Gauss-Jordan Elimination

To solve a system, we use a technique called Gauss-Jordan elimination. We can use this technique to determine if the system has a unique solution, infinite solutions, or no solution.

Echelon Form and Reduced Echelon Form:

1. Echelon Form

   A matrix is in **echelon form** if it has leading ones on the **main diagonal** and zeros **below** the leading ones. Here are some examples of matrices that are in echelon form.

   Examples: $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

2. Reduced Echelon Form

   A matrix is in **reduced echelon form** if it has leading ones on the **main diagonal** and zeros **above** and **below** the leading ones. Here are some examples of matrices that are in reduced echelon form.

   Examples: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Row Operations Involved In Gauss-Jordan:

1. Swap any two rows.

   Example: R2 ←→ R1
2. Multiply or divide any row by a nonzero constant.

   Example: -1/2R3 R2 – 2R1
3. Add or subtract one row to a multiple of another row.

   Example: R2 – 2R1

Gaussian Elimination:

Gaussian Elimination puts a matrix in echelon form.

Example: Solve the system by using Gaussian Elimination.

$$2x + 5y = 12$$
$$x - 3y = -5$$

1. Put the matrix in augmented matrix form.

$$\left[\begin{array}{cc|c} 2 & 5 & 12 \\ 1 & -3 & -5 \end{array}\right]$$

2. Use row operations to put the matrix in echelon form.

$$\left[\begin{array}{cc|c} 2 & 5 & 12 \\ 1 & -3 & -5 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|c} 1 & -3 & -5 \\ 2 & 5 & 12 \end{array}\right] \xrightarrow{R2-2R1} \left[\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 11 & 22 \end{array}\right] \xrightarrow{\frac{1}{11}R2} \left[\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 1 & 2 \end{array}\right]$$

3. Write the equations from the echelon form matrix and solve the equations.

$$\left[\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{c} x-3y=-5 \\ y=2 \end{array} \Rightarrow \begin{array}{c} x=1 \\ y=2 \end{array}$$

The solution to this system is x = 1 and y = 2.

<u>Gauss-Jordan Elimination:</u>

Gauss-Jordan Elimination puts a matrix in reduced echelon form.

Example: Solve the system by using Gauss-Jordan Elimination.

$$2x_1 - 5x_2 + 4x_3 = 8$$
$$2x_1 \qquad + 2x_3 = 4$$
$$-x_1 - 2x_2 + x_3 = 2$$

1. Put the matrix in augmented matrix form.

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & 8 \\ 2 & 0 & 2 & 4 \\ -1 & -2 & 1 & 2 \end{array}\right]$$

2. Use row operations to put the matrix in echelon form.

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & 8 \\ 2 & 0 & 2 & 4 \\ -1 & -2 & 1 & 2 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 2 & -5 & 4 & 8 \end{array}\right] \xrightarrow[R3-2R1]{R2-2R1} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & 0 & 4 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 2 \\ 0 & -1 & 0 & 4 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R2} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 4 & 0 & 0 \end{array}\right] \xrightarrow{R3-4R2} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

By the last row, we have the equation 0 = 8, which is a contradiction. Therefore, this system has no solution.

Example: Solve the system by using Gauss-Jordan Elimination.

$$x_1 - 2x_2 + x_3 - 3x_3 = 0$$
$$3x_1 - 6x_2 + 2x_3 - 7x_4 = 0$$

1. Put the matrix in augmented matrix form.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 3 & -6 & 2 & -7 & 0 \end{array}\right]$$

2. Use row operations to put the matrix in echelon form.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 3 & -6 & 2 & -7 & 0 \end{array}\right] \xrightarrow{R2-3R1} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{-R2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R1-R2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

3. Write the equations from the echelon form matrix and solve the equations.

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - 2x_2 - x_4 = 0 \\ x_3 - 2x_4 = 0 \\ x_2 = r \\ x_4 = s \end{array} \Rightarrow \begin{array}{l} x_1 - 2r - s = 0 \\ x_3 - 2s = 0 \\ x_2 = r \\ x_4 = s \end{array} \Rightarrow \begin{array}{l} x_1 = 2r + s \\ x_3 = 2s \\ x_2 = r \\ x_4 = s \end{array}$$

The solution to this system is x_1 = 2r + s, x_2 = 2s, x_3 = r, and x_4 = s.

Unique Solution, No Solution, or Infinite Solutions?

Here are some tips that will allow you to determine what type of solutions you have based on either the reduced echelon form.

1. If you have a leading one in every column, then you will have a **unique solution**.
2. If you have a **row of zeros** equal to a number for a **nonhomogeneous** system, ie a system where the equations equal different numbers, then the system has no solution.
3. If you don't have a leading one in every column in a **homogeneous** system, ie a system where all the equations equal zero, or a row or a row of zeros, then you have **infinite solutions**.