INITIAL VALUE PROBLEMS

REMEMBER
NUMERICAL DIFFERENTIATION

The derivative of $y(x)$ at x_0 is:

$$y'(x_0) = \lim_{h \to 0} \frac{y(x_0 + h) - y(x_0)}{h}$$

An approximation to this is:

$$y'(x_0) \approx \frac{y(x_0 + h) - y(x_0)}{h}$$

for small values of h.

Forward Difference Formula

$$y'(x_0) \approx \frac{y(x_0 + h) - y(x_0)}{h}$$

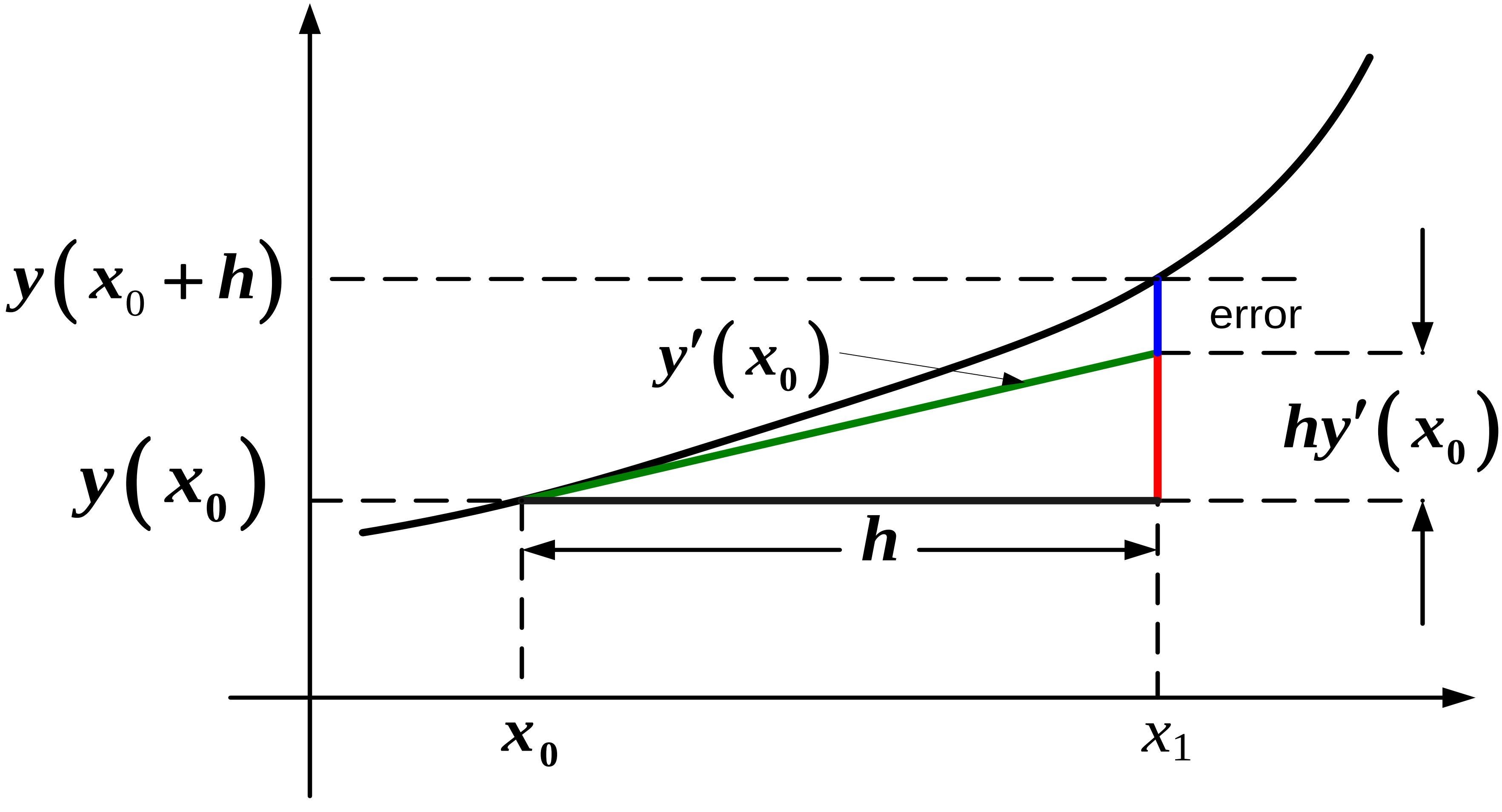
$$hy'(x_0) \approx y(x_0 + h) - y(x_0)$$

$$y(x_0) + hy'(x_0) \approx y(x_0 + h)$$

$$y(x_0 + h) \approx y(x_0) + hy'(x_0)$$

Provided h is small.

$$y(x_0 + h) \approx y(x_0) + hy'(x_0)$$



$$y\left(x_0 + h\right) \approx y(x_0) + hy'\left(x_0\right)$$

The value of a function at $x_0 + h$

can be approximated if we know its value

and its slope at an earlier point, x_0

provided h is small.

A sequence can be written as:

$$y_1 \approx y_0 + hy_0'$$

$$y_1 \approx y_0 + hy'_0$$

$$\boldsymbol{y_2 \approx y_1 + hy'_1}$$

$$\boldsymbol{y_{n+1} \approx y_n + hy'_n}$$

In an initial value problem, y'_n **is given**

and often written as: $f\left(x_n, y_n\right)$

Therefore, $\boldsymbol{y_{n+1} \approx y_n + hf\left(x_n, y_n\right)}$

Numerical Solution of Initial Value Problems

Euler's Method

$$y_{n+1} \approx y_n + hf\left(x_n, y_n\right)$$

$y_n = y(x_n)$ Value of y at $x = x_n$

$y_{n+1} = y(x_{n+1})$ Value of y at $x = x_{n+1}$

$$f(x_n, y_n) = y'(x_n)$$

First derivative of y **at** $x = x_n$

h **is a small number by which** ***x*** **is incremented.**

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h = x_0 + 2h$$

$$x_3 = x_2 + h = x_0 + 3h$$

$$x_n = x_0 + nh$$

So what are we doing in Euler's method?

We are approximating the value of the function y at $x = x_{n+1}$ based on the information that we obtained at the previous point, $x = x_n$.

The previous information are:

The value of y at $x = x_n$

The value of the first derivative of y at $x = x_n$

$$y_{n+1} \approx y_n + hf\left(x_n, y_n\right)$$

Consider the following initial value problem.

$$y' = y - x \quad , \quad y(0) = 0.5$$

Find the value of $y(0.8)$.

Let $h = 0.1$

Given:

$$f\left(x_n, y_n\right) = y'(\text{at } x = x_n) = y_n - x_n$$

$y(\text{at } x = x_0) = y_0 = 0.5$ **Our starting point.**

where $x_0 = 0$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$$

Based on the sequence generated by Euler's method we can write:

$$y_1 \approx y_0 + hf\left(x_0, y_0\right)$$

$$f(x_0, y_0) = y_0 - x_0 = 0.5 - 0 = 0.5$$

$$y_1 \approx 0.5 + 0.1 \times 0.5 = 0.55$$

$$y_2 \approx y_1 + hf\left(x_1, y_1\right)$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$f(x_1, y_1) = y_1 - x_1 = 0.55 - 0.1 = 0.45$$

$$y_2 \approx 0.55 + 0.1 \times 0.45 = 0.595$$

$$y_3 \approx y_2 + hf\left(x_2, y_2\right)$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$$

$$f(x_2, y_2) = y_2 - x_2 = 0.595 - 0.2 = 0.395$$

$$y_3 \approx 0.595 + 0.1 \times 0.395 = 0.6345$$

Approximate value of *y* at $x = x_3$ is 0.6345.

Therefore, we can write that

$$y(0.1) = y_1 \approx 0.55$$

$$y(0.2) = y_2 \approx 0.595$$

$$y(0.3) = y_3 \approx 0.6345$$

$$\cdot$$

$$\cdot$$

$$y(0.8) = y_8 \approx 0.728205$$

Modified Euler's Method (Heun's Method)

In the Modified Euler's method an average value of the slope is used.

$$y_{n+1} \approx y_n + h\left[\frac{1}{2}\left\{f\left(x_n, y_n\right) + f\left(x_{n+1}, y^*_{n+1}\right)\right\}\right]$$

Where y^*_{n+1} **is calculated using the Euler's method.**

$$y^*_{n+1} \approx y_n + hf\left(x_n, y_n\right)$$

$$y^*_{n+1} \approx y_n + hf(x_n, y_n)$$

y^*_{n+1} **as calculated using the Euler's method shown above is used to obtain the 2nd slope.**

$$f\left(x_{n+1}, y^*_{n+1}\right)$$

$$y_{n+1} \approx y_n + h\left[\frac{1}{2}\left\{f\left(x_n, y_n\right) + f\left(x_{n+1}, y^{*}_{n+1}\right)\right\}\right]$$

Consider the previous initial value problem.

$$y' = y - x \quad , \quad y(0) = 0.5$$

Find the value of ***y*(0.8).**

$$\text{Again, let } h = 0.1$$

$$f(x_0, y_0) = y_0 - x_0 = 0.5 - 0 = 0.5$$

$$y_1^* \approx 0.5 + 0.1 \times 0.5 = 0.55$$

$$f(x_1, y_1^*) = y_1^* - x_1 = 0.55 - 0.1 = 0.45$$

$$y_1 \approx y_0 + h\left[\frac{1}{2}\left\{f\left(x_0, y_0\right) + f\left(x_1, y_1^*\right)\right\}\right]$$

$$y_1 \approx y_0 + h\left[\frac{1}{2}\left\{f\left(x_0, y_0\right) + f\left(x_1, y_1^*\right)\right\}\right]$$

$$y_1 \approx 0.5 + 0.1\left[\frac{1}{2}\left\{0.5 + 0.45\right\}\right] = 0.5475$$

$$f(x_1, y_1) = y_1 - x_1 = 0.5475 - 0.1$$

$$= 0.4475$$

$$y_2^* \approx 0.5475 + 0.1 \times 0.4475$$

$$\approx 0.59225$$

$$f(x_2, y_2^*) = y_2^* - x_2 = 0.59225 - 0.2$$

$$= 0.39225$$

$$y_2 \approx y_1 + h\left[\frac{1}{2}\left\{f\left(x_1, y_1\right) + f\left(x_2, y_2^*\right)\right\}\right]$$

$$y_2 \approx 0.5475 + 0.1\left[\frac{1}{2}\left\{0.4475 + 0.39225\right\}\right]$$

$$\approx 0.589487$$

Taylor's Method of Order Two

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + R_2$$

R_2 is called Taylor's remainder.

If h is small then:

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2}{2!}y''(x)$$

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2}{2!} y''(x)$$

Taylor's method of order two can be written in the form of the following sequence.

$$y_{n+1} \approx y_n + hy'_n + \frac{h^2}{2} y''_n$$

Solve the following IVP by using Taylor's method of order two.

$$y' = -y + x + 1, \quad 0 \le x \le 1, \quad y(0) = 1$$

Solution:

$$y'' = -y' + 1 = -(-y + x + 1) + 1 = y - x$$

$$y_{n+1} \approx y_n + h y'_n + \frac{h^2}{2} y''_n$$

$$y'_n = -y_n + x_n + 1$$

$$y''_n = y_n - x_n$$

$$y_{n+1} \approx y_n + hy'_n + \frac{h^2}{2} y''_n$$

$$y_{n+1} \approx y_n + h(-y_n + x_n + 1) + \frac{h^2}{2}(y_n - x_n)$$

$$y_{n+1} = \left(1 - h + \frac{h^2}{2}\right) y_n + \left(h - \frac{h^2}{2}\right) x_n + h$$

Let *h* = 0.1

$$\boldsymbol{y}_{\boldsymbol{n}+1} = 0.905\boldsymbol{y}_{\boldsymbol{n}} + 0.095\boldsymbol{x}_{\boldsymbol{n}} + 0.1$$

Results with Taylor's Method of Order Two

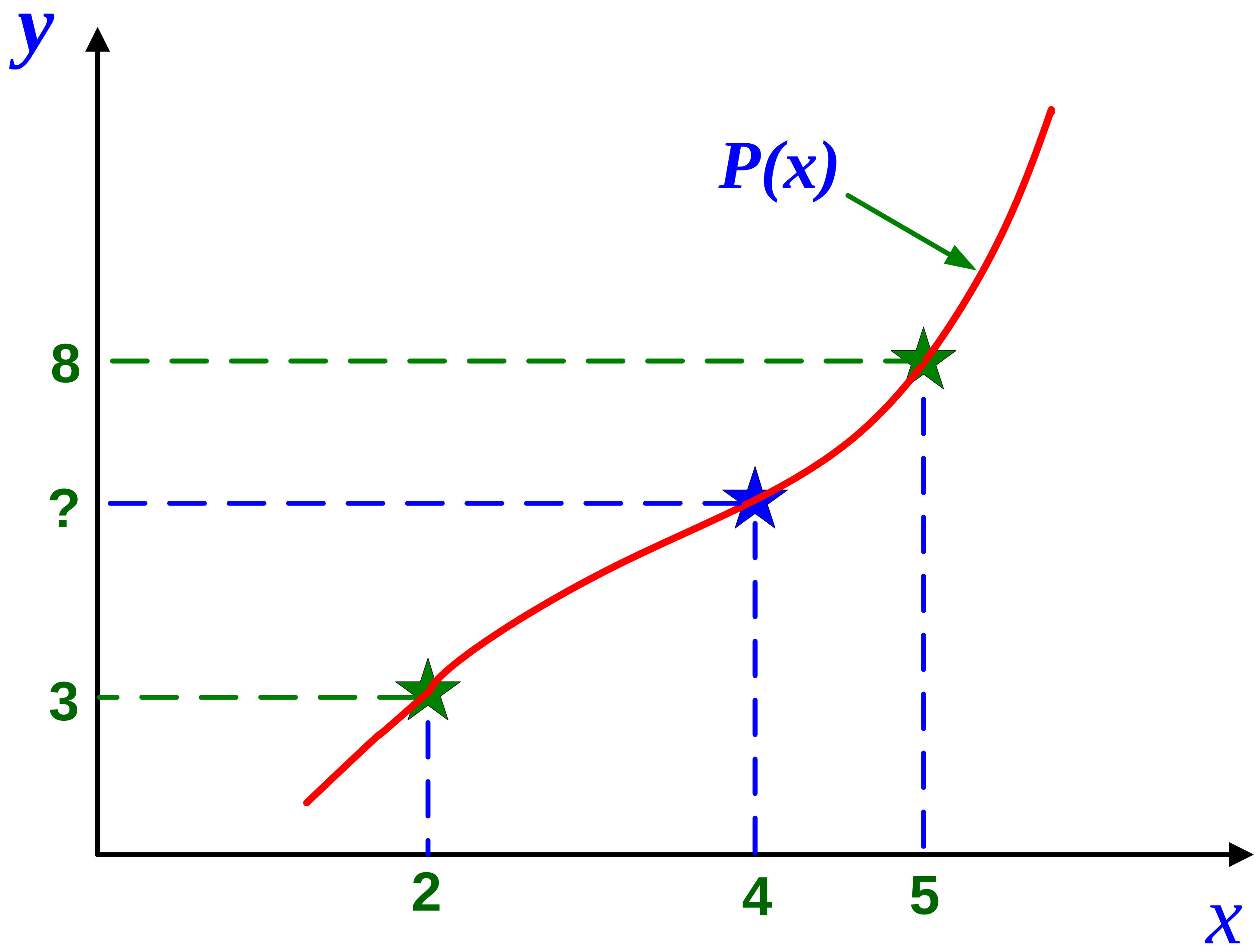
n	x_n	y_{n+1}
0	0.0	1.005
1	0.1	1.019025
2	0.2	1.041218
3	0.3	1.070802
4	0.4	1.107076

INTERPOLATION

Interpolation produces a function that matches the given data exactly. The function then should provide a good approximation to the data values at intermediate points.

Interpolation may also be used to produce a smooth graph of a function for which values are known only at discrete points, either from measurements or calculations.

- ✓ **Given data points**
- ✓ **Obtain a function, P(x)**
- ✓ **P(x) goes through the data points**
- ✓ **Use P(x)**
- ✓ **To estimate values at intermediate points**


y
P(x)
8
?
3
2
4
5
x

Assume that a function goes through three points:

$$\left(x_0, y\left(x_0\right)\right), \left(x_1, y\left(x_1\right)\right) \text{ and } \left(x_2, y\left(x_2\right)\right).$$

$$y(x) \approx P(x)$$

$$P\left(x\right) = L_0\left(x\right) y\left(x_0\right) + L_1\left(x\right) y\left(x_1\right) + L_2\left(x\right) y\left(x_2\right)$$

Lagrange Interpolating Polynomial

$$P(x) = L_0(x)\,y(x_0) + L_1(x)\,y(x_1) + L_2(x)\,y(x_2)$$

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,y(x_0)$$

$$+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,y(x_1)$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,y(x_2)$$

$$y'(x) \approx P'(x)$$

$$P'\left(x\right) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} y\left(x_0\right)$$

$$+ \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} y\left(x_1\right)$$

$$+ \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} y\left(x_2\right)$$

NUMERICAL INTEGRATION

$$\int_a^b y(x)dx =$$ **area under the curve *f(x)* between** $x = a$ **to** $x = b.$

In many cases a mathematical expression for *y(x)* is unknown and in some cases even if *y(x)* is known its complex form makes it difficult to perform the integration.

Simpson's Rule:

$$\int_{x_0}^{x_2} y(x)\,dx \approx \int_{x_0}^{x_2} P(x)\,dx$$

$$x_1 = x_0 + h \text{ and } x_2 = x_0 + 2h$$

$$\int_{x_0}^{x_2} P(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y(x_0)\,dx$$

$$+\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y(x_1)\,dx$$

$$+\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y(x_2)\,dx$$

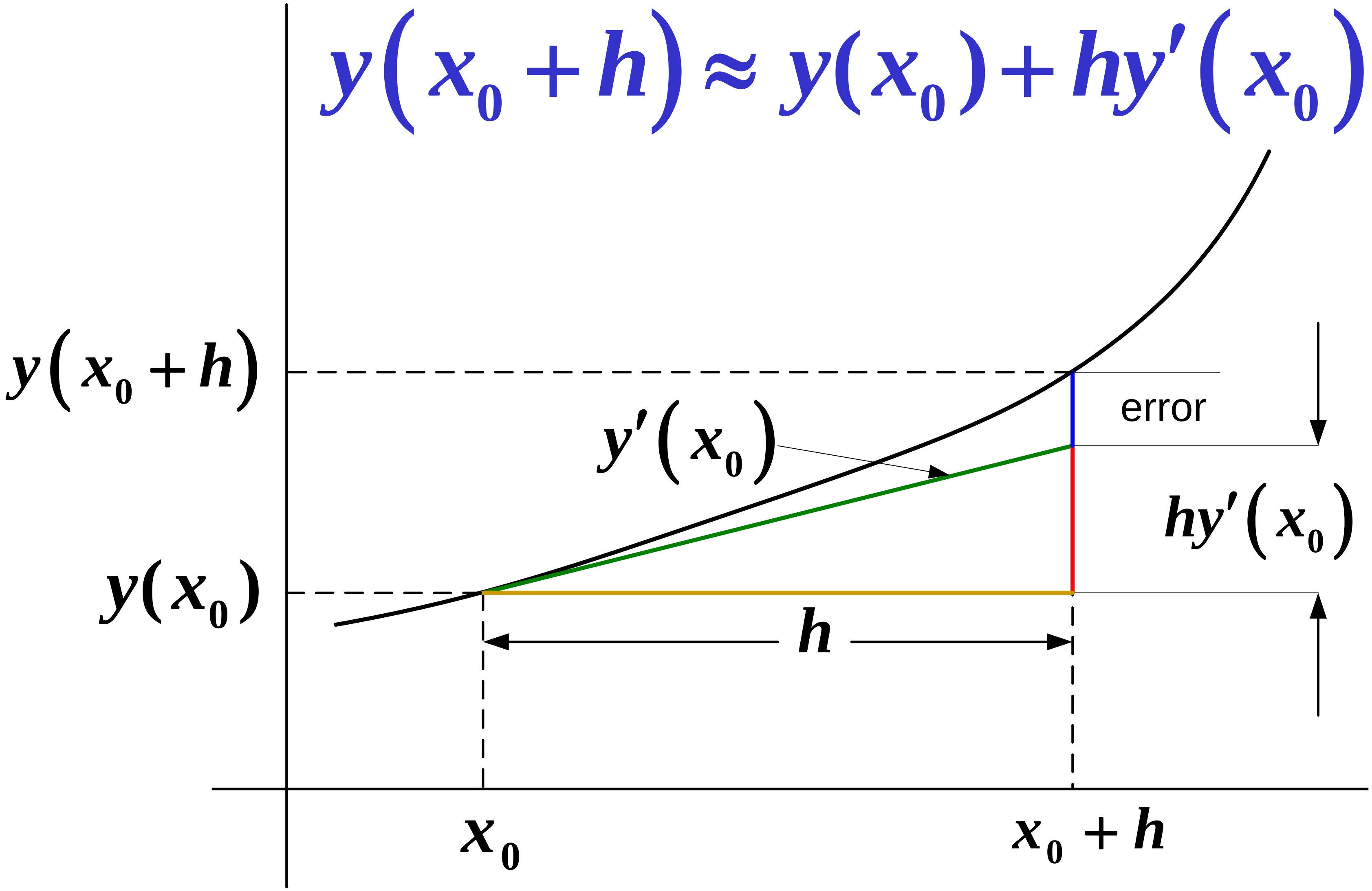
$$\int_{x_0}^{x_2} y(x)\,dx \approx \int_{x_0}^{x_2} P(x)\,dx$$

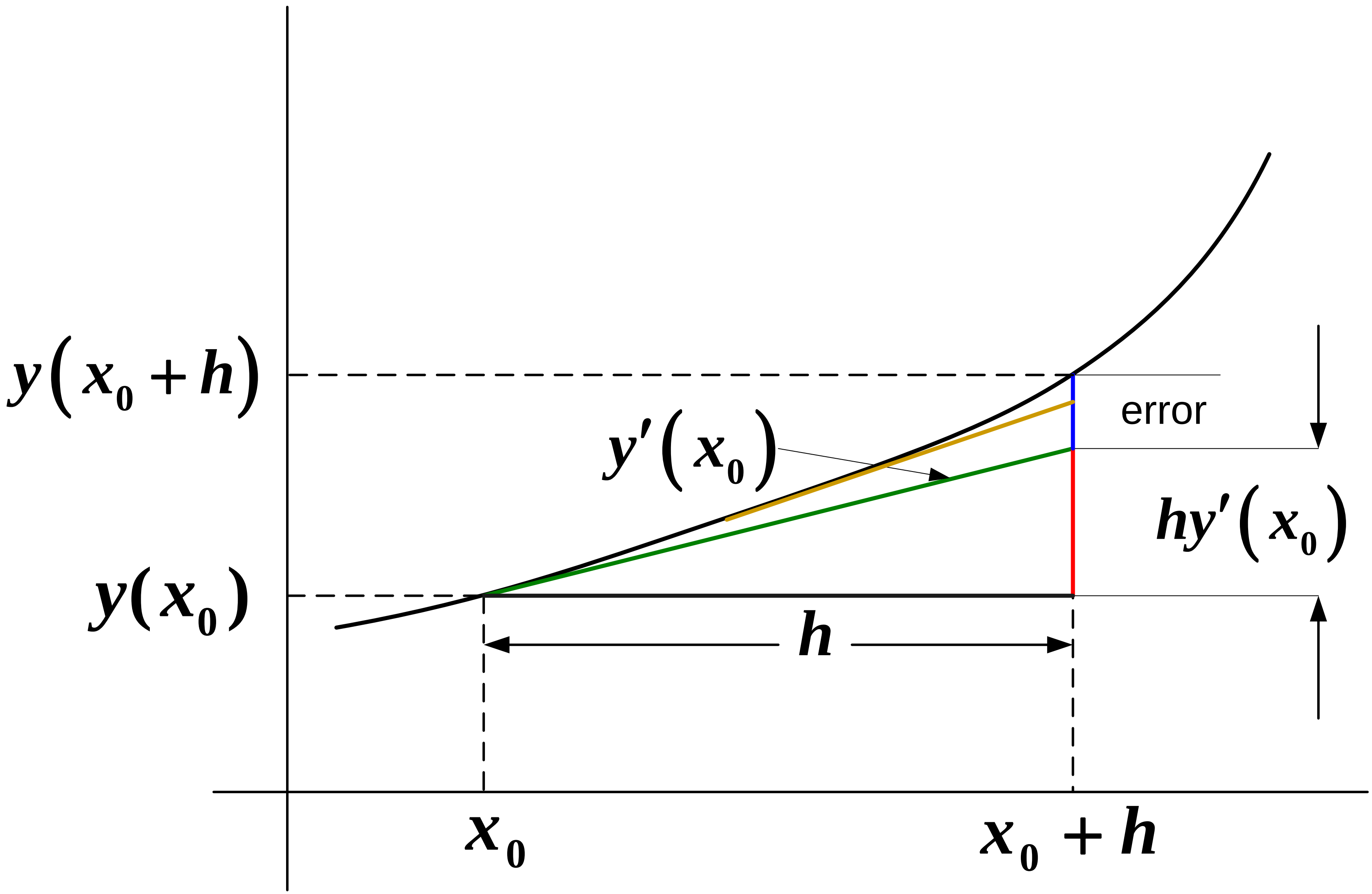
$$= \frac{h}{3}\left[y(x_0) + 4y(x_1) + y(x_2)\right]$$

Runge-Kutta Method of Order Four

Runge-Kutta method uses a sampling of slopes through an interval and takes a weighted average slope to calculate the end point. By using fundamental theorem as shown in Figure 1 we can write:

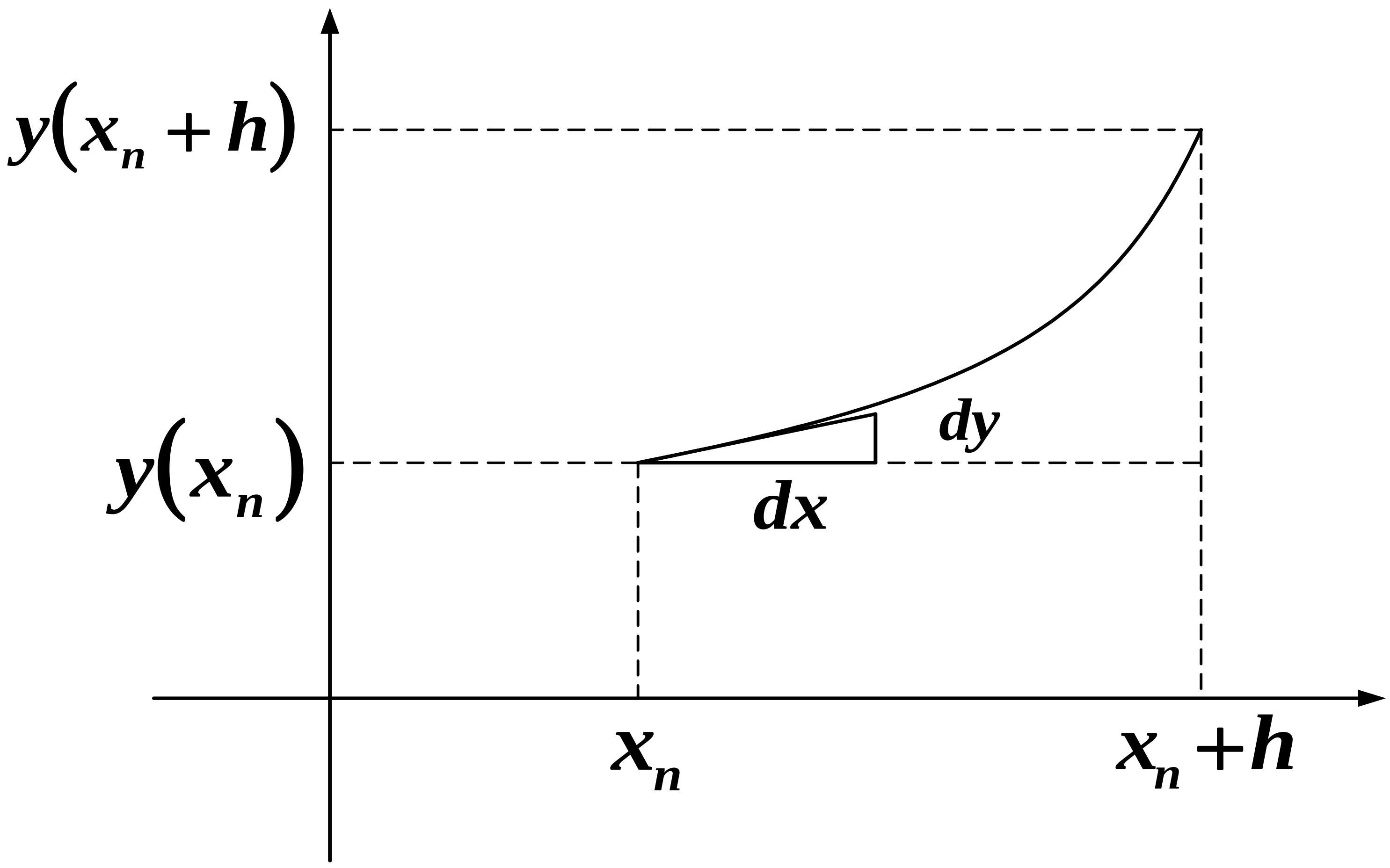
$$y\left(x_0+h\right)\approx y\left(x_0\right)+hy'\left(x_0\right)$$





$y(x_0+h)$
error
$y'(x_0)$
$hy'(x_0)$
$y(x_0)$
h
x_0
x_0+h

$$dy = y'(x)dx \qquad \cdots (1)$$



Integrating both sides of Eqn. (1) we get

$$\int dy = \int y'(x)dx \qquad \cdots \ (2)$$

Applying appropriate limits to Eqn. (2) we get

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'(x)dx \qquad \cdots \ (3)$$

$$y(x_n + h) - y(x_n) = \int_{x_n}^{x_n+h} y'(x)dx \qquad \cdots \ (4)$$

Let us now concentrate on the right-hand side of Eqn. (4).

$$\int_{x_n}^{x_n+h} y'(x)\,dx \approx \int_{x_n}^{x_n+h} P(x)\,dx \quad \cdots \ (5)$$

$P(x)$ can be generated by utilizing *Lagrange Interpolating Polynomial*. Assume that the only information we have about a function, $f(x)$ is that it goes through three points:

$$\left(x_0, y(x_0)\right), \left(x_1, y(x_1)\right) \text{ and } \left(x_2, y(x_2)\right).$$

$$P(x) \approx y(x)$$

$$P(x) = L_0(x)\,y(x_0) + L_1(x)\,y(x_1) + L_2(x)\,y(x_2)$$

$$P(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\,y(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\,y(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\,y(x_2)$$

Using Simpson's Integration,

$$\int_{x_0}^{x_2} y(x)\,dx \approx \int_{x_0}^{x_2} P(x)\,dx$$

$$x_1 = x_0 + h' \quad \text{and} \quad x_2 = x_0 + 2h'$$

$$\int_{x_0}^{x_2} P(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\, y(x_0)\,dx$$

$$+\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\, y(x_1)\,dx$$

$$+\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\, y(x_2)\,dx$$

$$\int_{x_0}^{x_2} y(x)\,dx \approx \int_{x_0}^{x_2} P(x)\,dx = \frac{h'}{3}\left[y(x_0)+4y(x_1)+y(x_2)\right]$$

..........................(6)

We can apply Simpson's Integration to Eqn. (6) with the following substitutions:

$$x_0 = x_n \;\; ; \;\; x_2 = x_n + h \;\; ;$$

$h' = h/2$ and the midpoint, $x_1 = x_n + \dfrac{h}{2}$

$$\int_{x_n}^{x_n+h} y'(x)dx \approx \frac{h}{6}\left[y'(x_n) + 4y'\left(x_n + \frac{h}{2}\right) + y'(x_n + h)\right]$$

....(7)

Compared to Euler's Formula, an average of six slopes is used in Eqn. (7) instead of just one slope. Actually, the slope at the midpoint has a weight of 4. The slope at the midpoint can be estimated in two ways.

$$\int_{x_n}^{x_n+h} y'(x)dx \approx \frac{h}{6}\left[y'(x_n) + 2y'\left(x_n + \frac{h}{2}\right) + 2y'\left(x_n + \frac{h}{2}\right) + y'(x_n + h)\right]$$

$$y(x_n + h) - y(x_n) \approx \frac{h}{6}\left[y'(x_n) + 2y'\left(x_n + \frac{h}{2}\right) + 2y'\left(x_n + \frac{h}{2}\right) + y'(x_n + h)\right]$$

$$y(x_n + h) \approx y(x_n) + \frac{h}{6}\left[y'(x_n) + 2y'\left(x_n + \frac{h}{2}\right) + 2y'\left(x_n + \frac{h}{2}\right) + y'(x_n + h)\right]$$

$$y(x_{n+1}) \approx y(x_n) + \frac{h}{6}\left[y'(x_n) + 2y'\left(x_n + \frac{h}{2} \right) + 2y'\left(x_n + \frac{h}{2} \right) + y'(x_{n+1}) \right]$$

………… (8)

The slopes can be estimated in the following manner:

$$y'(x_n) \approx k_1 = f(x_n, y_n)$$

$$y'\left(x_n + \frac{h}{2} \right) \approx k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1 \right)$$

$$y'\left(x_n + \frac{h}{2}\right) \approx k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$y'(x_n + h) \approx k_4 = f(x_n + h, y_n + hk_3)$$

Substituting the estimated slopes into Eqn. (8) gives the formula for Runge-Kutta Method of Fourth Order:

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$