principle of Moments
5. Moment of a Force about a Specified Axis
6. Moment of a Couple
7. Simplification of a Force and Couple System
8. Further Simplification of a Force and Couple System
9. Reduction of a Simple Distributed Loading

# 4.1 Moment of a Force– Scalar Formation

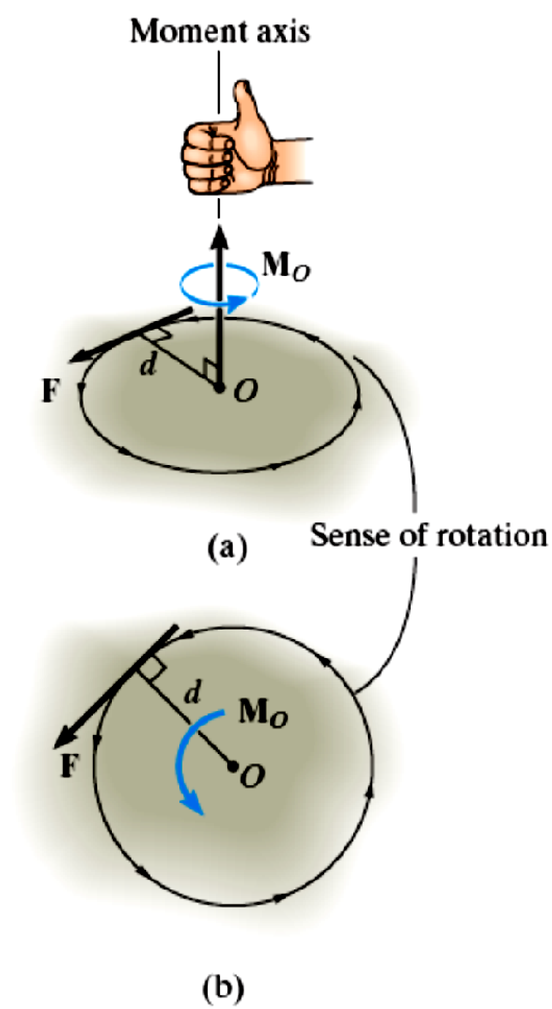
- When a force is applied to a body it will produce a tendency for the body to rotate about a point that is not on the line of action of the force.
- This tendency to rotate is called *torque*.
- But most often it is called *the moment of a force* or simply the *moment.*

The magnitude of the moment is directly proportional to the magnitude of **F** and the perpendicular distance or *moment arm d*.



- The moment $\mathbf{M}_O$ about point *O*, or about an axis passing through *O* and perpendicular to the plane, is a *vector quantity* since it has a specified magnitude and direction.

## ❑ Magnitude




- The magnitude of $\mathbf{M}_O$ is

$$M_O = F\,d$$

where $d$ = *moment arm*, or *perpendicular distance* from the axis at point *O* to the line of action of the force.

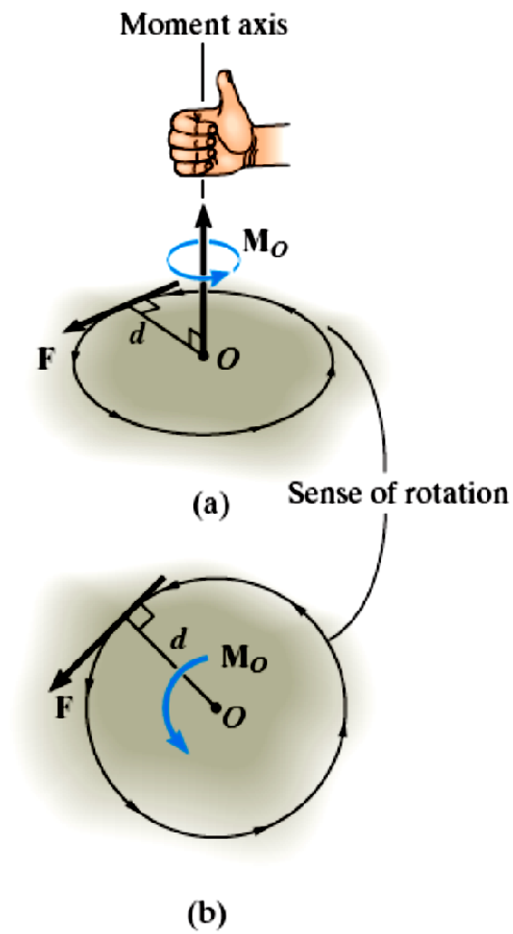
- Units of moment magnitude consist of force times distance, e.g., N·m or lb·ft.

## ❑ Direction

- The direction of $\mathbf{M}_O$ is defined by its ***moment axis***, which is perpendicular to the plane that contains the force **F** and its moment arm *d*.

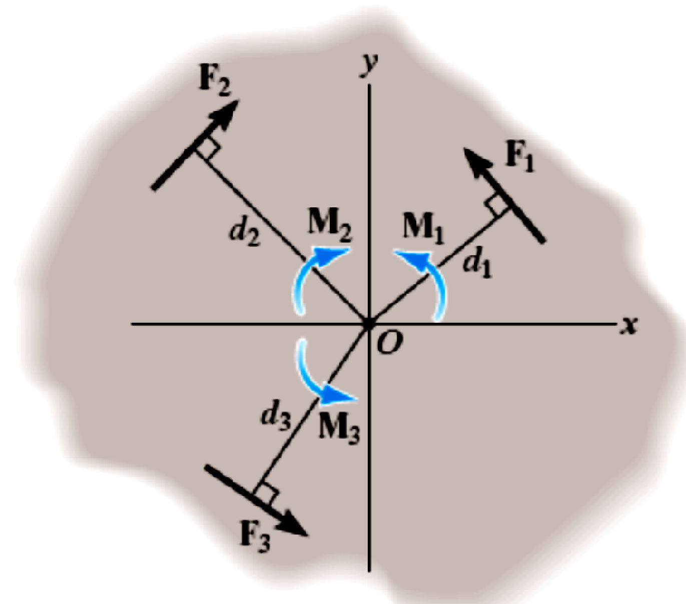
- The sense of direction of $\mathbf{M}_O$ can be established using the *right-hand rule*.

- Sign convention:
  - ➢ Counterclockwise moment is positive.
  - ➢ Clockwise moment is negative



## ❑ Resultant Moment

- For 2-D problems, where all the forces lie within the *x-y* plane, the resultant moment $(\mathbf{M}_R)_O$ about point *O* (the *z* axis) is the ***algebraic sum*** of the moments caused by all the forces in the system.



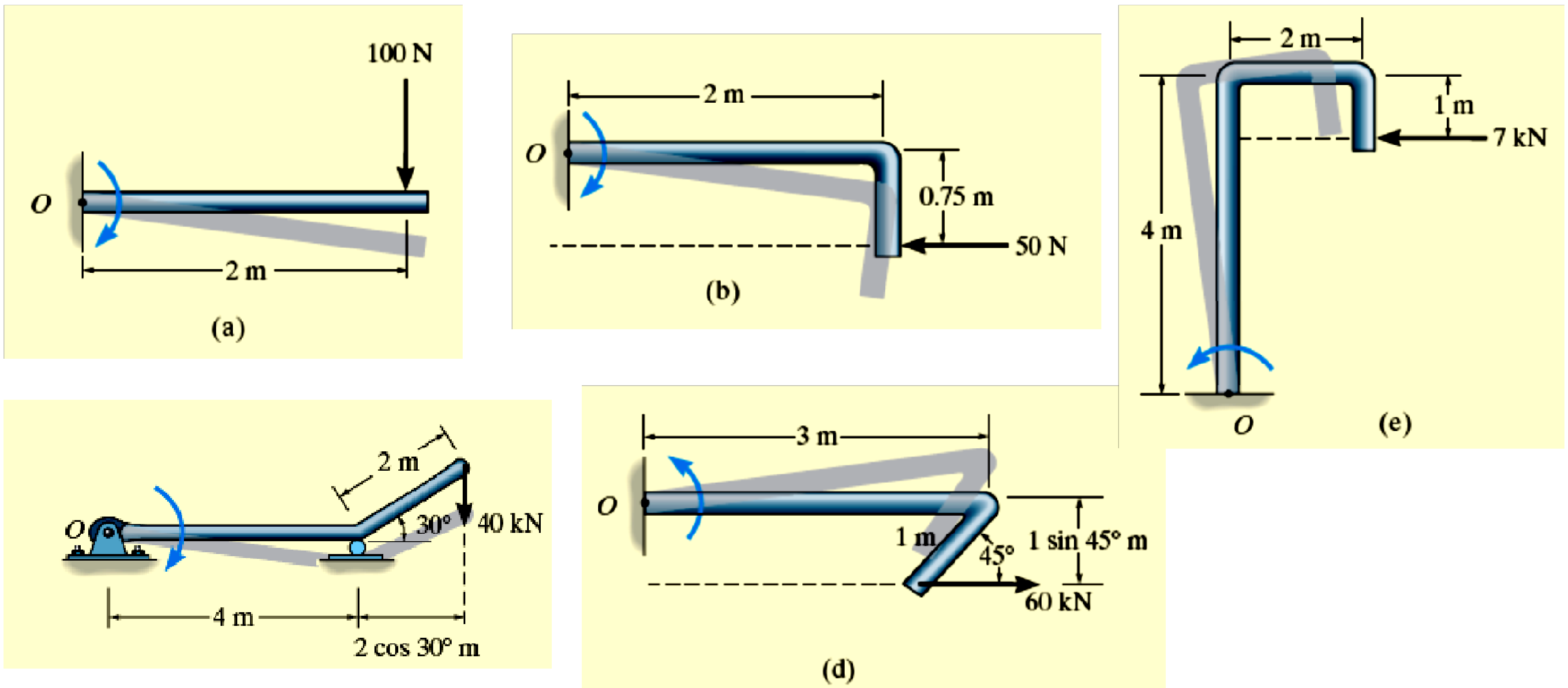

Resultant moment, $(\mathbf{M}_R)_O$ = moments of all the forces

↻+ $$(M_R)_O = \sum Fd$$

$$(M_R)_O = F_1 d_1 - F_2 d_2 + F_3 d_3$$

# Example 4.1

## Given :

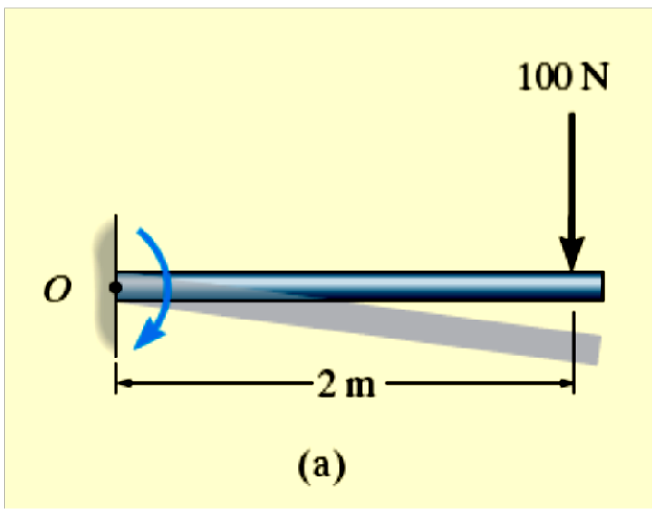


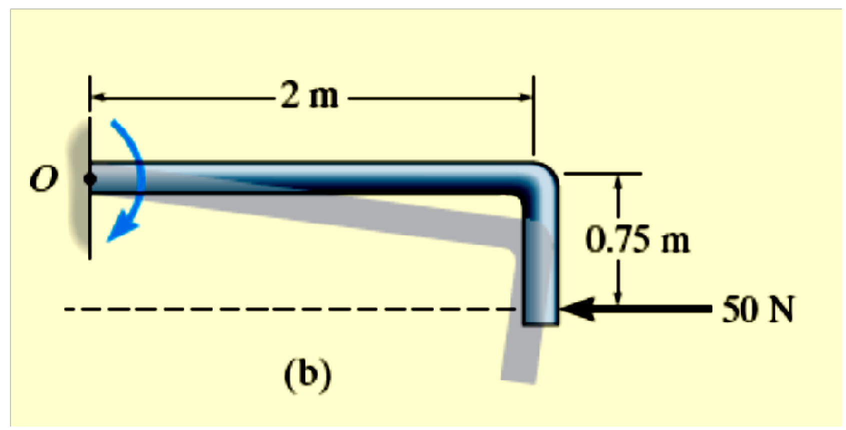

## Find :

For each case, determine the moment of the force about point **O**.

# Solution

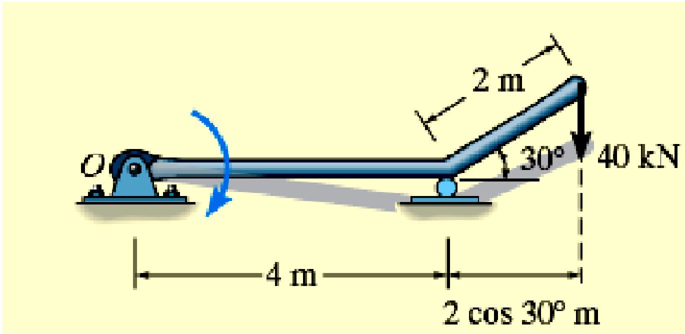
- Line of action is extended as a dashed line to establish moment arm *d*.
- Tendency to rotate is indicated and the orbit of the force about *O* is shown as a colored curl.



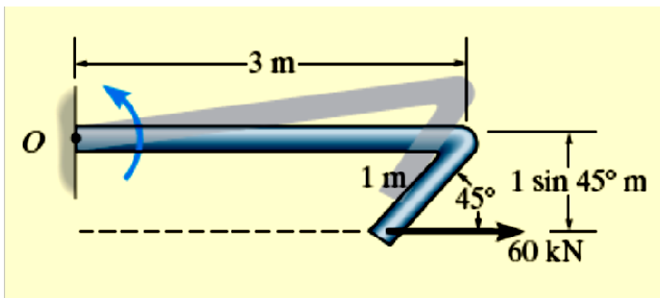




$$M_O = (100\text{ N})\,(2\text{ m}) = 200\text{ N}\cdot\text{m} \curvearrowright$$

$$M_O = (50\text{ N})\,(0.75)\text{ m}) = 37.5\text{ N}\cdot\text{m} \curvearrowright$$

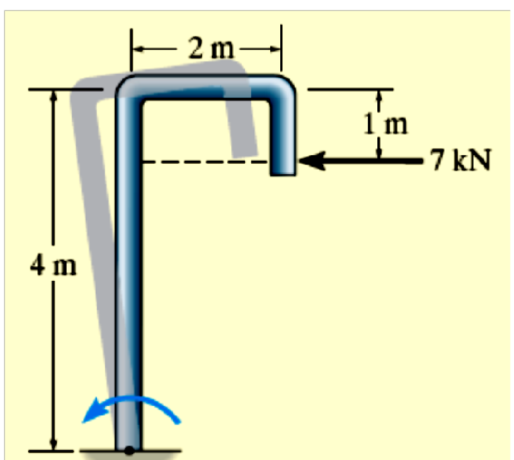



$$M_O = (40\ \text{kN})\ (4\text{m} + 2\cos 30°\ \text{m})$$
$$= 229\ \text{kN}\cdot\text{m} \ \circlearrowright$$



(d)

$$M_O = (60\ \text{kN})\ (1\sin 45°\ \text{m})$$
$$= 42.4\ \text{kN}\cdot\text{m} \ \circlearrowleft$$



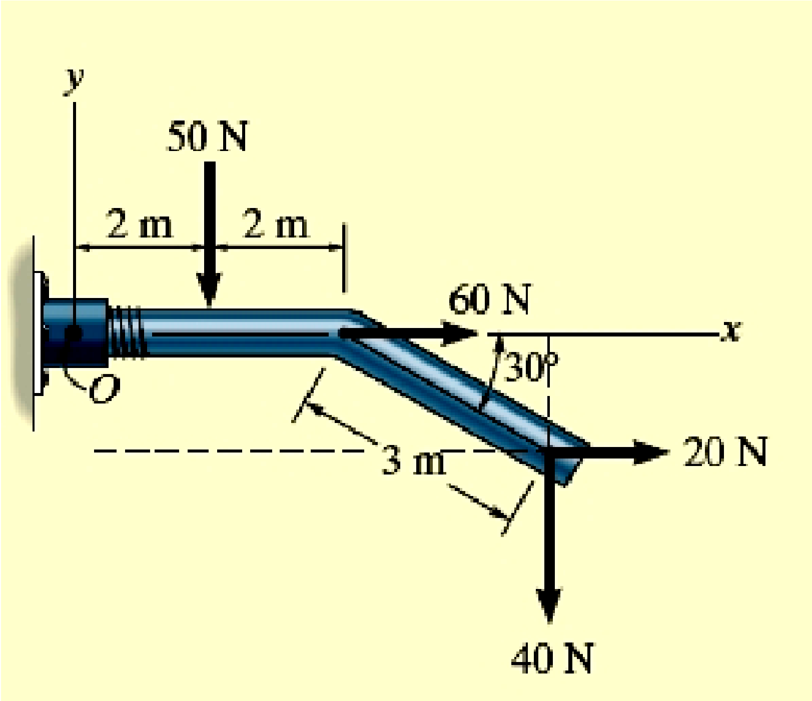
(e)

$$M_O = (7\ \text{kN})\ (4\ \text{m} - 1\ \text{m})$$
$$= 21.0\ \text{kN}\cdot\text{m} \ \circlearrowleft$$

# Example 4.3

## Given :

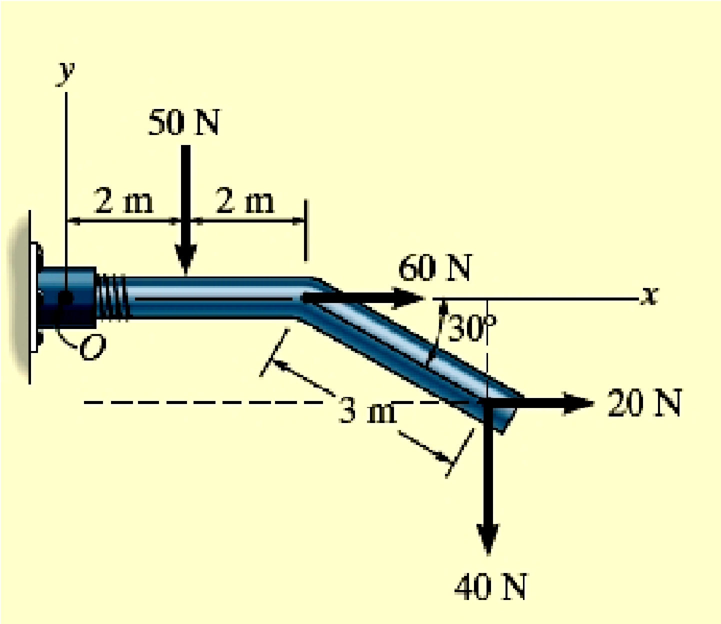
- Four forces act on the rod as shown.




## Find :

Determine the resultant moment of the four forces about point *O*.

# Solution




$$\circlearrowright + M_{R_O} = \sum F\,d$$

$$= -(50\text{ N})(2\text{ m})$$
$$+ (60\text{ N})(0\text{ m})$$
$$+ (20\text{ N})(3\sin 30^\circ\text{ m})$$
$$- (40\text{ N})(4 + 3\cos 30^\circ)\text{ m}$$

$$M_{R_O} = -334\text{ N}\cdot\text{m}$$
$$= 334\text{ N}\cdot\text{m}\ \circlearrowright$$

# 4.2 Cross Product

- Cross product of two vectors **A** and **B** yields the vector **C**, which is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

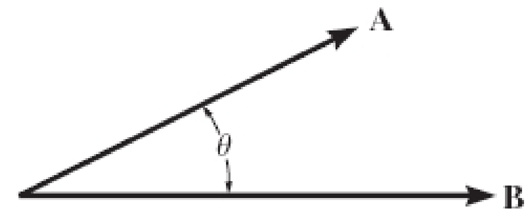
and is read "**C** equals **A** cross **B**."

- **Magnitude**

• The magnitude of **C** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle between their tails.

$$C = AB\sin\theta$$

where $0^\circ \leq \theta \leq 180^\circ$ .

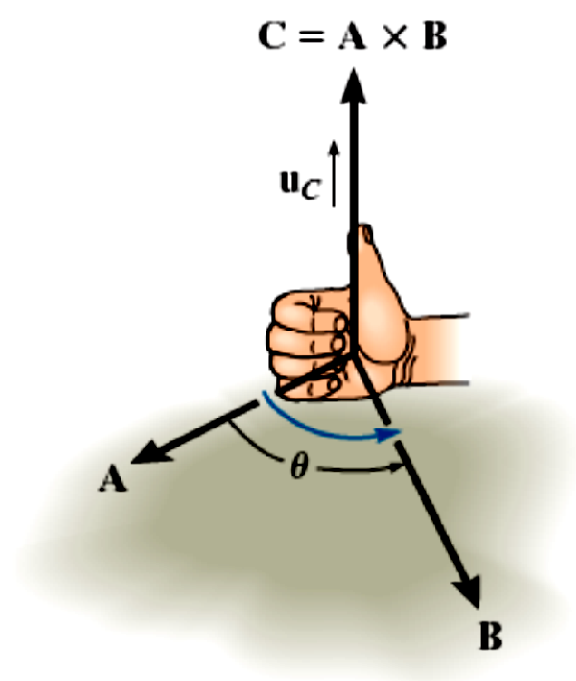


## ❑ Direction

- Vector **C** has a direction that is perpendicular to the plane containing **A** and **B**.

- The direction of **C** is specified by the right hand rule:

  *Curling the fingers of the right hand from vector* **A** *to vector* **B**, *the thumb points in the direction of* **C**.

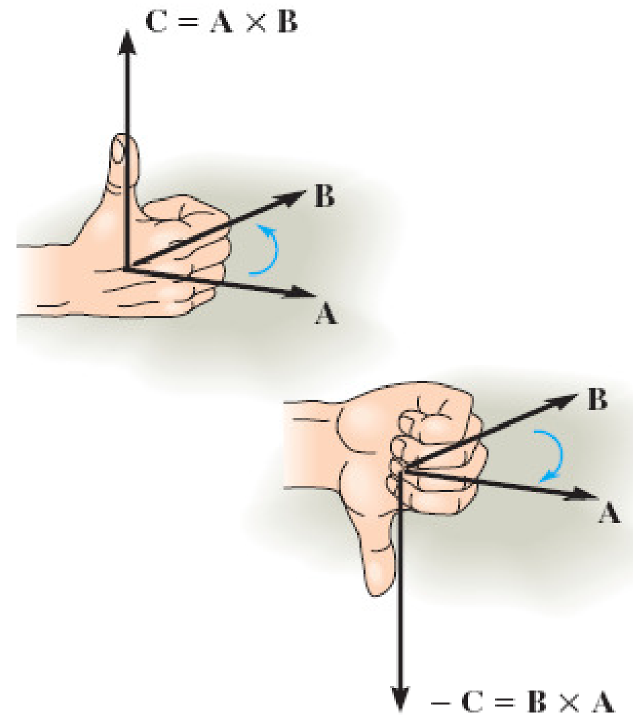



- Thus, vector **C** can be written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB\sin\theta)\,\mathbf{u}_C$$

where $\mathbf{u}_C$ is the unit vector in the direction of **C**.

## ❑ Laws of Operations




**1. Commutative law is not valid**

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

Rather,

$$\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$$

**2. Multiplication by a Scalar**

$$a\,(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})\,a$$

**3. Distributive Law**

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$$

**Note** : *Proper order of the cross products must be maintained since they are not commutative.*

## Cartesian Vector Formulation

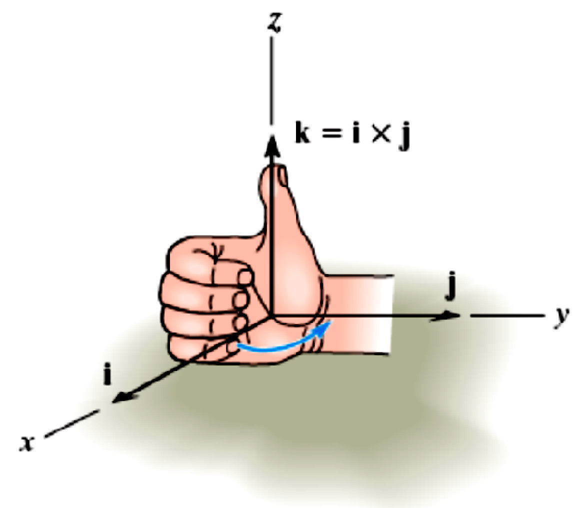
- Cross product of Cartesian unit vectors

(a) $|\mathbf{i} \times \mathbf{i}| = (1)(1) \sin 0^\circ = 0$

$\Rightarrow \mathbf{i} \times \mathbf{i} = \mathbf{0}$

(b) $|\mathbf{i} \times \mathbf{j}| = (1)(1) \sin 90^\circ = 1$

$\Rightarrow \mathbf{i} \times \mathbf{j} = (1)\,\mathbf{k}$




(c) $|\mathbf{j} \times \mathbf{k}| = (1)(1) \sin 90^\circ = 1$

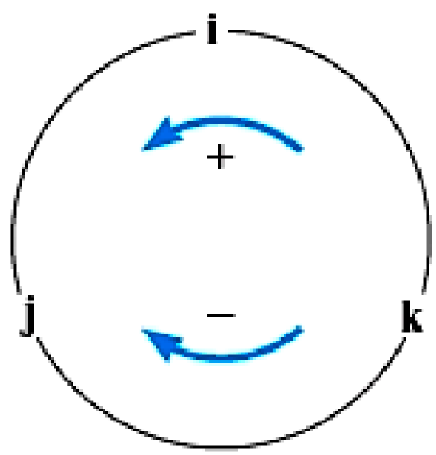
$\Rightarrow \mathbf{j} \times \mathbf{k} = (1)\,\mathbf{i}$

- In summary,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = 0$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{j} \times \mathbf{j} = 0$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{k} \times \mathbf{k} = 0$$



- If $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, then

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$= A_x B_x(\mathbf{i} \times \mathbf{i}) + A_x B_y(\mathbf{i} \times \mathbf{j}) + A_x B_z(\mathbf{i} \times \mathbf{k})$$

$$+ A_y B_x(\mathbf{j} \times \mathbf{i}) + A_y B_y(\mathbf{j} \times \mathbf{j}) + A_y B_z(\mathbf{j} \times \mathbf{k})$$

$$+ A_z B_x(\mathbf{k} \times \mathbf{i}) + A_z B_y(\mathbf{k} \times \mathbf{j}) + A_z B_z(\mathbf{k} \times \mathbf{k})$$

$$= A_x B_y\mathbf{k} - A_x B_z\mathbf{j} - A_y B_x\mathbf{k} + A_y B_z\mathbf{i} + A_z B_x\mathbf{j} - A_z B_y\mathbf{i}$$

$$= (A_y B_z - A_z B_y)\,\mathbf{i} - (A_x B_z - A_z B_x)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(determinant)

**Note :**

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \mathbf{i}\,(A_y B_z - A_z B_y) - \mathbf{j}\,(A_x B_z - A_z B_x) + \mathbf{k}\,(A_x B_y - A_y B_z)$$