

Isentropic flow through Nozzles

- Conservation of momentum for a inviscid and steady flow

$$dp + \frac{1}{2} \rho d(V^2) + \gamma dz = 0$$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

- conservation of mass

$$\dot{m} = \rho AV = \text{const.}$$

Since $\dot{m} = \rho AV = c$, $\therefore \ln \rho + \ln A + \ln V = c'$

differentiation $\rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

$$\rightarrow -\frac{dV}{V} = \frac{d\rho}{\rho} + \frac{dA}{A} \quad \left(= \frac{dp}{\rho V^2} \right)$$

$$\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho} = \frac{dp}{\rho V^2} \left(1 - \frac{d\rho}{\rho} \cdot \frac{\rho V^2}{dp}\right)$$

$$= \frac{dp}{\rho V^2} \left(1 - \frac{V^2}{dp/d\rho}\right)$$

Since $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$, and $\text{Ma} = \frac{V}{c}$

$$\therefore \frac{dp}{\rho V^2} (1 - \text{Ma}^2) = \frac{dA}{A}$$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

$$\frac{dp}{\rho V^2} = \frac{dA}{A} \frac{1}{1 - \text{Ma}^2} = \frac{dV}{V}$$

$$-\frac{dp}{\rho V^2} = -\frac{dA}{A} \frac{1}{1 - Ma^2} = \frac{dV}{V}$$

diverging duct

$$\frac{dA}{A} = \frac{dP}{\rho V^2} (1 - Ma^2)$$

Flow →



converging duct

Flow →



Subsonic flow
($Ma < 1$)

$$dA > 0$$

$$dV < 0$$

Supersonic flow
($Ma > 1$)

$$dA > 0$$

$$dV > 0$$

(a)

$$-\frac{dp}{\rho V^2} = -\frac{dA}{A} \frac{1}{1 - Ma^2} = \frac{dV}{V}$$

$$dA < 0$$

$$dV > 0$$

$$dA < 0$$

$$dV < 0$$

(b)



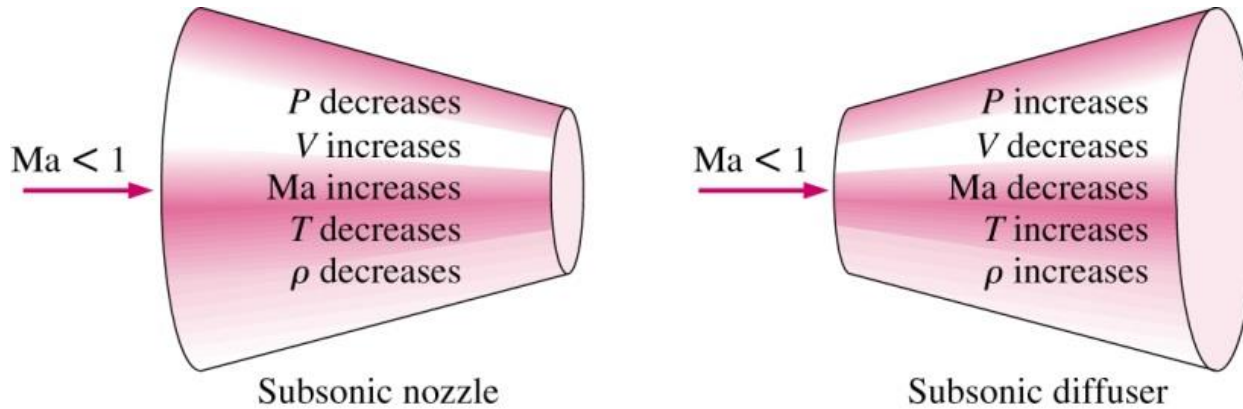
One-Dimensional Isentropic Flow

Variation of Fluid Velocity with Flow Area

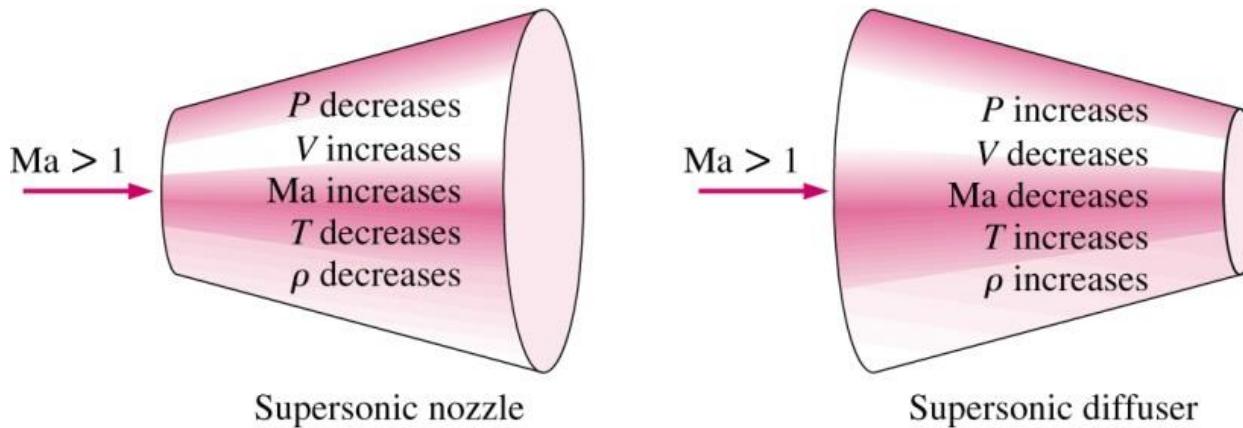
$$\frac{dA}{A} = \frac{dP}{\rho V^2} (1 - Ma^2) \quad -\frac{dp}{\rho V^2} = -\frac{dA}{A} \frac{1}{1 - Ma^2} = \frac{dV}{V}$$

- This is an important relationship
 - For $Ma < 1$, $(1 - Ma^2)$ is positive \Rightarrow dA and dP have the same sign and dV has opposite sign.
 - Pressure of fluid must increase as the flow area of the duct increases, and must decrease as the flow area decreases
 - For $Ma > 1$, $(1 - Ma^2)$ is negative \Rightarrow dA and dP have opposite signs and dV has same sign.
 - Pressure must increase as the flow area decreases, and must decrease as the area increases

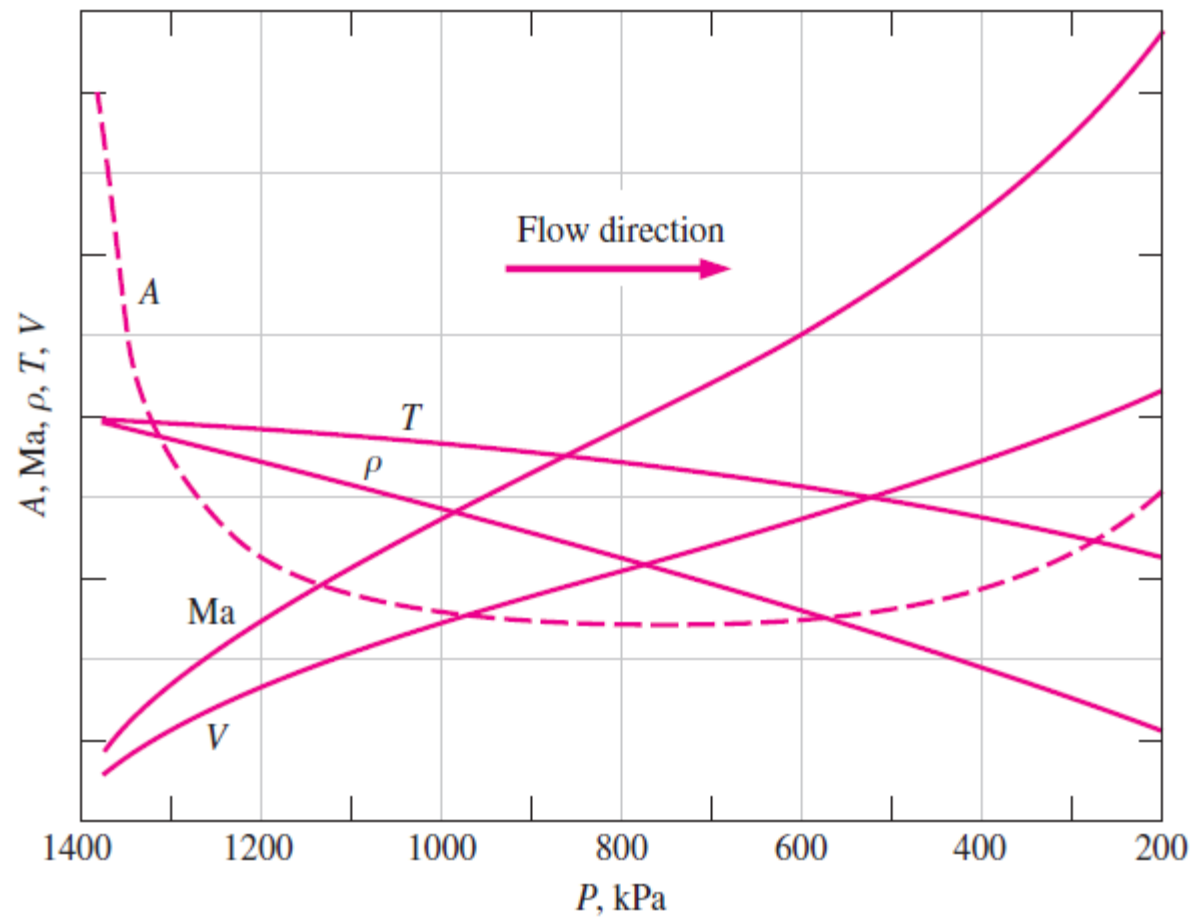
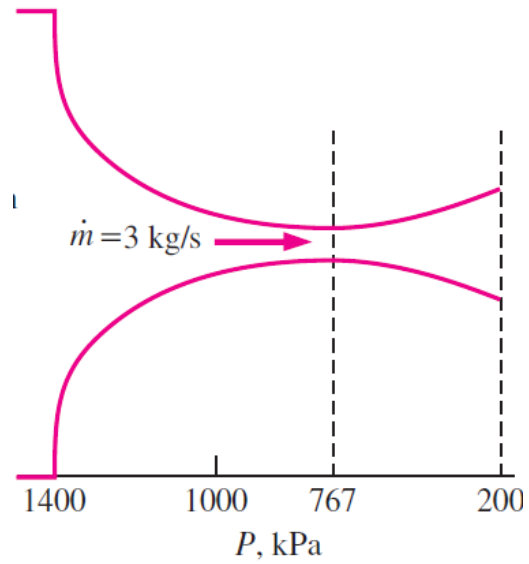
Comparison of flow properties in subsonic and supersonic nozzles and diffusers

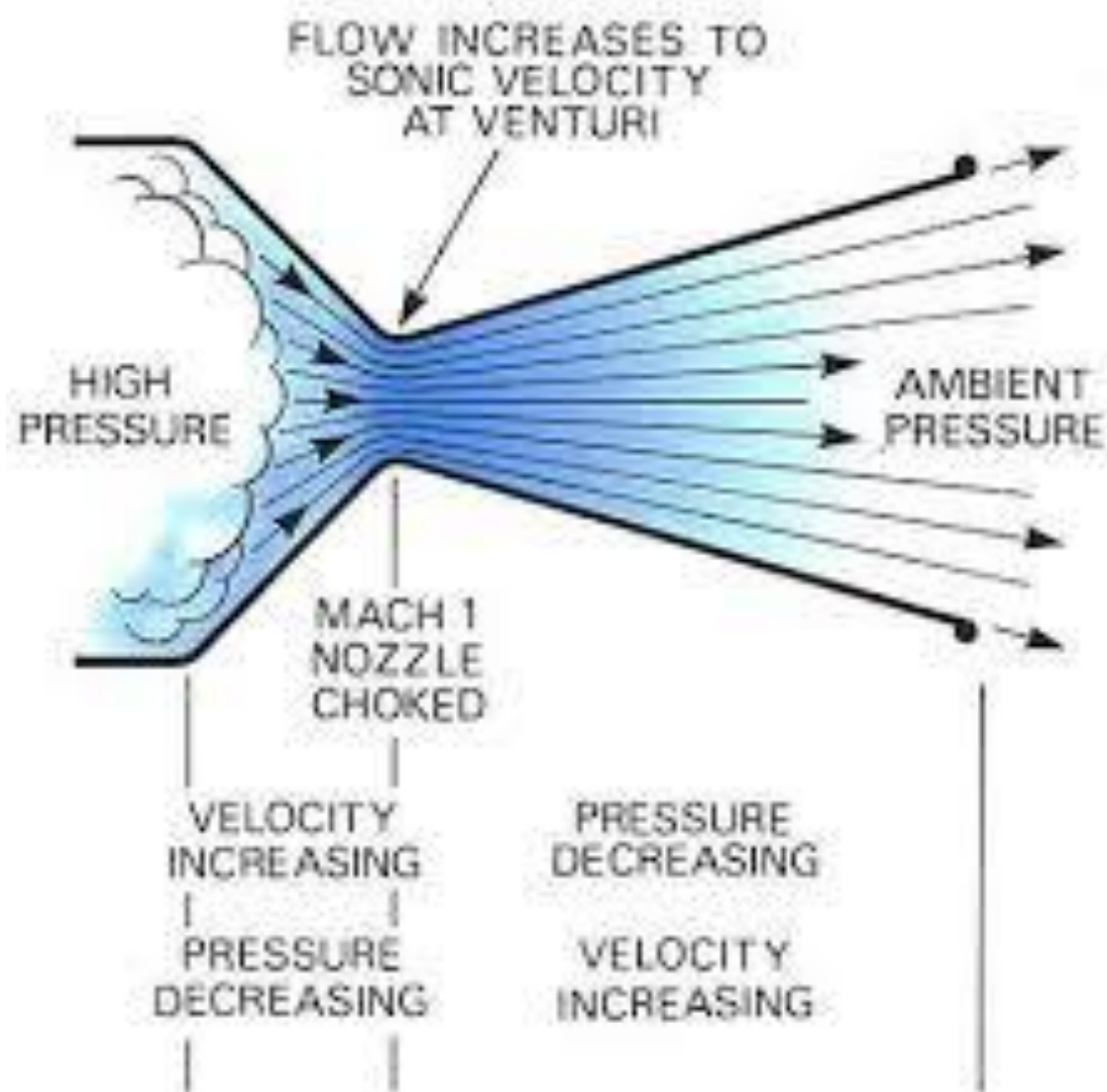


(a) Subsonic flow



(b) Supersonic flow

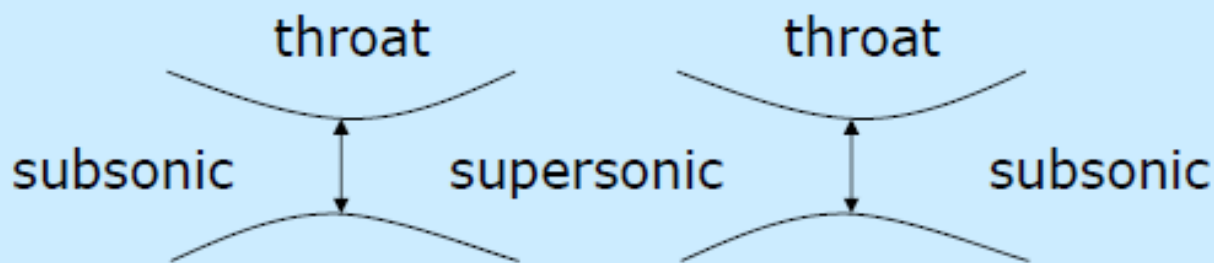




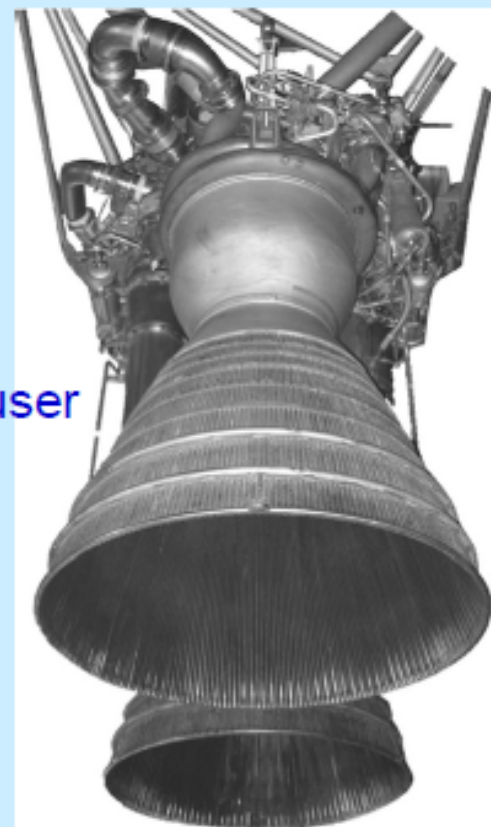


Therefore the sonic conduction $Ma=1$ can be obtained in a converging-diverging duct at the minimum area location.

- For subsonic flow \rightarrow converging diverging **nozzle**
- For supersonic flow \rightarrow converging diverging **diffuser**



Converging-diverging **nozzle** Converging-diverging **diffuser**



Ratio A/A^*

$$\rho AV = \rho^* A^* V^* \quad \text{or} \quad \frac{A}{A^*} = \frac{\rho^* V^*}{\rho V}$$

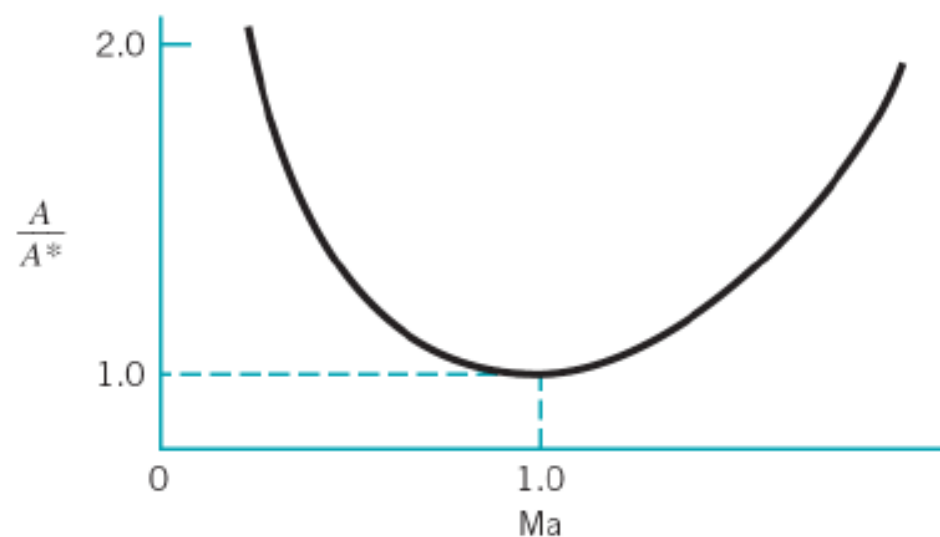
$$\text{with } V^* = \sqrt{kRT^*} \quad \text{and} \quad V = \text{Ma} \sqrt{kRT}$$

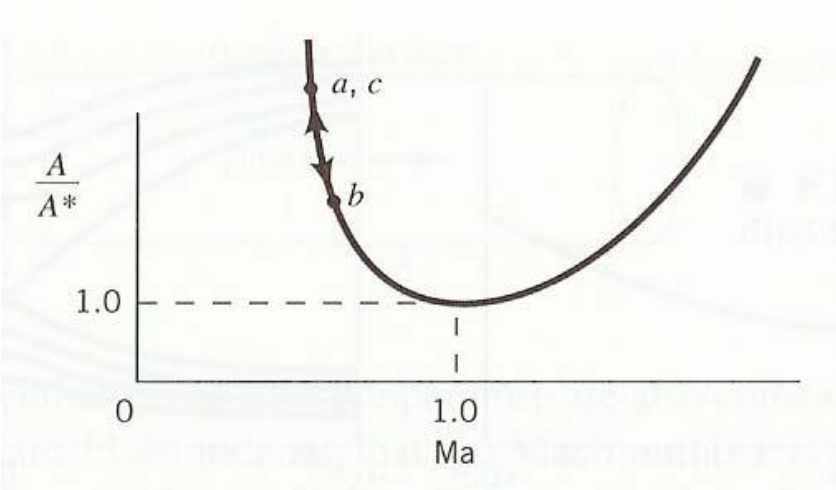
$$\begin{aligned} \rightarrow \frac{A}{A^*} &= \frac{\rho^* \sqrt{kRT^*}}{\rho \sqrt{kRT} \text{Ma}} = \frac{1}{\text{Ma}} \frac{\rho^*}{\rho_0} \sqrt{\frac{T^*/T_0}{T/T_0}} \\ &= \frac{1}{\text{Ma}} \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \left[1 + \frac{k+1}{2} \text{Ma}^2\right]^{\frac{1}{k-1}} \left[\frac{2/k+1}{1/1 + [(k-1)/2] \text{Ma}^2}\right]^{\frac{1}{2}} \end{aligned}$$

$$\Rightarrow \frac{A}{A^*} = \frac{1}{\text{Ma}} \left[\frac{1 + [(k-1)/2] \text{Ma}^2}{1 + [(k-1)/2]}\right]^{\frac{k+1}{2(k-1)}}$$

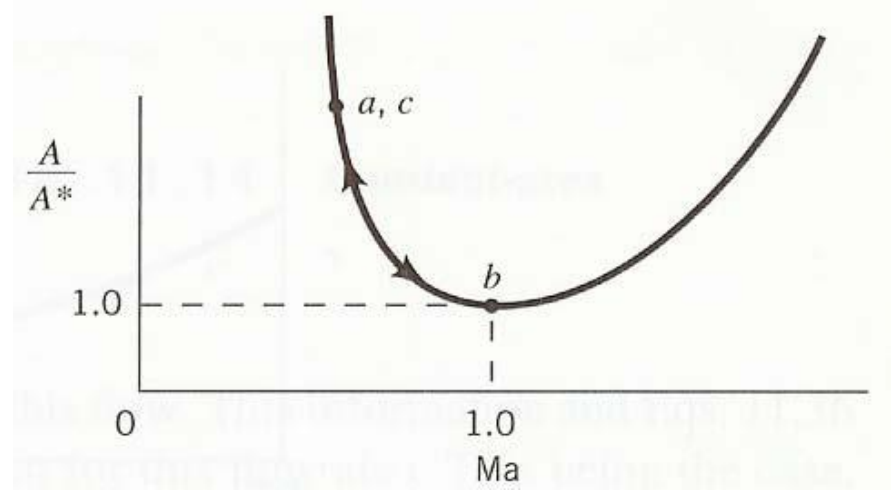
Figure 11.10 (p. 639)

The variation of area ratio with Mach number for **isentropic flow** of an ideal gas ($k = 1.4$, linear coordinate scales).

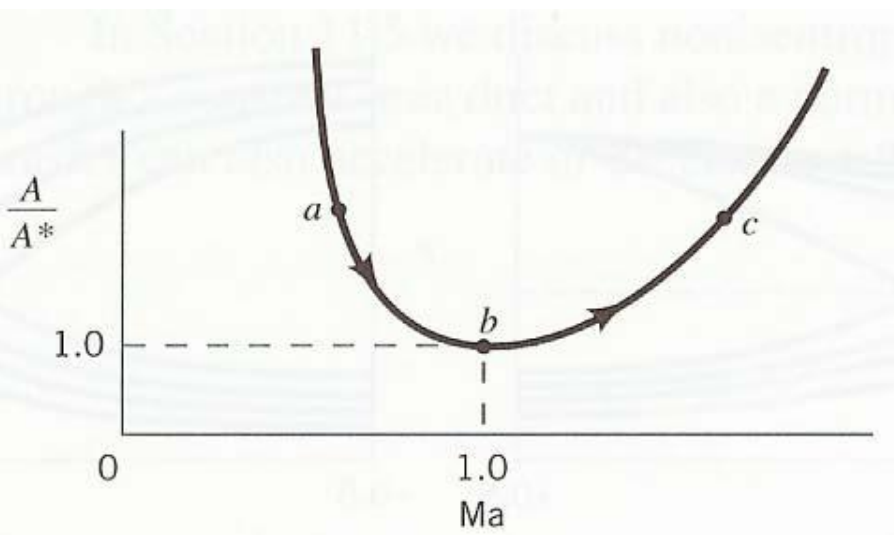




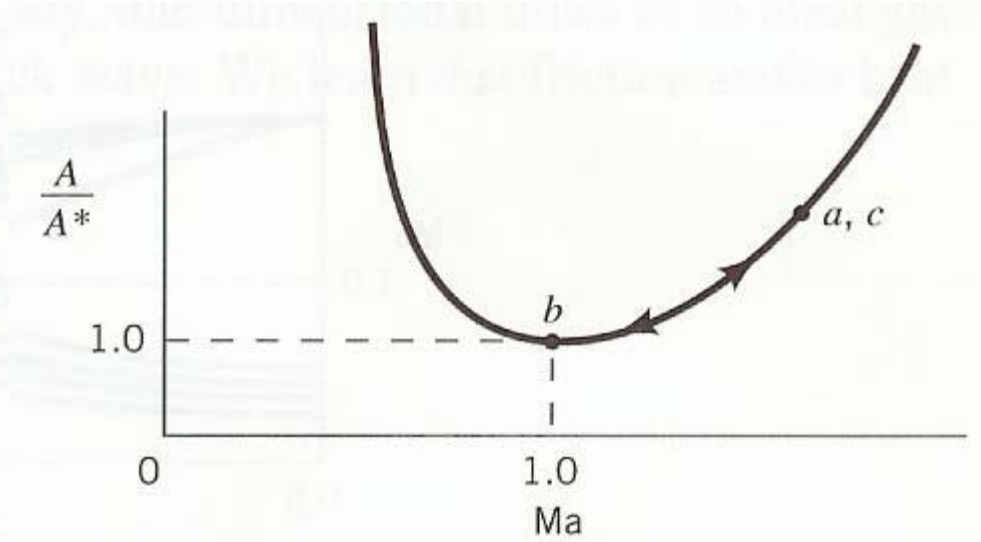
Subsonic -Subsonic



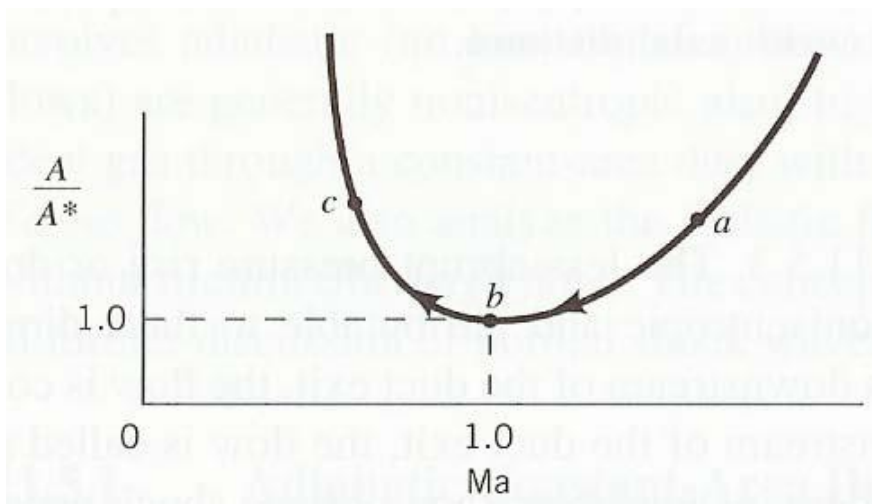
Subsonic- Subsonic (Chocked)



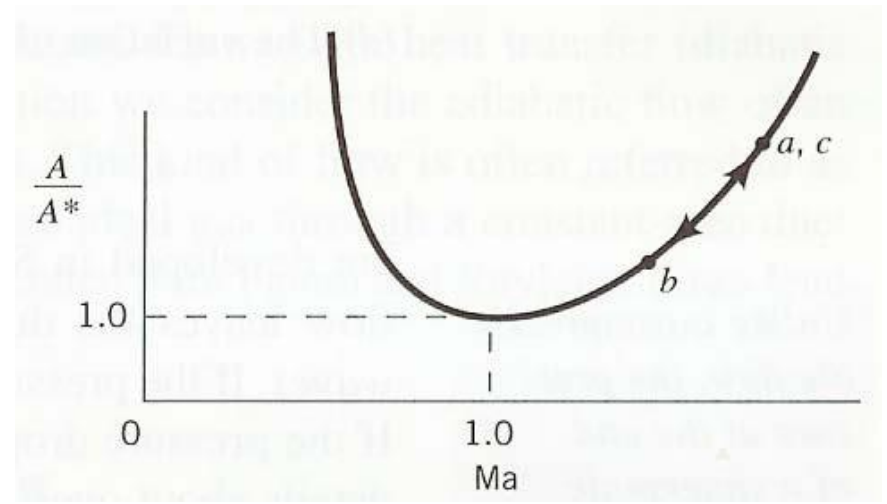
Subsonic – Supersonic (Chocked)



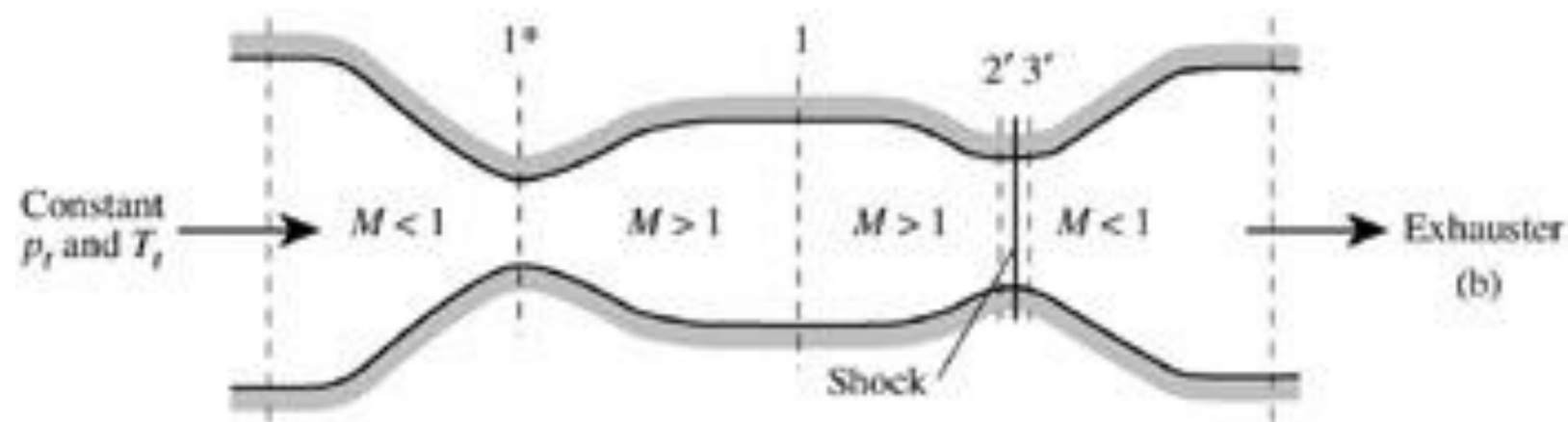
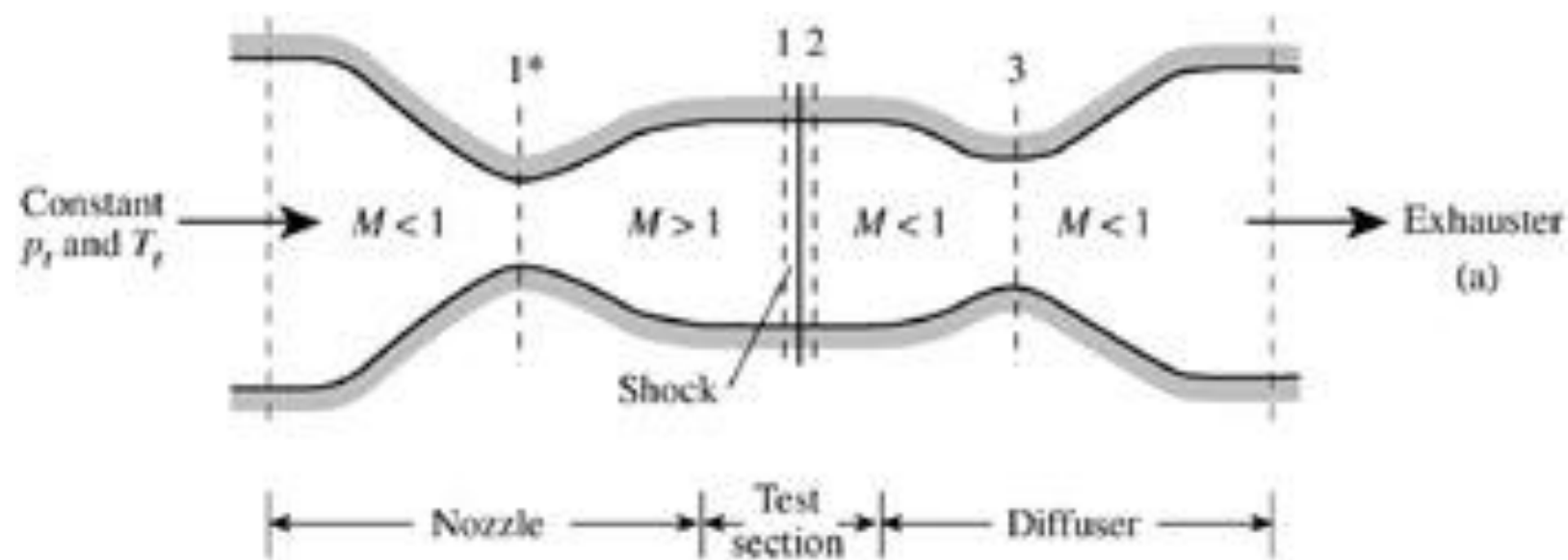
Supersonic-Supersonic (chocked)



Supersonic – Subsonic (Choked)



Supersonic-supersonic (not-choked)



Isentropic Flow Through Nozzles

Converging Nozzles

- Under steady flow conditions, mass flow rate is

cor

$$\dot{m} = \rho AV = \left(\frac{P}{RT} \right) A \left(Ma \sqrt{kRT} \right) = P A Ma \sqrt{\frac{k}{RT}}$$

- Substituting T and P from the expressions on slides 23 and 24 gives

$$\dot{m} = \frac{A Ma P_0 \sqrt{k/(RT_0)}}{[1 + (k - 1)Ma^2/2]^{(k+1)/[2(k-1)]}}$$

- Mass flow rate is a function of stagnation properties, flow area, and Ma

Isentropic Flow Through Nozzles

Converging Nozzles

- The maximum mass flow rate through a nozzle with a given throat area A^* is fixed by the P_0 and T_0 and occurs at $Ma = 1$

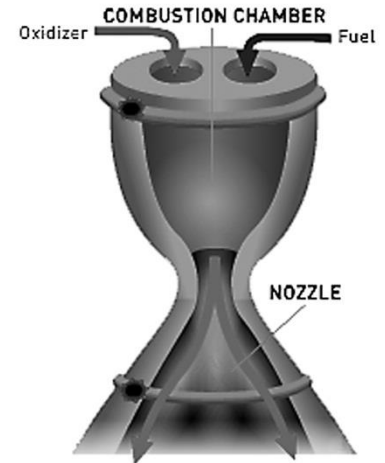
$$\dot{m} = A^* P_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1} \right)^{(k+1)/[2(k-1)]}$$

- This principal is important for chemical processes, medical devices, flow meters, and anywhere the mass flux of a gas must be known and controlled.

Isentropic Flow Through Nozzles

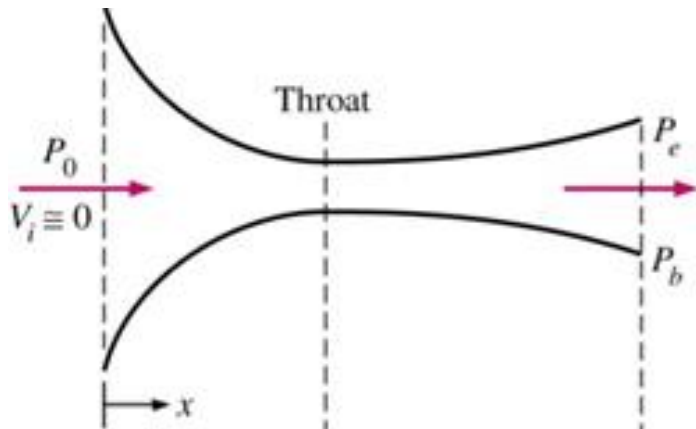
Converging-Diverging Nozzles

- The highest velocity in a converging nozzle is limited to the sonic velocity ($Ma = 1$), which occurs at the exit plane (throat) of the nozzle
- Accelerating a fluid to supersonic velocities ($Ma > 1$) requires a diverging flow section
 - Converging-diverging (C-D) nozzle
 - Standard equipment in supersonic aircraft and rocket propulsion
- Forcing fluid through a C-D nozzle does not guarantee supersonic velocity
 - Requires proper back pressure P_b



Isentropic Flow Through Nozzles

Converging-Diverging Nozzles



$$P_0 > P_b > P_c$$

- Flow remains subsonic, and mass flow is less than for choked flow. Diverging section acts as diffuser

$$P_b = P_c$$

- Sonic flow achieved at throat. Diverging section acts as diffuser. Subsonic flow at exit. Further decrease in P_b has no effect on flow in converging portion of nozzle

