

## Low Reynolds number flows

The *Reynolds number* is the most important dimensionless number in fluid mechanics. It is defined by:

$$\text{Re} = \frac{\rho UL}{\mu} \quad (1)$$

In which  $U$  is a characteristic velocity scale,  $L$  is a characteristic length scale,  $\rho$  is the density of the fluid and  $\mu$  is its dynamic viscosity. The “characteristic” velocity and length scales are different for different problems. For a relatively simple and well defined flow, such as the flow through a cylindrical tube, the characteristic scales are easily defined:  $U$  is the mean flow velocity in the pipe, and  $L$  is the pipe diameter. For more complex problems, the definition of characteristic scales may be more difficult, sometimes, even, the problem cannot be described by just one single Reynolds number.

The Reynolds number represents the ratio of the importance of inertial effects in the flow, to viscous effects in the flow.

“Inertia” is the property of an object to remain at a constant velocity, unless an outside force acts on it. An object with a large inertia will resist strongly to a change in velocity, in other words it is difficult to start or stop its movement. An object with a small inertia, on the other hand, will almost instantaneously start or stop when acted upon by some external or internally generated force. As we will see later when looking at the mathematics, inertia of fluid flows is caused by non-linear interactions within the flow field. These non-linearities may cause instabilities in the flow to grow, and therefore the flow can get turbulent when inertial effects are dominant, that is, for large Reynolds numbers. For small Reynolds number, on the other hand, the flow will always be laminar. For pipe flow, the critical Reynolds number above which turbulence may exist, is about 2000.

“Viscosity” is the resistance of a fluid to flow under the influence of an applied external force. It is the source of drag on objects moving through the fluid. For such an object, inertia hence strives to keep the object going, whereas viscosity tries to stop it.

**Table 1: A spectrum of Reynolds numbers for self-propelled organisms (after Vogel<sup>1</sup>).**

	Reynolds number
A large whale swimming at 10 m/s	300,000,000
A tuna swimming at the same speed	30,000,000
A duck flying at 20 m/s	300,000
A large dragon fly going 7 m/s	30,000
A copepod in a speed burst of 0.2 m/s	300
Flapping wings of the smallest flying insects	30
An invertebrate larva, 0.3 mm long, at 1 mm/s	0.3
A sea urchin sperm advancing the species at 0.2 mm/s	0.03
A bacterium, swimming at 0.01 mm/s	0.00001

Let us look at some characteristic Reynolds numbers for self-propelled organisms from nature, shown in Table 1. The characteristic length scale is the size of the organism, the characteristic velocity is its swimming speed. The density and viscosity are those of water, i.e.  $\rho=1000 \text{ kg/m}^3$  and  $\mu=1 \text{ mPa s}$  or air, i.e.  $\rho=1.2 \text{ kg/m}^3$  and  $\mu=0.018 \text{ mPa s}$ . We can see that the Reynolds numbers range from very large values, for a swimming whale, to extremely small values for swimming bacteria. That means that for a whale, inertial effects dominate, and thus, after stopping to swim, the whale will continue to move further, or “coast” for a substantial distance and time. For a bacterium, on the other hand, inertial effects will not be important at all and viscous effects dominate, so that a bacterium will stop almost instantaneously.

Now, the mathematics! Fluid flow is described by the Navier-Stokes equation, that describes the evolution of the velocity vector field:

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \bar{u} \quad (2)$$

In this equation,  $\bar{u}$  is the velocity vector and  $p$  is the pressure. On the left-hand side, the first term is the unsteady (time-dependent) inertial component (in fact, it is the acceleration of the flow). The second term is the non-linear inertial term. On the right-hand side, the first term is the driving pressure gradient, and the final term represents viscous dissipation.

We are going to estimate the relative importance of the various terms in the Navier-Stokes equation by scaling the terms using a characteristic velocity  $U$  and a characteristic length  $L$ . Hence:

The velocity  $\bar{u}$  scales with  $U$ ,

The spatial derivative  $\nabla$  scales with  $1/L$ ,

Time  $t$  scales with  $L/U$  (this is called the “advective” time scale).

<sup>1</sup> S. Vogel, *Life in moving fluids*, Princeton University Press, 1994.

This scaling implies the following for the various terms:

Unsteady inertia scales with  $U^2/L$ , i.e.  $\frac{\partial \vec{u}}{\partial t}$  is of the order  $\mathbf{O}\left(\frac{U^2}{L}\right)$ ,

Non-linear inertia scales also with  $U^2/L$ , i.e.  $(\vec{u} \cdot \nabla)\vec{u}$  is of the order  $\mathbf{O}\left(\frac{U^2}{L}\right)$ ,

The viscous term scales with  $\mu U/\rho L^2$ , i.e.  $\frac{\mu}{\rho} \nabla^2 \vec{u}$  is of the order  $\mathbf{O}\left(\frac{\mu U}{\rho L^2}\right)$ .

From this scaling, it follows that the ratio of the inertia terms to the viscous term scales as:

$$\text{Ratio of inertial terms to viscous term: } \frac{\rho U L}{\mu} = \text{Re}$$

which is the previously defined Reynolds number. If the Reynolds number is very small, i.e. much smaller than one,  $\text{Re} \ll 1$ , the inertial terms can be neglected in the Navier-Stokes equation, which in that case may be written as:

$$0 = -\nabla p + \mu \nabla^2 \vec{u} \quad (3)$$

In this form, the equation is known as the *Stokes equation*. Low Reynolds number flow is also called *Stokes flow*. Equation (3) implies that there must be a balance between the pressure term and the viscous term, and therefore, the pressure must scale as  $p = \mathbf{O}\left(\frac{\mu U}{L}\right)$ .

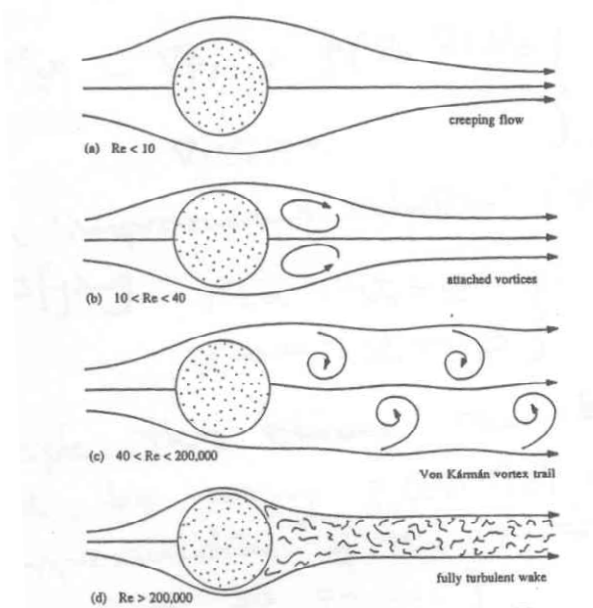
The Stokes equation (3) has the following properties. It is:

- instantaneous: A Stokes flow has no dependence on time other than through time-dependent boundary conditions. This means that, given the boundary conditions of a Stokes flow, the flow can be found without knowledge of the flow at any other time;
- linear: The Stokes Equation is linear, which means that the response of a Stokes flow will be proportional to the forces applied to it. Linearity also allows superposition of solutions;
- time-reversible: An immediate consequence of instantaneity, time-reversibility means that a time-reversed Stokes flow solves the same equations as the original Stokes flow. This property can sometimes be used (in conjunction with linearity and symmetry in the boundary conditions) to derive results about a flow without solving it fully.

In the case of flow due to a periodic motion, such as in (pulsating) blood flow, or the periodic oscillation of cilia or flagella in micro-organism propulsion [see the webpage on Cilia in nature], the scaling of the Navier-Stokes equation should be done somewhat differently. Instead of using the advective time scale  $L/U$ , we use the period  $T$  of the motion as the characteristic time scale, and we define the characteristic velocity as  $U = LT$ , with  $L$  the characteristic length scale. Then, in equation (2), the inertial terms, on the left-hand side, scale as  $L/T^2$ , and the viscous term, i.e. the second term on the right, scales as  $\mu/\rho LT$ . The

ratio between the inertial terms and the viscous terms, therefore, scales as  $\rho L^2/\mu T$ . This can be interpreted as the Reynolds number for periodic flows,  $Re_T$ . The number is also known, however, as the *Stokes number*,  $St$ . Again, if  $St$ , or  $Re_T$  is very small, i.e. much smaller than one, the inertial effects in the periodically driven flow can be neglected in the Navier-Stokes equation, and the Stokes equation (3) is obtained again. The balance between the viscous stresses and the pressure gradient, now imply that the

pressure must scale as  $p = \mathbf{O}\left(\frac{\mu}{T}\right)$ .

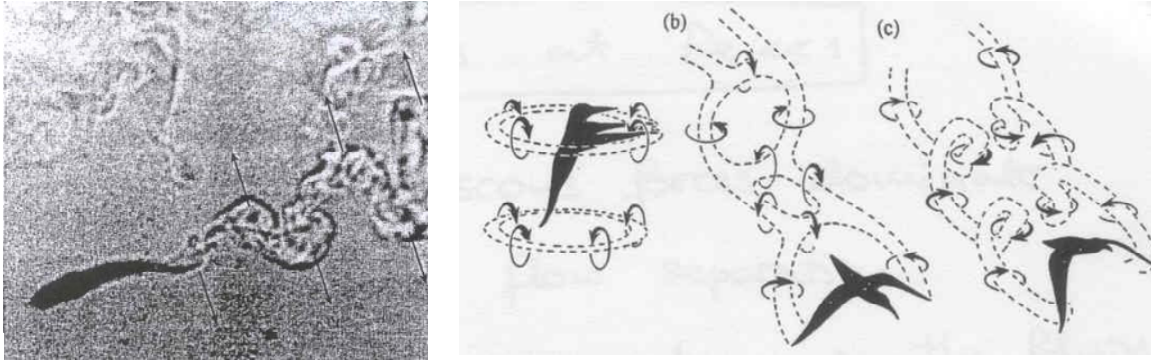


**Figure 1: Patterns in fluid flow around a cylinder as a function of the Reynolds number.**

It is now, hopefully, clear, that the nature of a fluid flow can be completely different depending on the Reynolds number. This can be illustrated with some practical examples. Figure 1, for example, shows the flow patterns that are caused by the flow around a

cylinder. The Reynolds number, in this case, is defined as  $Re = \frac{\rho UR}{\mu}$ , where  $U$  is the

flow velocity,  $R$  is the radius of the cylinder, and  $\rho$  and  $\mu$  are the fluid properties. For small Reynolds numbers, when inertia is not important, the flow is smooth and laminar. For higher Reynolds numbers, inertia begins to play a role and two stationary vortices are present behind the cylinder. At still higher  $Re$ , where inertia is dominant, vortex shedding starts, i.e. the vortices are not stationary anymore but detach from the top and bottom of the cylinder. This happens in an alternating fashion and this flow pattern is known as the Von Karman vortex street. At this stage, the flow is still not turbulent: although the flow shows vortices, they are laminar there is no chaotic motion if the fluid. At the highest Reynolds numbers, turbulence sets in and the wake behind the cylinder is filled with chaotic fluid motion.



**Figure 2: A fish swimming, generating eddies with its moving tail, and the vortices behind flying birds (hovering, flying slowly forward, and flying rapidly forward).<sup>2</sup>**

Figure 2 shows the flow patterns due to a swimming fish and flying birds. As we have seen already in Table 1, these flows are characterized by relatively large Reynolds numbers. This is evidenced by the existence of vortices, that shed from the tail or the wings of the animals, and the formation of which is essential for the propulsion.

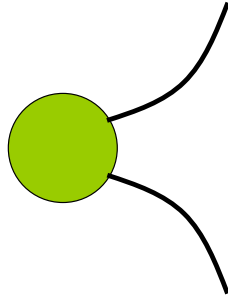
Table 2 summarizes the general differences between low ( $\ll 1$ ) and high ( $\gg 1$ ) Reynolds number flows.

**Table 2: Characteristics of low- and high-Reynolds number flows.**

High Reynolds number flow ( $\gg 1$ )	Low Reynolds number flow ( $Re \ll 1$ )
Inertial forces dominate	Viscous forces dominate
Flow separation (e.g. vortex shedding)	No flow separation
May be turbulent	Always laminar
Not reversible	Reversible
Non-linear	Linear
Large momentum (vortices in wakes)	Small momentum (no vortices in wakes)
Coasting	No coasting

Now let us look at a hypothetical swimming organism, as sketched in Figure 3, and sketch its swimming behavior as a function of the Reynolds number. It has, by the way a lot of resemblance to the real micro-organism *Chlamydomonas* [see webpage on cilia in nature]. The swimmer has a main body and two arms that it can use to propel itself. A Reynolds number can be defined in various ways for this case. Here, we define it on the basis of the velocity  $V$  of the stroke of the arms, and the length of the arms  $L$ . If we denote the duration of a stroke by  $T$ , then  $V \sim L/T$ .

<sup>2</sup> S. Vogel, *Life in moving fluids*, Princeton University Press, 1994.



**Figure 3: A hypothetical swimmer, having a spherical main body and two arms that it can use to propel itself.**

In Figure 4, our swimmer moves its arms back and forth in a symmetric way. That is, the shape of the arms is the same during the forward stroke as during the backward stroke. If the duration of the forward stroke equals that of the backward stroke, the swimmer will not swim far: irrespective of the value of the Reynolds number, it will just oscillate around the same fixed position. Hence, the reciprocal movement of the arms does not result in a net swimming speed. Now what if forward stroke is much faster than the backward stroke? The force applied during the forward stroke will be larger than during the backward stroke. Will the swimmer actually swim?

The answer depends on the Reynolds number regime, in other words, whether inertia or viscous effects dominate the flow. First, it is important to realize that, for the effectiveness of a stroke of the arms, not the applied force  $F$  is important, but the impulse, which is equal to:

$$I = \int_0^T F dt \quad (4)$$

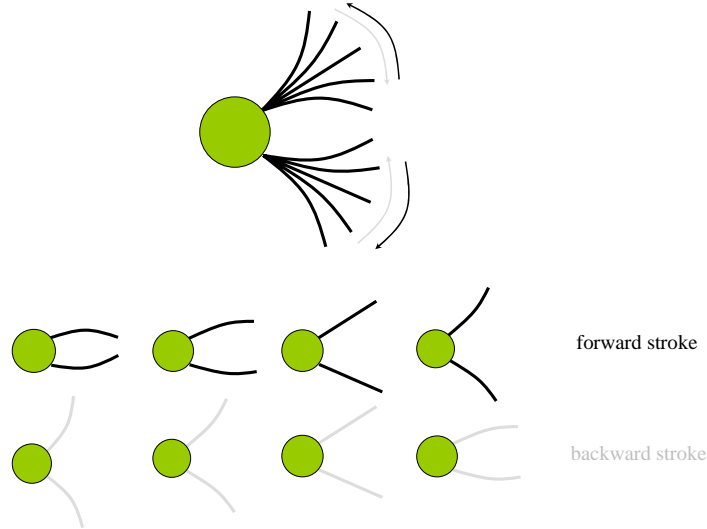
and  $T$  is the duration of the stroke.

If the organism in Figure 4 is large, or of the stroke velocity is very fast, then the Reynolds number is high and inertia plays a determining role. In that case, the force the arm applies on the fluid is dominated by inertia (the term  $(\vec{u} \cdot \nabla)\vec{u}$  in the Navier-Stokes equation (2)), which scales with the square of the stroke velocity  $V$ , hence  $F \sim V^2$ . In that case, the impulse scales as:

$$I \approx V^2 T \approx V \quad (\text{large Re}) \quad (5)$$

The last step follows from the fact that the stroke velocity  $V$  scales with the reciprocal stroke duration,  $1/T$ . From equation (5) we see that, if the stroke velocity is larger, then the impulse is larger. That means that moving the arms fast in one direction, but slowly in the other indeed helps to swim, even though the shape of the movement is not asymmetric. If we suppose that our swimmer of Figure 4 has a fast forward stroke, and a

slow backward stroke, then it will travel a net distance to the right, since the impulse applied during the forward stroke is larger than that during the backward stroke. A net “thrust” is produced.

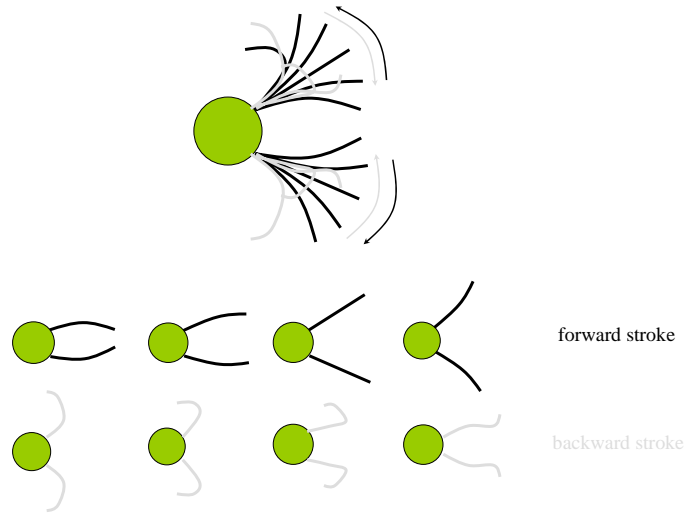


**Figure 4: Our hypothetical swimmer moves its arms in a symmetric way: the shape during the forward stroke is the same as during the backward stroke. Depending on the Reynolds number regime, the swimmer will indeed swim, or just oscillate around the same position and travel no net distance.**

At low Reynolds numbers, which we would have for very small swimmers, the situation becomes different. Now, inertia does not play a role anymore and viscous forces are dominant. That means that the force applied by the arms is determined by the viscous term in the Navier-Stokes equation, i.e.  $\frac{\mu}{\rho} \nabla^2 \vec{u}$ , which scales linearly with velocity, that is  $F \sim V$ . The impulse then scales as:

$$I \approx VT \approx \text{constant} \quad (\text{small Re}) \quad (6)$$

Thus the impulse is independent of the stroke velocity! That means that it does not matter how fast the arms are moving, the impulse applied during one stroke is unaffected (so long as  $\text{Re} \ll 1$ ). The swimmer of Figure 4 is just moving back and forth, and does not have any net displacement. Seen from the perspective of the fluid, after one cycle of moving back and forth of the arms, the fluid ends up exactly at the same location as it started from, and the flow is just oscillatory. This is a consequence of the time-reversibility of the Stokes equation (3).



**Figure 5: This hypothetical swimmer moves its arms in an asymmetric way: during the forward stroke they are more extended, and the effect on the surrounding fluid is maximized, whereas during the backward stroke the arms are more bent, minimizing the effect on the fluid. The swimmer will move to the right.**

To produce net flow, or thrust, in low Reynolds number situations, there must be an asymmetry in *shape*. An asymmetry in time, as we have just seen, is not sufficient. The swimmer in Figure 5 moves its arms asymmetrically. During the forward stroke, its arms are extended, maximizing the effect on the surrounding fluid. During the backward stroke, the arms are folded, minimizing their influence on the fluid. This difference in *shape* from the two directions of movement results in a difference in drag force, and indeed this swimmer will swim (to the right) also in low-Reynolds number situations.

The ciliated micro-organisms such as Paramecium swim under low-Reynolds numbers conditions. [see the webpage on Cilia in nature] In other words, the flow they are producing is a Stokes flow. Since the movement of the cilia has a strong asymmetry in shape, the micro-organisms are able to propel themselves. If they would move their cilia in a symmetric way, they would just move back and forth around a fixed position.