6.6 Euler's Method

Leonhard Euler made a huge number of contributions to mathematics, almost half after he was totally blind.

(When this portrait was made he had already lost most of the sight in his right eye.)

Leonhard Euler 1707 - 1783

→

It was Euler who originated the following notations:

$f(x)$ (function notation)

e (base of natural log)

π (pi)

i $\sqrt{-1}$

$\sum$ (summation)

Δy (finite change)

Leonhard Euler 1707 - 1783

→

There are many differential equations that can not be solved. We can still find an approximate solution.

We will practice with an easy one that can be solved.

$$\frac{dy}{dx} = 2x$$

Initial Value: $(0, 1)$

But first… Suppose we couldn't solve this conventionally…

→

Remember how it all started? $$f'(x) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

Suppose we took the limit out.
How would that change this?

$$f'(x) \approx \frac{f(x+dx) - f(x)}{dx}$$

for very small dx values.

Now let's do a little algebra

$$f'(x)\,dx \approx f(x+dx) - f(x)$$

$$f(x+dx) \approx f(x) + f'(x)\,dx$$

Which brings us to…

Euler's Method
of
Approximations

$$f(x+dx) \approx f(x) + f'(x)\,dx$$

So if we know $f'(x)$, we have a value for dx,

and we have an initial value $f(a)$

Then by Euler's Method:

$$f(a+dx) \approx f(a) + f'(a)\,dx$$

BUT WHAT DOES IT ALL MEAN!?

There are many differential equations that can not be solved.
We can still find an approximate solution.

We will practice with an easy one that can be solved.

$$\frac{dy}{dx} = 2x$$

Initial Value: $(0, 1)$

But first… Suppose we couldn't solve this conventionally…

→

$$\frac{dy}{dx} = 2x \qquad f(0) = 1 \qquad dx = 0.5$$

$$f(a + dx) \approx f(a) + f'(a)\,dx$$

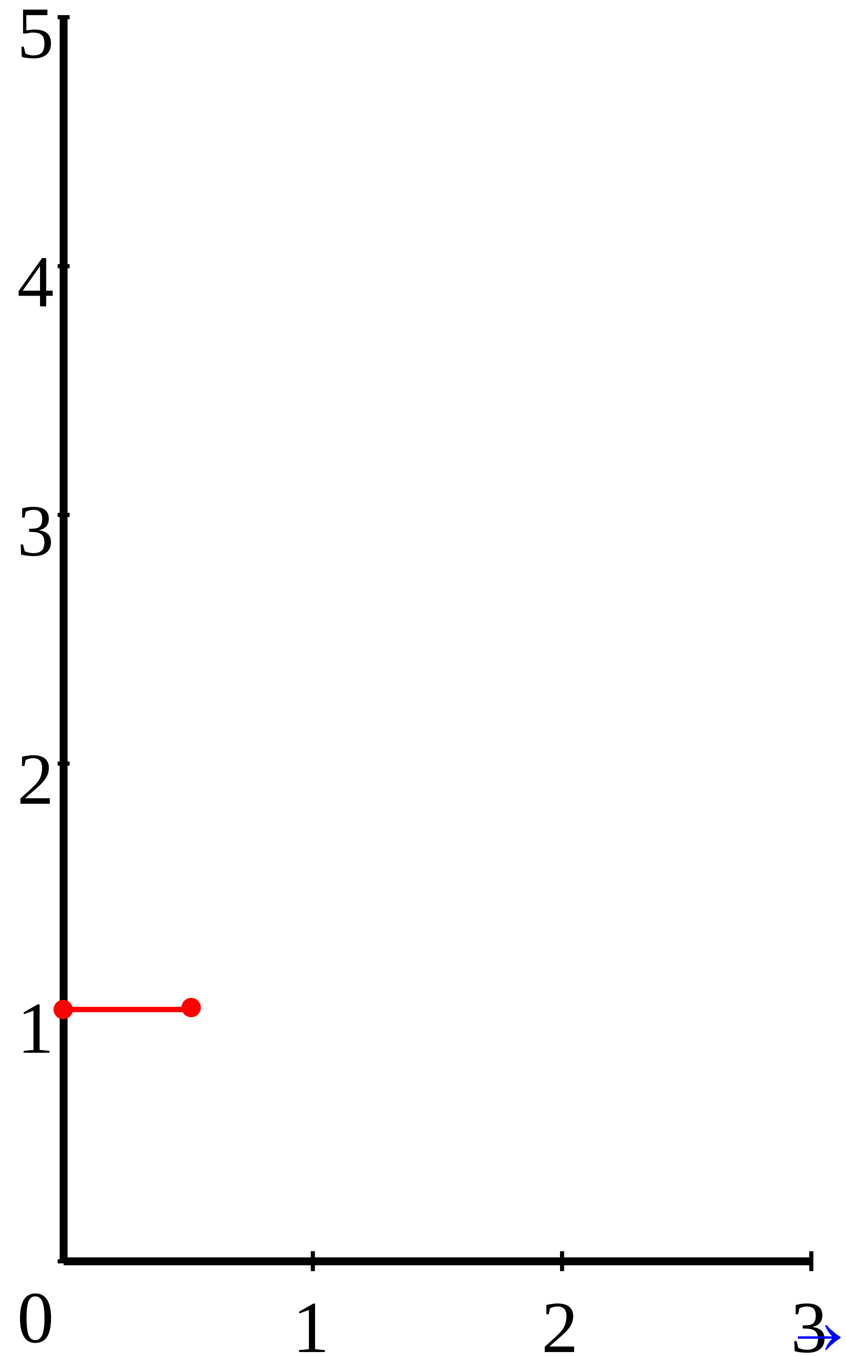
$$f(0 + 0.5) \approx f(0) + f'(0)\,0.5$$

$$f(0.5) \approx f(0) + f'(0)\,0.5$$

$$f(0.5) \approx 1$$

So we have an approximate coordinate $(0.5, 1)$

How could we use this to approximate $f(1)$?



$$\frac{dy}{dx} = 2x \qquad f(0) = 1 \qquad dx = 0.5$$

$$f(a + dx) \approx f(a) + f'(a)\,dx$$

$$f(0.5 + 0.5) \approx f(0.5) + f'(0.5)\,0.5$$

$$f(1) \approx f(0.5) + f'(0.5)\,0.5$$

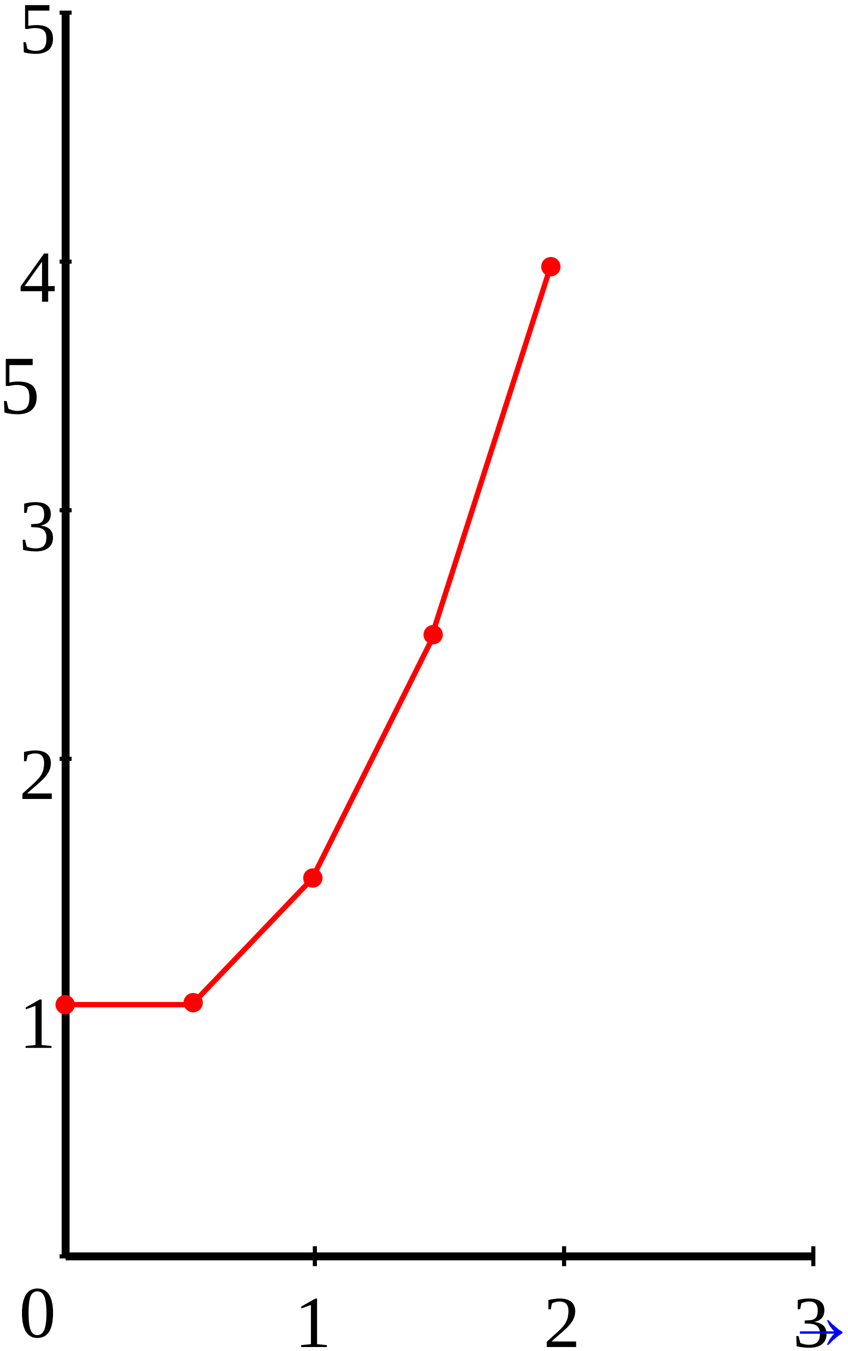
$$f(1) \approx 1 + (1) \cdot 0.5$$

$$f(1) \approx 1.5$$

$$f(1.5) \approx ? \qquad 2.5$$

$$f(2) \approx ? \qquad 4$$

$$f'(0.5) = 2(0.5) = 1$$



$$\frac{dy}{dx} = 2x \qquad f(0) = 1 \qquad dx = 0.5$$

So how would this compare to the real solution?

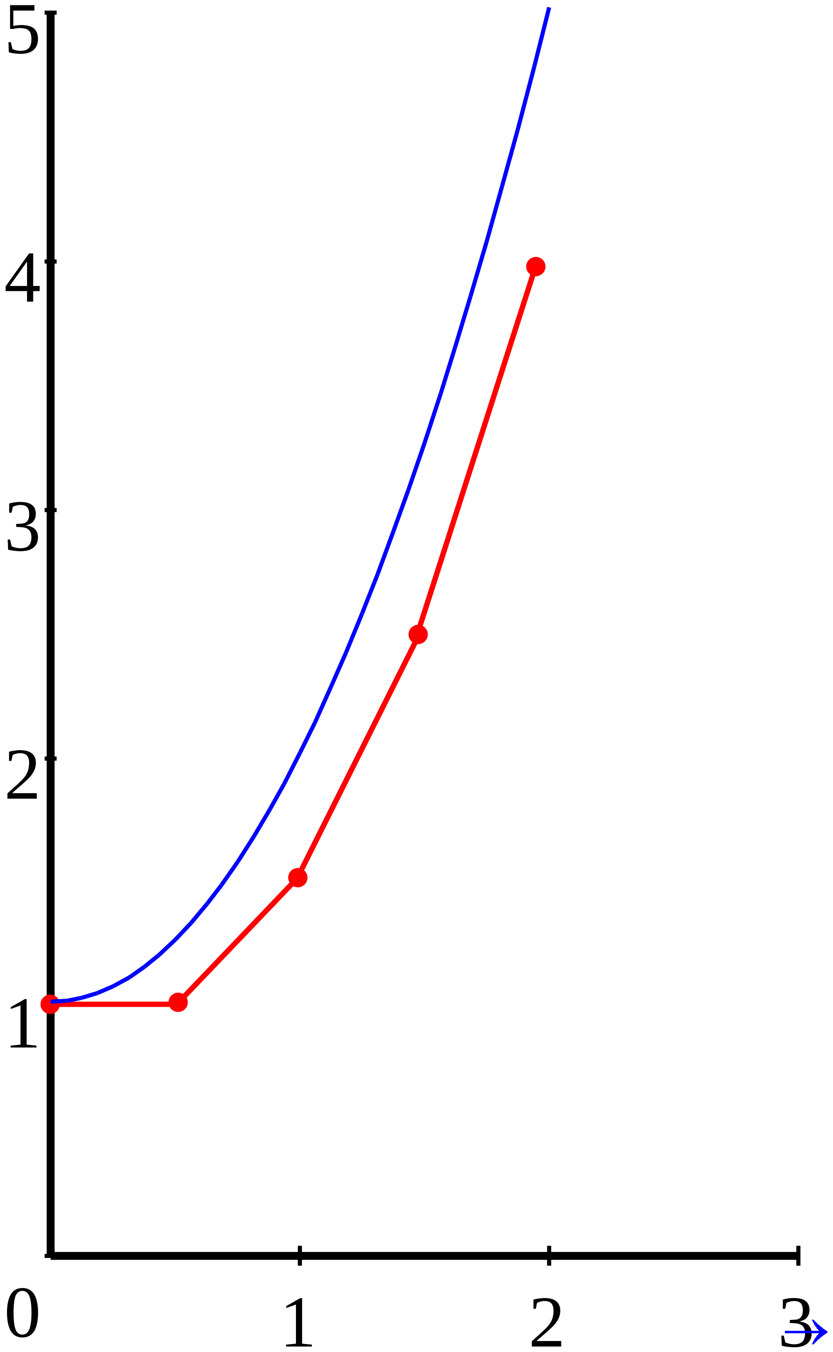
$$y = x^2 + 1$$

How could we make the approximation better?

Answer: A smaller dx value

$$dx = 0.1$$

Try that in the Euler program on your calculator.



Again, the real use of Euler's Method is for differential equations that we can't immediately solve.

Like, say, this one:

$$y' = x + y \qquad y(1) = 2$$

$$dx = 0.2$$

Approximate $y(2)$

Remember:

$$f(a+dx) \approx f(a) + f'(a)\,dx$$

$$y' = x + y$$

x	y'	y
1	3	2
1.2	3.8	2.6
1.4	4.76	3.36
1.6	5.91	4.312
1.8	7.29	5.494
2	8.95	6.953

$\rightarrow$

The actual solution to this is:

$$y = 4e^{x-1} - x - 1$$

Don't worry about how?

Like, say, this one:

$$y' = x + y \qquad y(1) = 2$$

$$dx = 0.2$$

Approximate $y(2)$

Remember:

$$f(a+dx) \approx f(a) + f'(a)\,dx$$

$$y' = x + y$$

x	y'	y
1	3	2
1.2	3.8	2.6
1.4	4.76	3.36
1.6	5.91	4.312
1.8	7.29	5.494
2	8.95	6.953

Use this solution to test your approximation on your calculator

→

The book refers to an “Improved Euler’s Method”. We will not be using it, and you do not need to know it.

The calculator also contains a similar but more complicated (and more accurate) formula called the Runge-Kutta method.

You don’t need to know anything about it other than the fact that it is used more often in real life.

This is the RK solution method on your calculator.

π