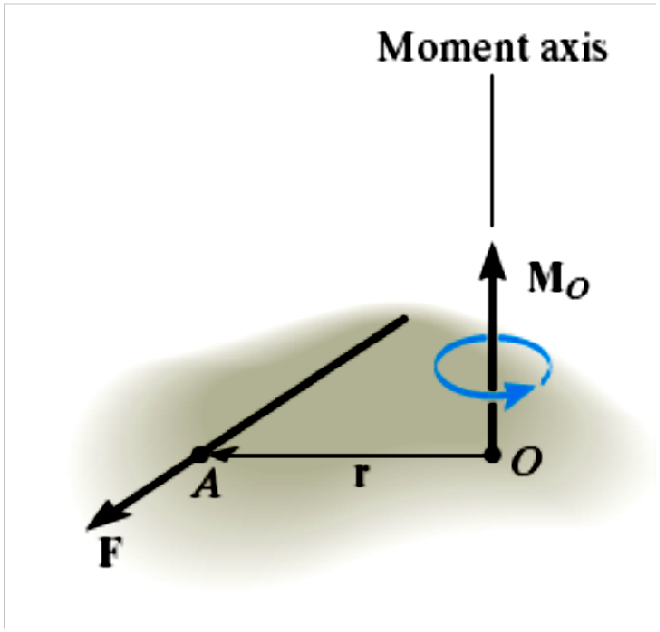
4.3 Moment of a Force – Vector Formulation

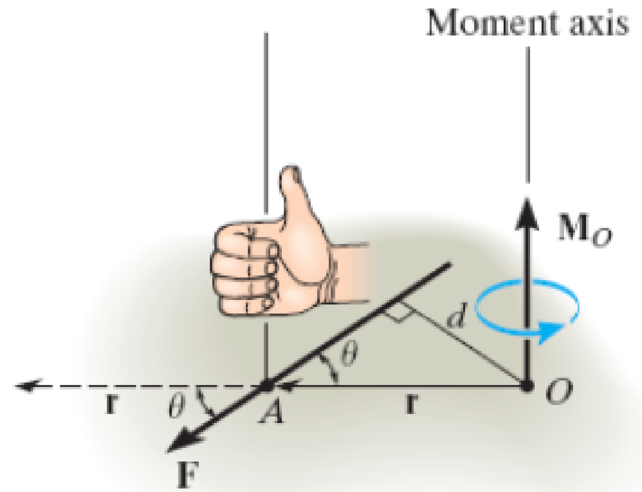
- The moment of a force **F** about point *O* (or actually about the moment axis passing through *O*) can be expressed using cross product, namely,



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where **r** = a position vector directed *from point O to* ***any*** *point on the line of action* **F**.

❑ Magnitude



- The magnitude of $\mathbf{M}_O$ is defined as

$$M_O = r\,F \sin\theta$$

where θ is measured between the *tails* of **r** and **F** (**r** is treated as *a sliding vector*).

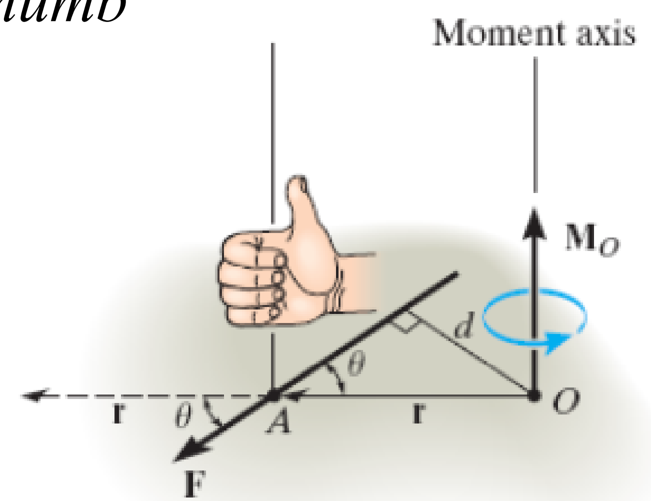
- Since the moment arm $d = r \sin\theta$, we have

$$M_O = r\,F \sin\theta = F\,(r \sin\theta) = F\,d$$

❑ Direction

- The direction & sense of $\mathbf{M}_O$ is determined by the right hand rule:

Sliding $\mathbf{r}$ *to the dashed position & curling the right hand fingers from* $\mathbf{r}$ *toward* $\mathbf{F}$, *the thumb points in the direction of* $\mathbf{M}_O$.



Note:

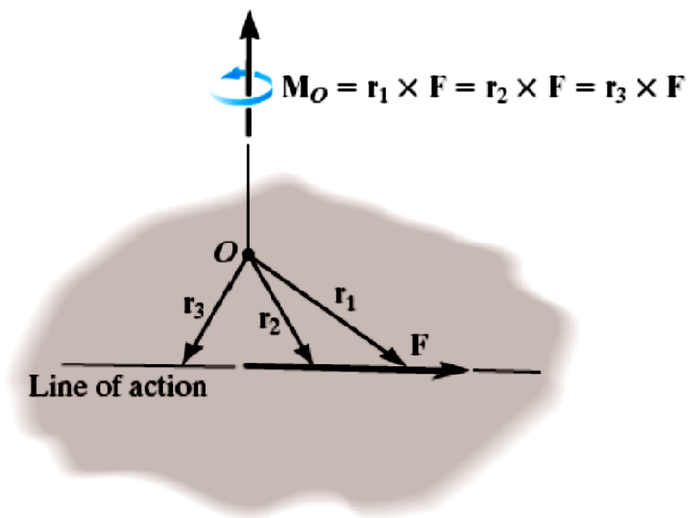
- The "curl" of the fingers indicates the sense of rotation caused by the force.

- The order of $= \mathbf{r} \times \mathbf{F}$ must be maintained as the cross product does not obey the commutative law.

❑ Principle of Transmissibility

- Using the vector cross product, there is no need to find the moment arm when determining the moment of a force.

- As can be seen from the figure, we can use any position vector **r** measured from point *O* to the line of action of the force **F**.



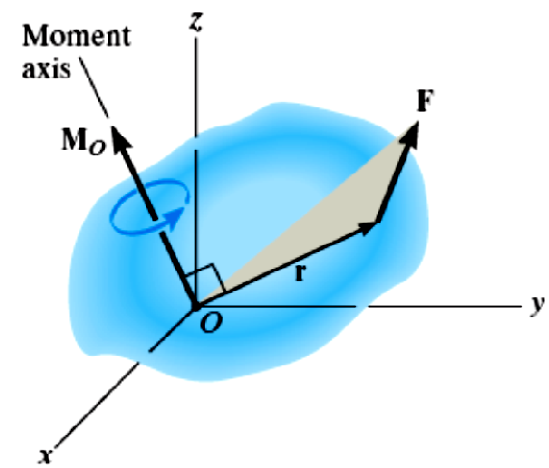
$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

- Thus, **F** can be considered as a *sliding vector* because it can be applied at any point along its line of action and still create the same moment about point *O*.

- This property is called the ***principle of transmissibility*** of a force.

❑ Cartesian Vector Formulation

- Using Cartesian coordinate system (x, y, z), the position vector **r** and force **F** can be expressed as Cartesian vectors.



- Therefore,

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \qquad (1)$$

where

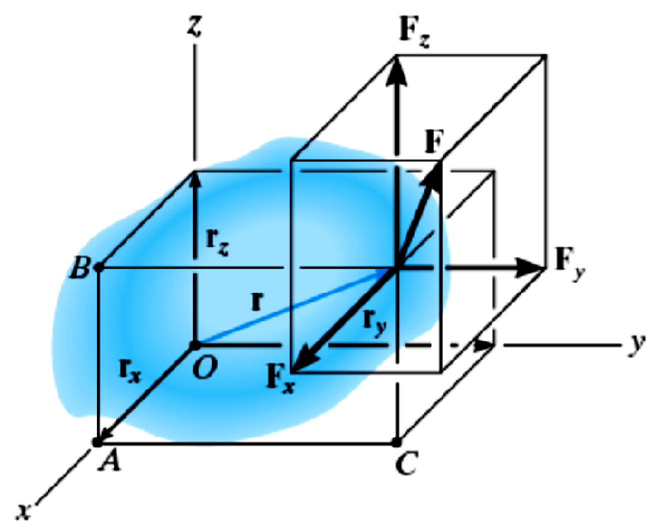
r_x, r_y, r_z are the x, y, z components of the position vector from point O to *any point* on the line of action **F**

F_x, F_y, F_z are the x, y, z components of the force vector

Note: $\mathbf{M}_O$ is perpendicular to the plane containing **r** and **F**.

- Expanding the determinant in Eq.(1) yields

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} - (r_x F_z - r_z F_x)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}$$



or

$$\mathbf{M}_O = (M_O)_x\,\mathbf{i} + (M_O)_y\,\mathbf{j} + (M_O)_z\,\mathbf{k}$$

where

$$(M_O)_x = (r_y F_z - r_z F_y)$$

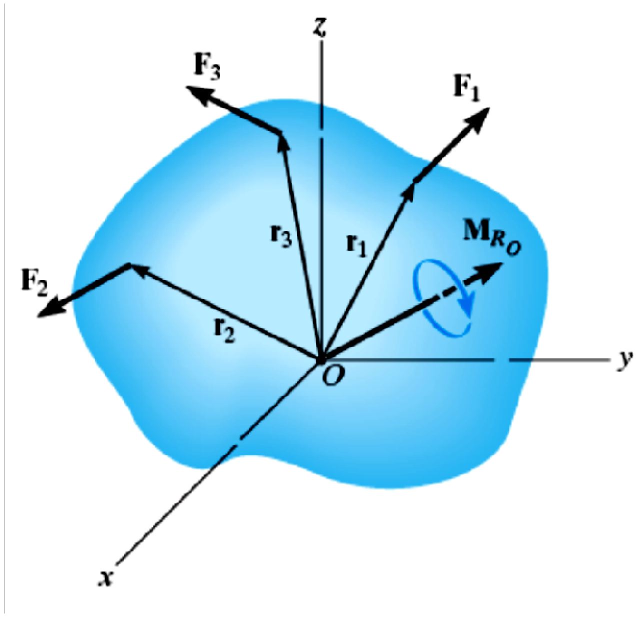
$$(M_O)_y = -(r_x F_z - r_z F_x)$$

$$(M_O)_z = (r_x F_y - r_y F_x)$$

❑ Resultant Moment of a System of Forces

The resultant moment of a system of forces about point *O* can be determined by vector addition of the moment of each force.

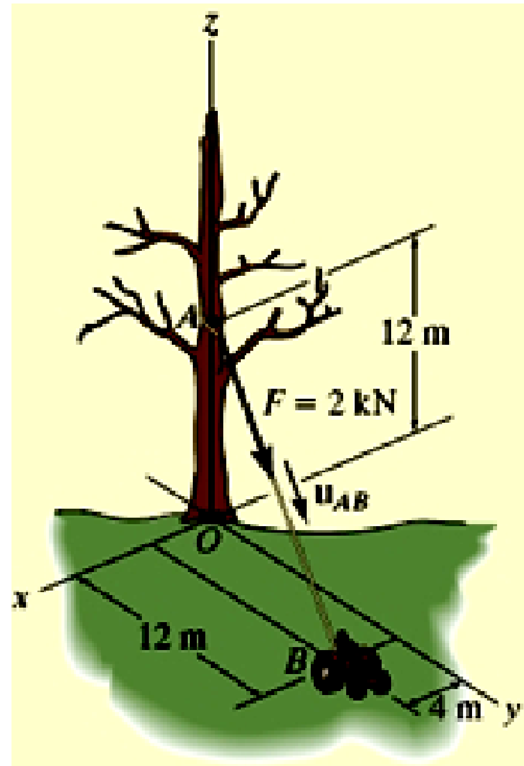
$$\mathbf{M}_{R_O} = \Sigma\,(\mathbf{r} \times \mathbf{F})$$



Example 4.3

Given :

- The force $F = 2\text{kN}$ acts on the tree as shown in the figure.

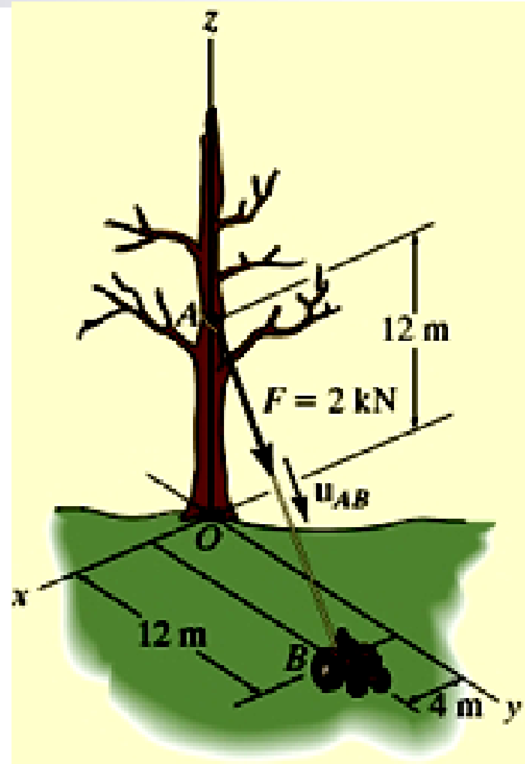


Find :

Determine the moment produced by the force **F** about point *O*. Express the result as a Cartesian vector.

Solution

Express the force as a Cartesian vector



- Coordinates: $A\ (0, 0, 12)$ m

  $B\ (4, 12, 0)$ m

- Position vector of B relative to A

$$\mathbf{r}_{AB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

$$= (4 - 0)\mathbf{i} + (12 - 0)\mathbf{j} + (0 - 12)\mathbf{k}$$

$$= \{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\ \text{m}$$

$$r_{AB} = \sqrt{(4)^2 + (12)^2 + (-12)^2} = \sqrt{304}\ \text{m}$$

$$\mathbf{F} = F\,\mathbf{u}_{AB} = F_B\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = (2\ \text{kN})\left(\frac{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}}{\sqrt{304}}\right)$$

$$= \{0.4588\mathbf{i} + 1.376\,\mathbf{j} - 1.376\,\mathbf{k}\}\ \text{kN}$$

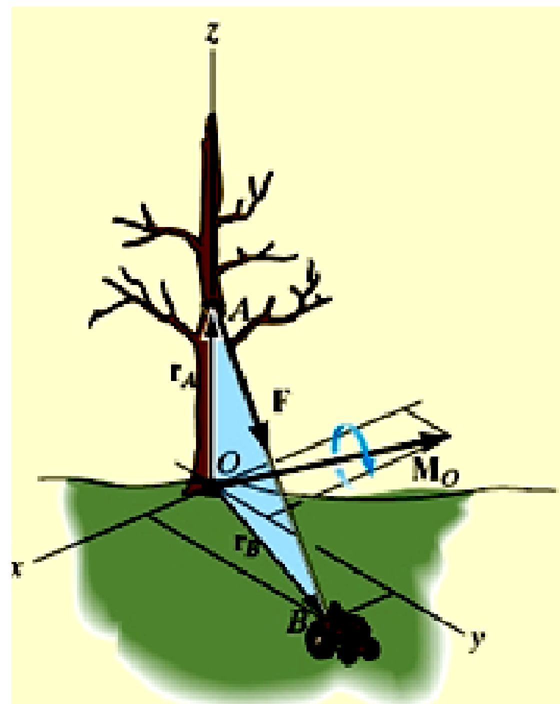
Moment

- The moment of a force **F** about point *O* is

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where **r** is a position vector directed *from point O to any point on the line of action of* **F**.

- Either $\mathbf{r}_A$ of $\mathbf{r}_B$ can be used.



- Using $\mathbf{r} = \mathbf{r}_A$ we have

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= \mathbf{i}\,[(0)(-1.376) - (12)(1.376)]$$
$$-\,\mathbf{j}\,[(0)(-1.376) - (12)(0.4588)]$$
$$+\,\mathbf{k}\,[(0)(1.376) - (0)(0.4588)]$$

$$= \{-16.5\,\mathbf{i} + 5.51\,\mathbf{j}\}\ \text{kN·m}$$

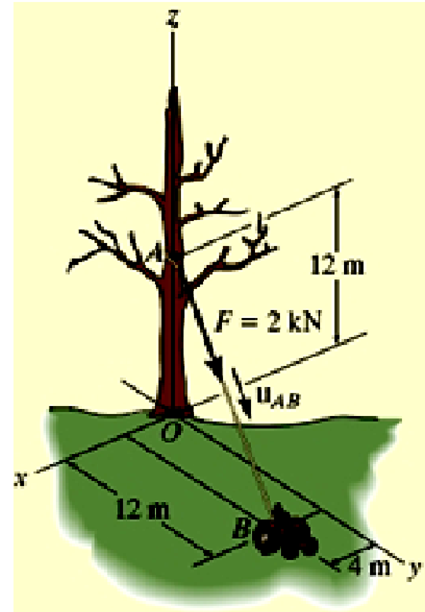
- Using $\mathbf{r} = \mathbf{r}_B$, we have

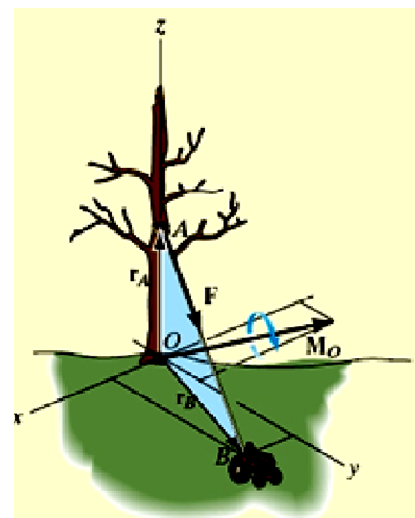
$$\mathbf{M}_O = \mathbf{r}_B \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= \mathbf{i}\,[(12)(-1.376) - (0)(1.376)]$$
$$-\mathbf{j}\,[(4)(-1.376) - (0)(0.4588)]$$
$$+\mathbf{k}\,[(4)(1.376) - (12)(0.4588)]$$

$$= \{-16.5\,\mathbf{i} + 5.51\,\mathbf{j}\}\ \text{kN}\cdot\text{m}$$

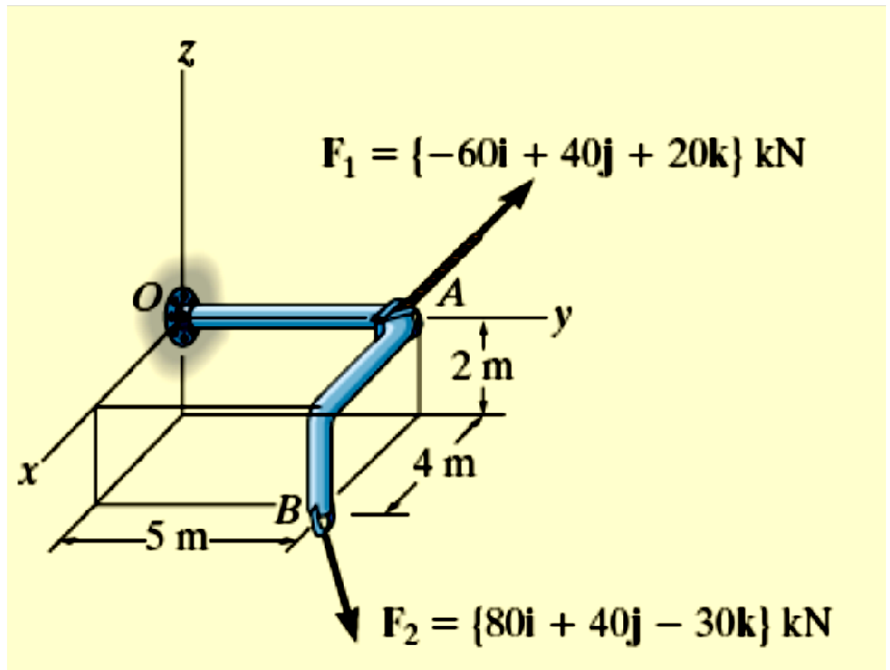




Example 4.4

Given :

- Two forces act on the rod as shown.

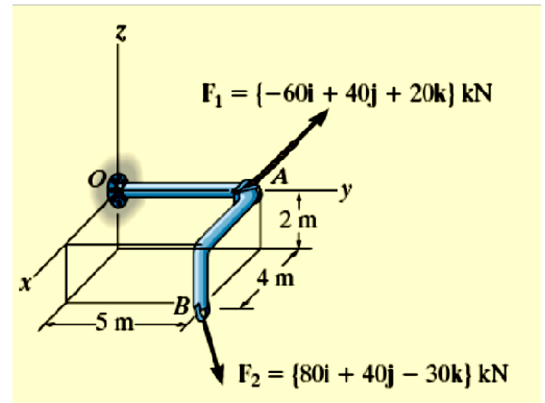


Find :

Determine the resultant moment they create about the flange at *O*. Express the result as a Cartesian vector.

Solution

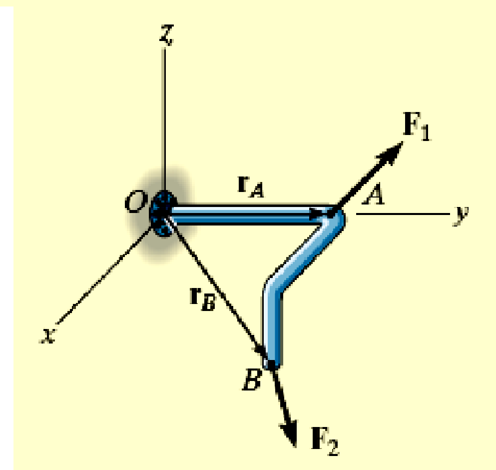
- Position vectors directed from point O to each force are:



$$\mathbf{r}_A = \{\ 5\mathbf{j}\}\ \text{m}$$

$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j}\ \ -2\mathbf{k}\ \}\ \text{m}$$

- The resultant moment about O is



$$\mathbf{M}_{R_O} = \Sigma\ (\mathbf{r} \times \mathbf{F})$$

$$= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= \mathbf{i}\,[5(\ 20) - \ 0\,(40)\ ] \ - \mathbf{j}\,[0 - 0]\ + \mathbf{k}\,[0 - (5)(-\,60)]$$
$$+ \ \mathbf{i}\,[5(-30) - (-2)(40)] - \mathbf{j}[4(-30) - (-2)(80)] + \mathbf{k}[4(40) - (5)(80)]$$

$$= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}\ \text{kN}\cdot\text{m}$$

Note: The coordinate direction angles of $\mathbf{M}_O$ can be determined from the unit vector for $\mathbf{M}_O$.

- The magnitude of $\mathbf{M}_O$

$$M_{R_O} = \sqrt{(30)^2 + (-40)^2 + (60)^2} = 78.10\ \text{kN}\cdot\text{m}$$

- Unit vector in the direction of $\mathbf{M}_O$

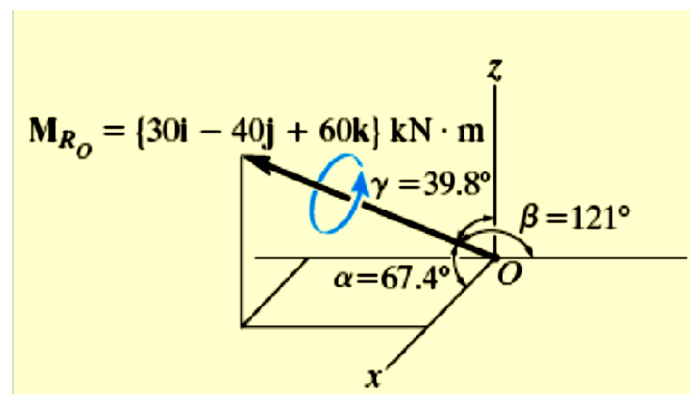
$$\mathbf{u}_{M_{R_O}} = \frac{\mathbf{M}_{R_O}}{M_{R_O}} = \frac{\{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}}{78.1} = 0.3841\mathbf{i} - 0.5122\mathbf{j} + 0.7682\mathbf{k}$$

- The coordinate direction angles for $\mathbf{M}_O$

$$\alpha = \cos^{-1}(0.3841) = 67.4^\circ$$

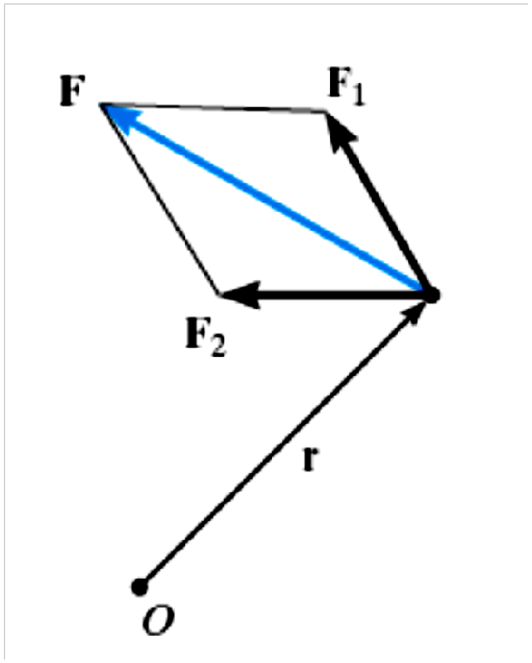
$$\beta = \cos^{-1}(-0.5122) = 121^\circ$$

$$\gamma = \cos^{-1}(0.7682) = 39.8^\circ$$



4.4 Principles of Moments

- It is also known as *Varignon's* Theorem
- It states that *the moment of a force about a point is equal to the sum of the moments of the components of the force about the point*.
- **Proof**



• Consider the force **F** shown in the figure.

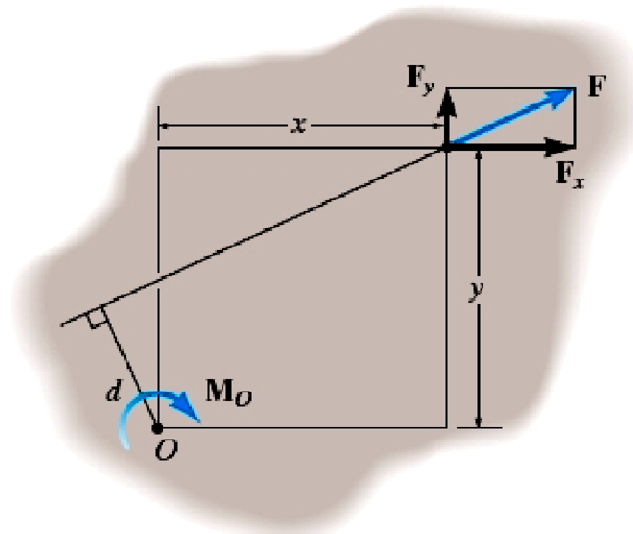
• Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$
$$= \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$
$$= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

■ Applications

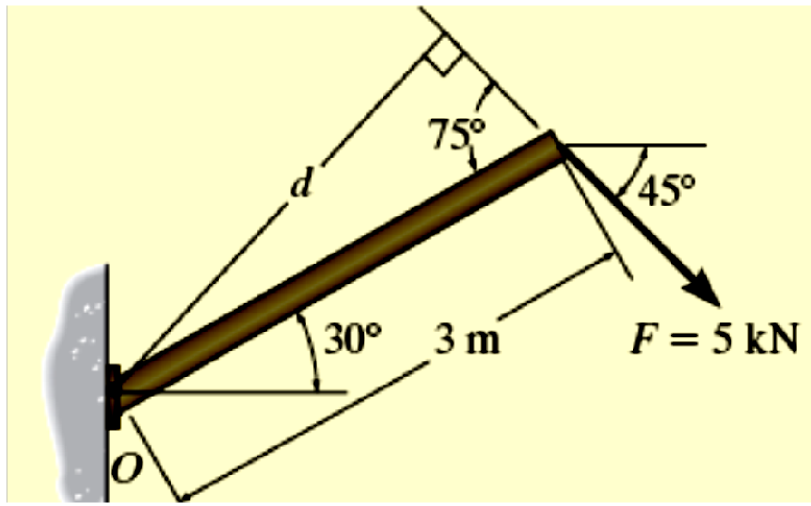
• For 2-D problems, the principle of moments allows us to determine the moment using a scalar analysis by resolving the force into its rectangular components.

$$+\circlearrowright \quad M_O = F_x y - F_y x$$



Example 4.5

Given :

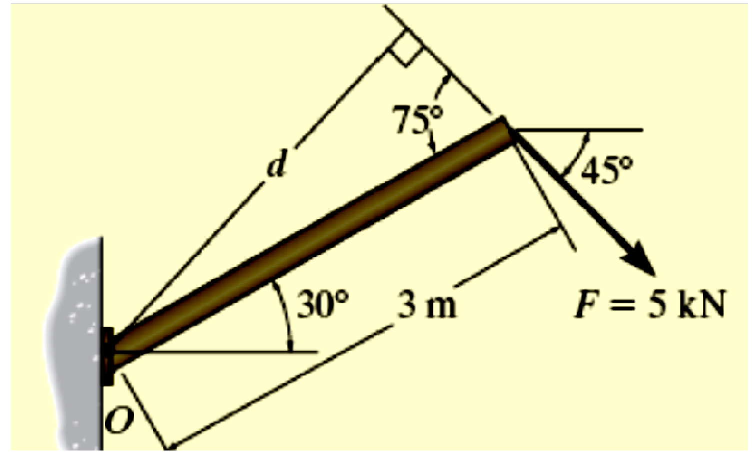


Find :

Determine the moment of the force about point *O*.

Solution

Method I



From trigonometry,

$$d = (3\ \text{m})\sin 75^\circ = 2.898\ \text{m}$$

Thus,

$$M_O = F\,d$$

$$= (5\ \text{kN})\,(2.898\ \text{m})$$

$$= 14.5\ \text{kN}\cdot\text{m} \quad \curvearrowright$$

Since the force tends to rotate or orbit clockwise about point *O*, the moment is directed into the page.

Method II

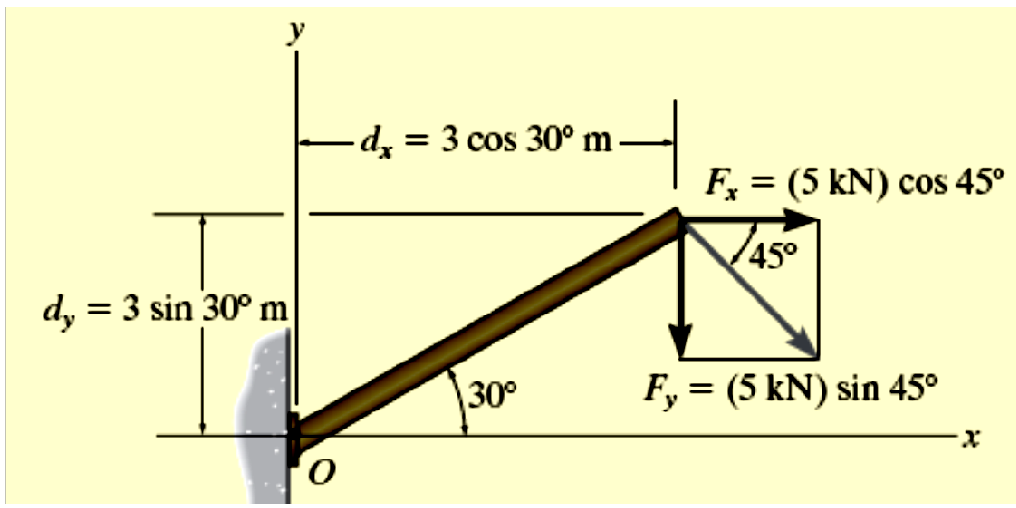
Applying the principle of moments, we have

$$\circlearrowright + \quad M_O = -F_x\, d_y - F_y\, d_x$$

$$= \{ -(5 \cos 45^{\circ})(3 \sin 30^{\circ}) - (5 \sin 45^{\circ})(3 \cos 30^{\circ}) \} \text{ kN·m}$$

$$= -14.5 \text{ kN·m}$$

$$= 14.5 \text{ kN·m} \quad \circlearrowright$$



Method III

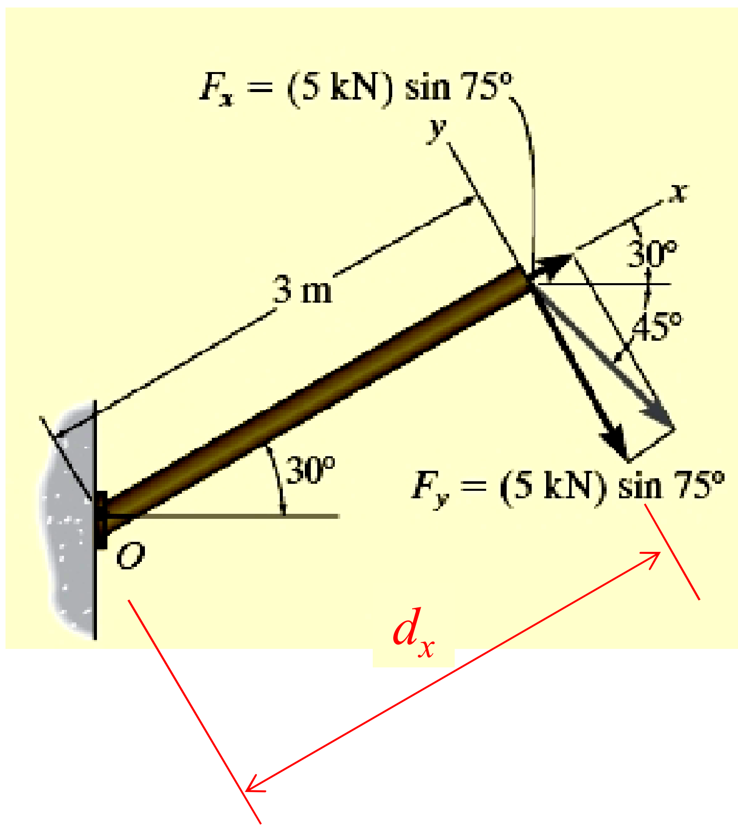
- Let x and y axes be parallel and perpendicular to the rod's axis.
- $\mathbf{F}_x$ produces no moment about O since its line of action passes through O.

$$\circlearrowleft + \; M_O = -F_y\, d_x$$

$$= -(5 \sin 75^{o}\,\text{kN})\,(3\ \text{m})$$

$$= -14.5\ \text{kN·m}$$

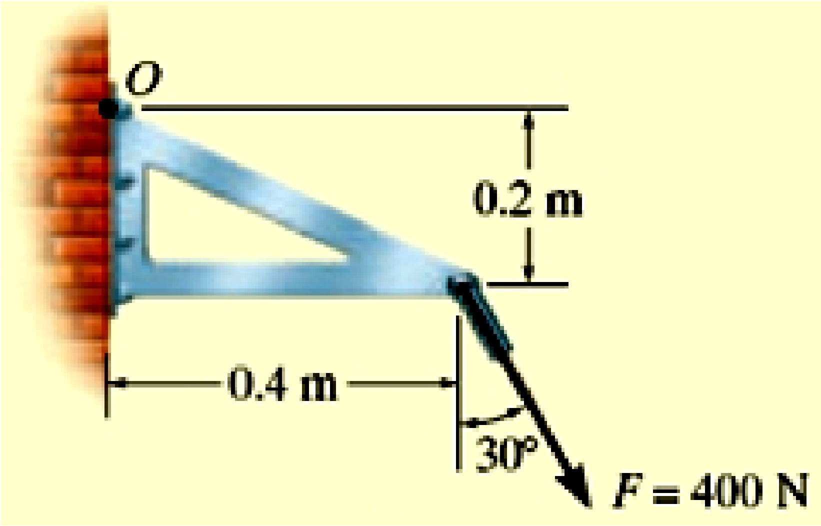
$$= 14.5\ \text{kN·m} \;\; \circlearrowright$$



Example 4.6

Given :

- Force **F** acts at the end of the angle bracket as shownin the figure.



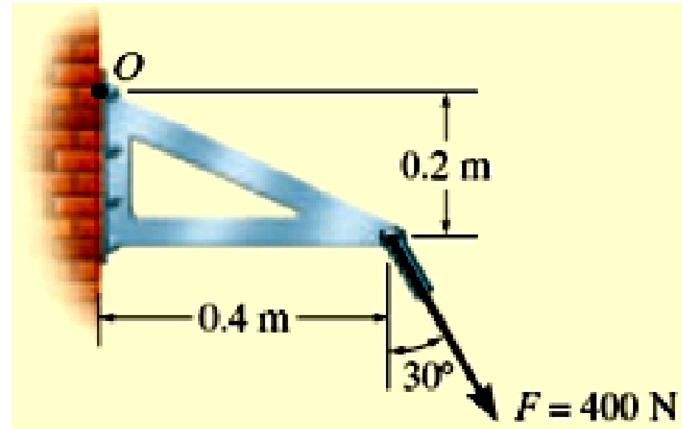
Find :

Determine the moment of the force about point *O*.

Solution

Method I (Scalar Analysis)

- Resolve the force into its x and y components.

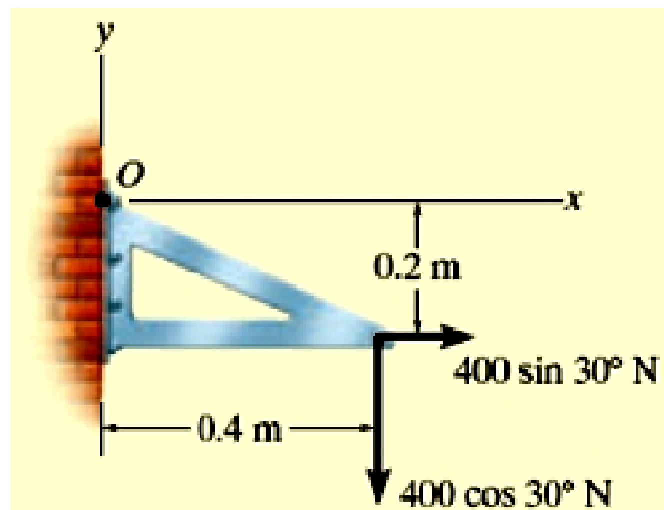


- The moment is then given by

$$\curvearrowleft + M_O = (400 \sin 30^\circ \text{ N})(0.2 \text{ m}) - (400 \cos 30^\circ \text{ N})(0.4 \text{ m})$$

$$= -98.6 \text{ N·m}$$

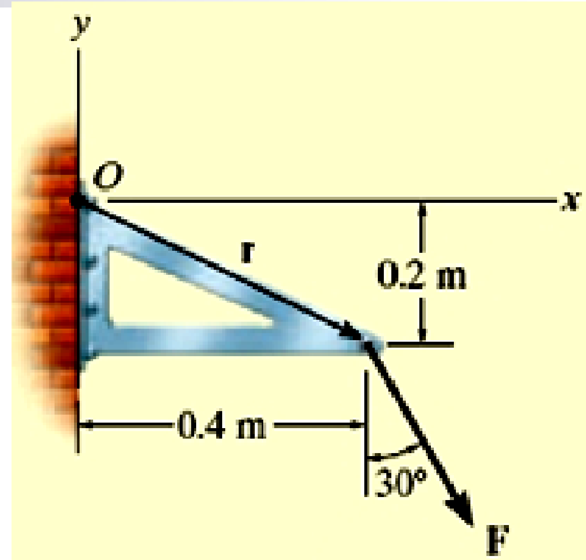
$$= 98.6 \text{ N·m} \quad \curvearrowright$$



or

$$\mathbf{M}_O = \{-98.6\mathbf{k}\} \text{ N·m}$$

Method II (Vector Analysis)



- Find the position vector

  $\mathbf{r} = \{0.4\mathbf{i} - 0.2\,\mathbf{j}\}$ m

- Express the force as a Cartesian vector.

  $\mathbf{F} = \{400 \sin 30^{\circ}\,\mathbf{i} - 400 \cos 30^{\circ}\,\mathbf{j}\}$ N

  $= \{200.0\,\mathbf{i} - 346.4\,\mathbf{j}\}$ N

- The moment is

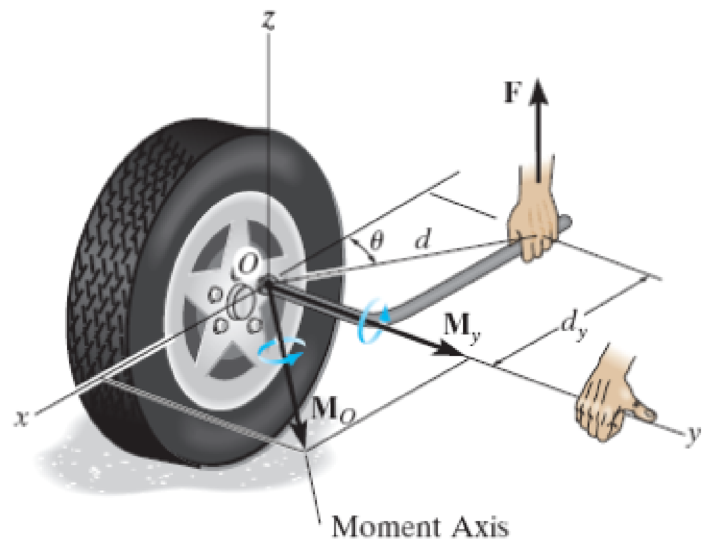
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$

$$= \mathbf{i}\,(0) - \mathbf{j}\,(0) + \mathbf{k}\,[\,(0.4)(-346.4) - (-0.2)(200.0)]$$

$$= \{-98.6\mathbf{k}\}\ \text{N·m}$$

4.5 Moment of a Force about a Specified Axis

- In the figure, the moment produced by the force **F** creates a tendency for the nut to rotate about the *moment axis* passing through *O*.
- However, the nut can only rotate about the *y* axis.

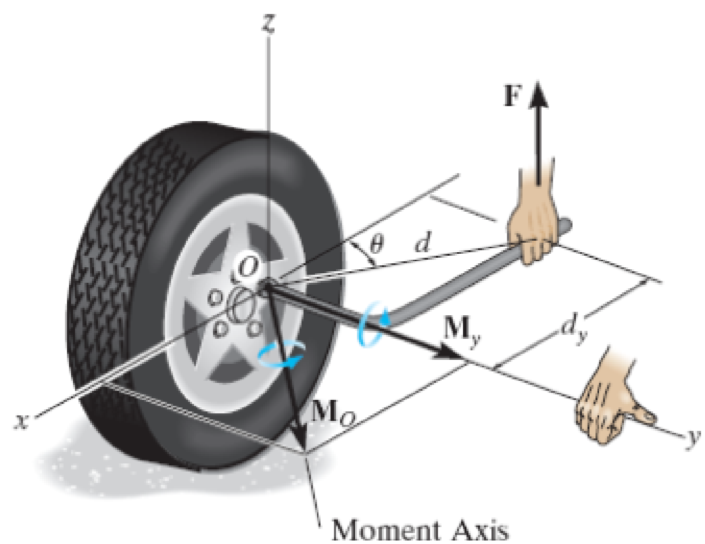


- Therefore, to determine the turning effect, only the *y* component of the moment is needed, and the total moment is not important.
- To find the component of the moment about a specified axis (in this case, the *y* axis) that passes through *O*, a scalar or vector analysis can be used.

❑ Scalar Analysis

- In the figure, the moment of **F** about the *y* axis is

$$M_y = F\, d_y$$
$$= F\,(d \cos\theta)$$



- According to the right-hand rule, M_y is directed along the positive *y* axis.

- In general, the moment of a force **F** about a specified axis *a* is

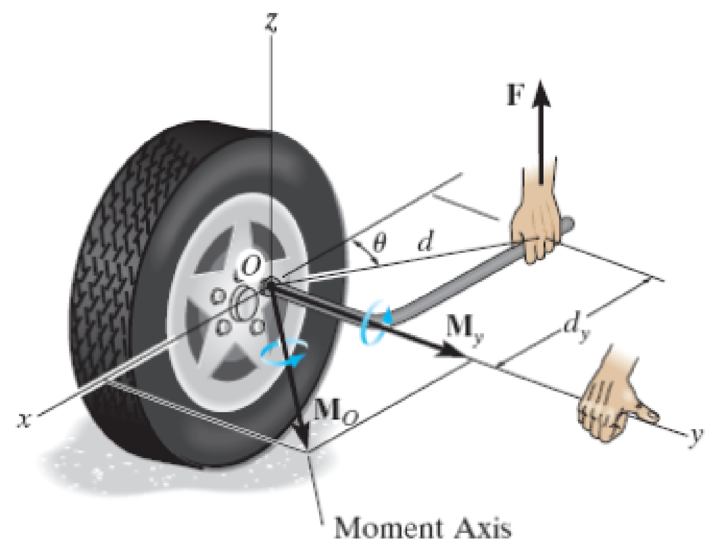
$$M_a = F\, d_a$$

where *d* is the perpendicular distance from the line of action of the force to the axis *a*.

❑ Vector Analysis

- In the figure, the moment of **F** about O is

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

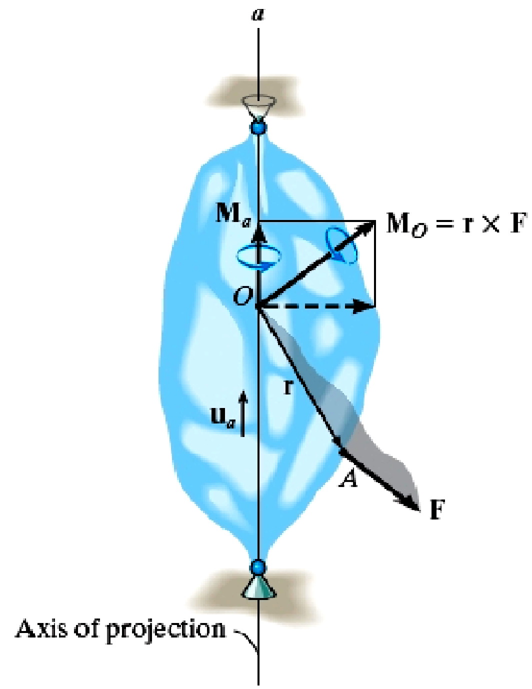


- The component M_y along the y axis is the projection of $\mathbf{M}_O$ onto the y axis.

$$M_y = \mathbf{j} \bullet \mathbf{M}_O = \mathbf{j} \bullet (\mathbf{r} \times \mathbf{F})$$

where **j** is the unit vector for the y axis.

- In general, if $\mathbf{u}_a$ is the unit vector that specifies the direction of the a axis shown in the adjacent figure, then the moment of $\mathbf{F}$ about the a axis is



$$M_a = \underbrace{\mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})}_{\text{Scalar triple product}} \qquad (1)$$

Scalar triple product

- In Cartesian form, Eq.(1) becomes

$$M_a = \left(u_{a_x}\mathbf{i} + u_{ay}\mathbf{j} + u_{a_z}\mathbf{k}\right) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(u_{a_x}\mathbf{i} + u_{ay}\mathbf{j} + u_{a_z}\mathbf{k}\right) \cdot \left[(r_y F_z - r_z F_y)\mathbf{i} - (r_x F_z - r_z F_x)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}\right]$$

$$= u_{a_x}\left(r_y F_z - r_z F_y\right) - u_{ay}\left(r_x F_z - r_z F_x\right) + u_{a_z}\left(r_x F_y - r_y F_x\right) \qquad \cdots (2)$$

- Eq.(2) can be written in the form of a determinant as

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

where

u_{a_x}, u_{a_y}, u_{a_z} are the x, y, z components of the unit vector defining the direction of the a axis

r_x, r_y, r_z are the x, y, z components of the position vector extended ***from any point*** *O* on the *a* axis **to *any*** *point A on the line of action* of the force

F_x, F_y, F_z are the x, y, z components of the force vector

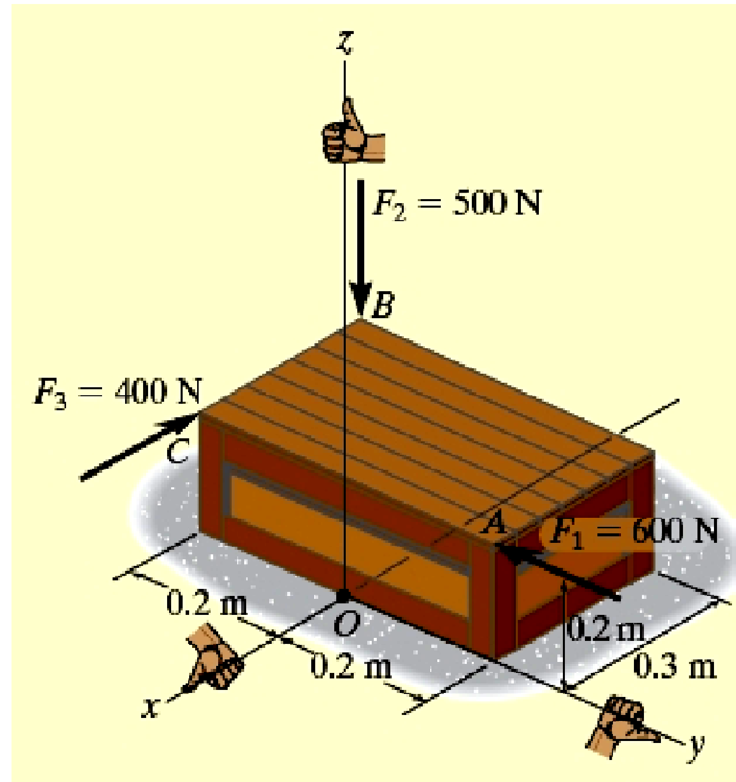
- The moment of **F** about the a axis in Cartesian vector form is

$$\mathbf{M}_a = M_a \, \mathbf{u}_a$$

Example 4.7

Given :

- Three forces are acting in the direction as shown in the figure.

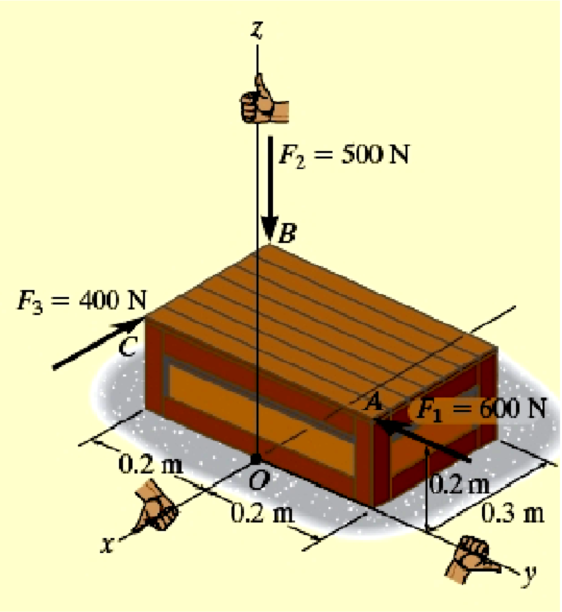


Find :

Determine the resultant moment of the forces about the *x* axis, the *y* axis, and the *z* axis.

Solution

- A force will not produce a moment about an axis if its line of action is parallel to or passes through that axis.



$$\circlearrowleft + \; M_x = F_1\,(0.2\text{ m}) + F_2\,(0.2\text{ m})$$
$$= (600\text{ N})\,(0.2\text{ m}) + (500\text{ N})\,(0.2\text{ m})$$
$$= 220\text{ N}\cdot\text{m}$$

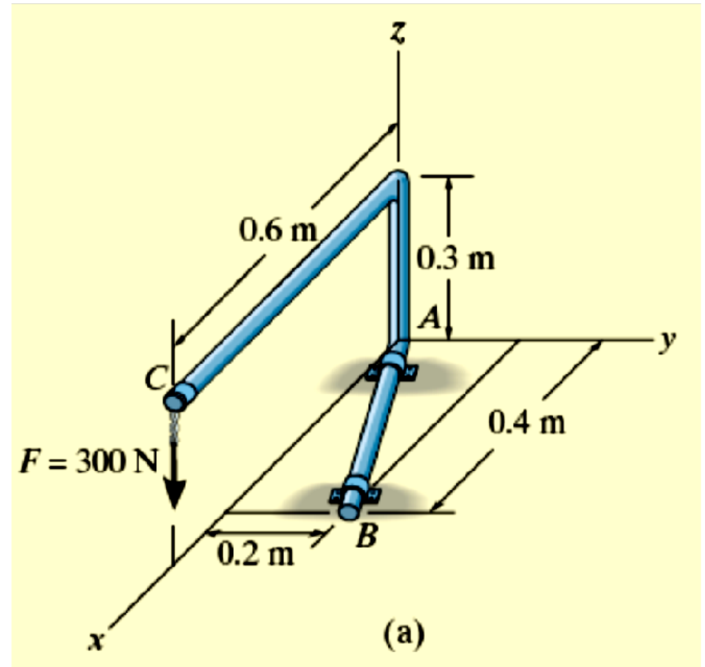
$$\circlearrowleft + \; M_y = -F_2\,(0.3\text{ m}) - F_3\,(0.2\text{ m})$$
$$= -(500\text{ N})\,(0.3\text{ m}) - (400\text{ N})\,(0.2\text{ m})$$
$$= -230\text{ N}\cdot\text{m}$$

$$\circlearrowleft + \; M_z = -F_3\,(0.2\text{ m})$$
$$= -(400\text{ N})\,(0.2\text{ m})$$
$$= -80\text{ N}\cdot\text{m}$$

Example 4.8

Given :

- The force $F = 300$ N acts on the rod as shown.



Find :

Determine the moment $\mathbf{M}_{AB}$ produced by the force $\mathbf{F}$ which tends to rotate the rod about the AB axis.

Solution

- The moment M_{AB} is given by

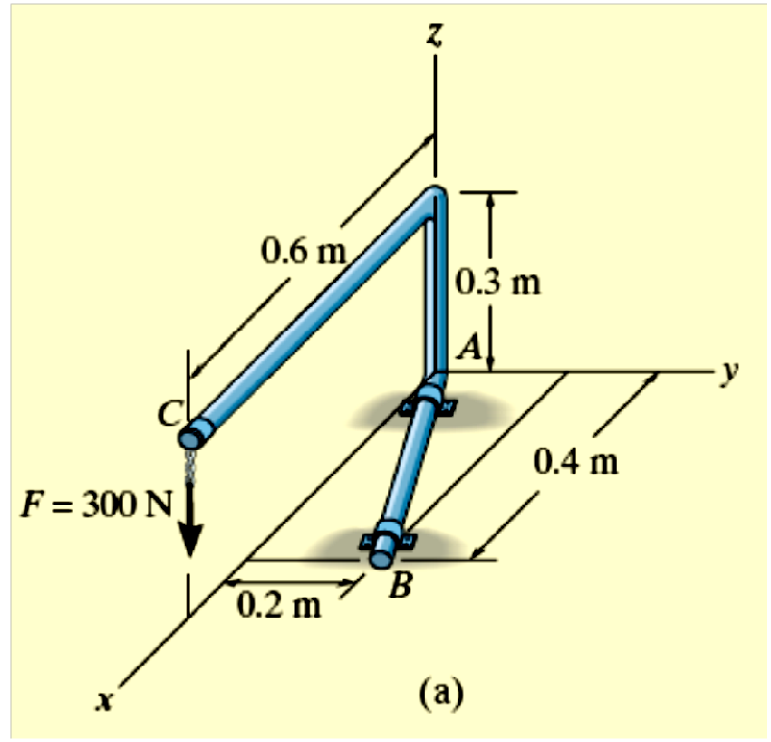
$$M_{AB} = \mathbf{u}_B \bullet (\mathbf{r} \times \mathbf{F})$$

- The unit vector which defines the direction of the *AB* axis is

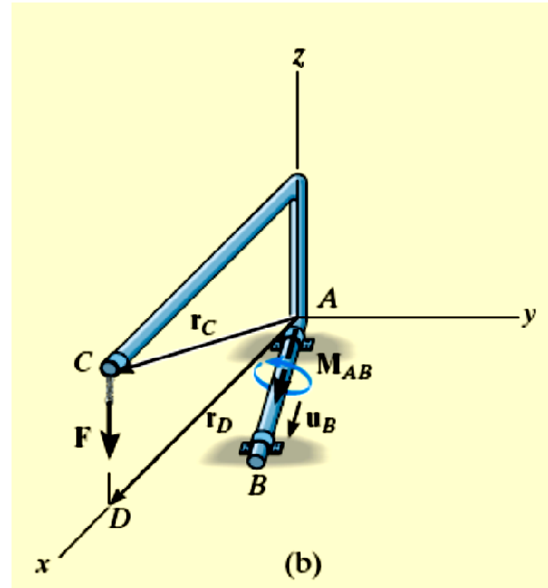
$$u_B = \frac{\mathbf{r}_B}{r_B}$$

$$= \frac{0.4\mathbf{i} + 0.2\mathbf{j}}{\sqrt{0.4^2 + 0.2^2}}$$

$$= 0.8944\ \mathbf{i} + 0.4472\ \mathbf{j}$$



- Since **r** is a position vector extending ***from any point on the AB axis to any point on the line of action*** of **F**, either $\mathbf{r}_C$, $\mathbf{r}_D$, $\mathbf{r}_{BC}$ or $\mathbf{r}_{BD}$ can be used.



For simplicity, take $\mathbf{r} = \mathbf{r}_D$ where

$$\mathbf{r}_D = \{0.6\ \mathbf{i}\}\ \text{m}$$

- The force is $\mathbf{F} = \{-300\ \mathbf{k}\}$ N

- Therefore,

$$M_{AB} = \mathbf{u}_B \cdot (\mathbf{r}_D \times \mathbf{F}) = \begin{vmatrix} 0.8944 & 0.4472 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & -300 \end{vmatrix}$$

$$= 0.8944\,[0 - 0] - 0.4472\,[(0.6)(-300) - 0] - 0$$

$$= 80.50\ \text{N·m}$$

- Hence, the moment $\mathbf{M}_{AB}$ produced by the force **F** which tends to rotate the rod about the *AB* axis.

$$\mathbf{M}_{AB} = M_{AB}\,\mathbf{u}_B$$

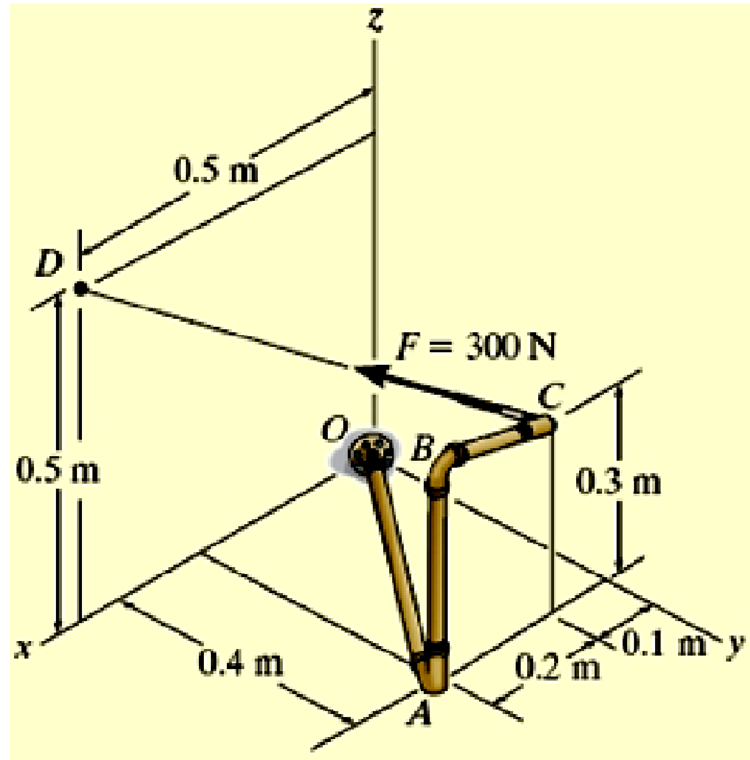
$$= (80.50\ \text{N·m})\,(0.8944\,\mathbf{i} + 0.4472\,\mathbf{j})$$

$$= \{72.0\mathbf{i} + 36.0\mathbf{j}\}\ \text{N·m}$$

Example 4.9

Given :

The force $F = 300$ N acts on the pipe assembly as shown.



Find :

Determine the magnitude of the moment of force **F** about segment *OA*.

Solution

- The moment of **F** about the *OA* axis, is given by

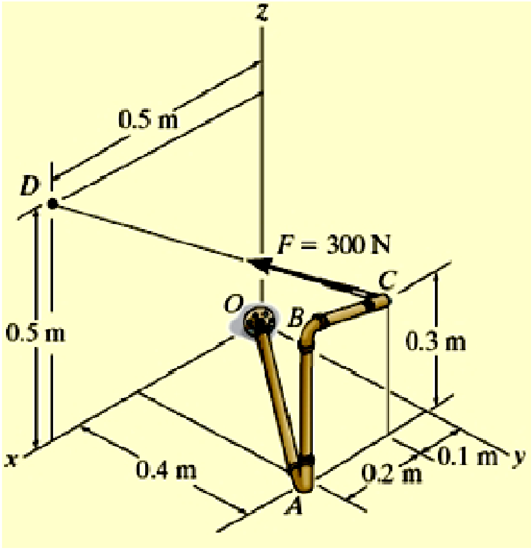
$$M_{OA} = \mathbf{u}_{OA} \bullet (\mathbf{r} \times \mathbf{F})$$

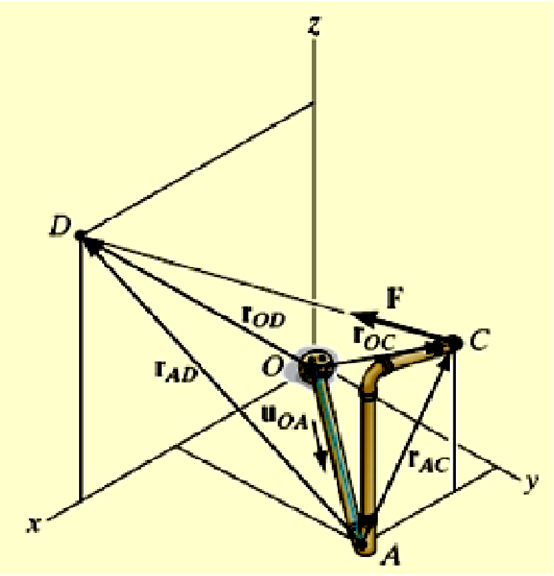
- The unit vector which defines the direction of the *OA* axis is

$$\mathbf{u}_{OA} = \frac{\mathbf{r}_{OA}}{r_{OA}}$$

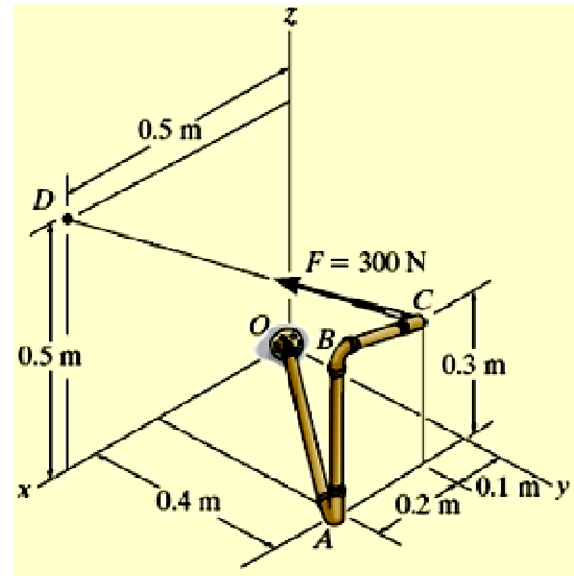
$$= \frac{0.3\mathbf{i} + 0.4\mathbf{j}}{\sqrt{(0.3)^2 + (0.4)^2}}$$

$$= 0.6\mathbf{i} + 0.8\mathbf{j}$$





- Since $\mathbf{r}$ is a position vector extending ***from any point on the OA axis to any point on the line of action*** of $\mathbf{F}$, either $\mathbf{r}_{OC}$, $\mathbf{r}_{OD}$, $\mathbf{r}_{AC}$ or $\mathbf{r}_{AD}$ can be used.



For simplicity, take $\mathbf{r} = \mathbf{r}_{OD}$ where

$$\mathbf{r}_{OD} = \{0.5\mathbf{i} + 0.5\mathbf{k}\} \text{ m}$$

- The force $\mathbf{F}$ in Cartesian vector form is

$$\mathbf{F} = F\,\mathbf{u}_{CD} = F\left(\frac{\mathbf{r}_{CD}}{r_{CD}}\right)$$

$$= F\left(\frac{(x_D - x_C)\mathbf{i} + (y_D - y_C)\mathbf{j} + (z_D - z_C)\mathbf{k}}{\sqrt{(x_D - x_C)^2 + (y_D - y_C)^2 + (z_D - z_C)^2}}\right)$$

$$= (300)\left(\frac{(0.5-0.1)\mathbf{i} + (0-0.4)\mathbf{j} + (0.5-0.3)\mathbf{k}}{\sqrt{(0.5-0.1)^2 + (0-0.4)^2 + (0.5-0.3)^2}}\right) = (300)\left(\frac{0.4\mathbf{i} - 0.4\mathbf{j} + 0.2\mathbf{k}}{0.6}\right)$$

$$= \{200\mathbf{i} - 200\mathbf{j} + 100\,\mathbf{k}\} \text{ N}$$

- Therefore,

$$M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r}_{OD} \times \mathbf{F}) = \begin{vmatrix} 0.6 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 200 & -200 & 100 \end{vmatrix}$$

$$= 0.6\,[0 - (0.5)(-200)] - 0.8\,[(0.5)(100) - (0.5)(200)] + 0$$

$$= 100\ \text{N}\cdot\text{m}$$