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Numerical Integration

Trapezoidal Rule
Simpson's Rule

Numerical Integration

- Many definite integrals of interest can't be evaluated analytically.
- Probably the best-known example is the integral that gives the area under the standard bell-shaped curve, given by

$$F(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2} dz$$

- This integral appears very frequently in probability and statistics and is extensively tabulated.

Numerical Integration

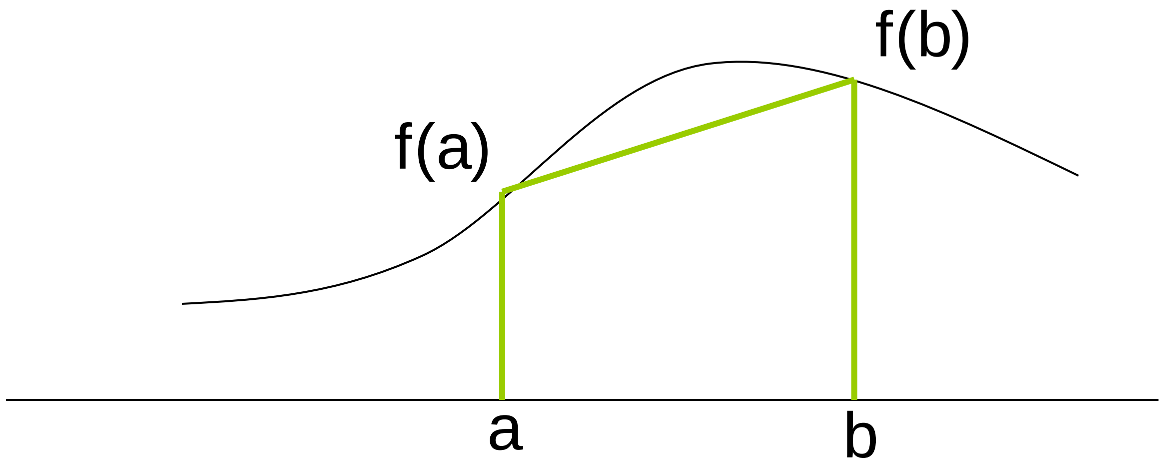
- The integral is tabulated because it is a fact that there is no way to express the antiderivative of $\exp(-z^2)$ in terms of elementary functions.
- Because of cases like this we need methods to perform approximate integration; for other cases it may be more convenient to use a numerical method than a symbolic one.
- Often we're given values of f(x) at various points but not a formula for f and so have no choice but to use a numerical method (for example, data from an experiment)

Numerical Integration

- The Newton-Cotes formula
  - Trapezoidal Rule
  - Simpson’s Rule
- Romberg Integration
- Gaussian Quadrature

Trapezoidal Rule

- The trapezoidal rule is equivalent to approximating the area of the trapezoid under the straight line connecting f(a) and f(b).



Trapezoidal Rule

- One way to improve the accuracy of the trapezoid rule is to divide the integration interval from a to b into a number of segments (N).
- The areas of individual segment can then be added to yield the integral for the entire interval.
- There are N+1 equally spaced nodes

$$x_0,\ x_1,\ \ldots.,\ x_N$$

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Trapezoidal Rule

- Let, $x_0 = a$ and $x_N = b$
- Assume

$$x_k = x_0 + kh, \quad k = 0,1,2,\ldots,N$$

are equally spaced nodes

$$h = \frac{b-a}{N}$$

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Trapezoidal Rule

- Therefore

$$\int_a^b f(x)\ \ dx = \int_{x_0}^{x_1} f(x)\ \ dx + \int_{x_1}^{x_2} f(x)\ \ dx + \ldots + \int_{x_{N-1}}^{x_N} f(x)\ \ dx$$

with

$$\int_{x_k}^{x_{k+1}} f(x)\ \ dx \approx \int_{x_k}^{x_{k+1}} p_n(x)\ \ dx$$

Trapezoidal Rule

- Let

$$p_n(x) = f_k + r\Delta f_k$$

- with

$$x = x_k + rh \qquad (dx = h\,dr)$$

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Trapezoidal Rule

$$\int_{x_k}^{x_{k+1}} f(x) \quad dx \approx \int_{x_k}^{x_{k+1}} p_n(x) \quad dx$$

$$= \int_{x_k}^{x_k+h} \left(f_k + r\Delta f_k \right) dx$$

$$= h\int_{0}^{1} \left(f_k + r\Delta f_k \right) dr$$

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Trapezoidal Rule

$$= h\left(rf_k + \frac{r^2}{2}\Delta f_k\right)\Bigg|_0^1 \qquad = h\left(f_k + \frac{1}{2}\Delta f_k\right)$$

$$= h\left[f_k + \frac{1}{2}(f_{k+1} - f_k)\right]$$

$$= \frac{h}{2}(f_k + f_{k+1})$$

Trapezoidal Rule

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f_0 + f_1\right) + \frac{h}{2}\left(f_1 + f_2\right) + \ldots + \frac{h}{2}\left(f_{N-1} + f_N\right)$$

$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{N-1} + f_N\right)$$

$$\int_a^b f(x)\ dx = \frac{h}{2}\left(f_0 + f_N + 2\sum_{i=1}^{N-1} f_i \right)$$

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Example

- Approximate the following integral using the Trapezoidal rule with h=0.5 and h=0.25.

$$\int_1^4 \frac{x}{\sqrt{x+4}} \; dx$$

prepared by Razana Alwee

Example

- For h=0.5, a=1 and b=4

$$N = \frac{b-a}{h} = \frac{4-1}{0.5} = 6$$

prepared by Razana Alwee

Example

i	x_i	$f(x_i)=\dfrac{x_i}{\sqrt{x_i+4}}$	
0	1.0	0.4472	
1	1.5		0.6396
2	2.0		0.8165
3	2.5		0.9806
4	3.0		1.1339
5	3.5		1.2780
6	4.0	1.4142	
	Total	1.8614	4.8486

Example

$$\int_1^4 \frac{x}{\sqrt{x+4}}\,dx = \int_1^4 f(x)\,dx$$

$$= \frac{h}{2}\left[f_0 + f_6 + 2\left(f_1 + f_2 + f_3 + f_4 + f_5\right)\right]$$

$$= \frac{0.5}{2}\left[1.8614 + 2\left(4.8486\right)\right]$$

$$= 2.8896$$

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Example

- For h=0.25, a=1 and b=4

$$N = \frac{b-a}{h} = \frac{4-1}{0.25} = 12$$

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i	x_i	$f(x_i)=\dfrac{x_i}{\sqrt{x_i+4}}$	
0	1.00	0.4472	
1	1.25		0.5455
2	1.50		0.6396
3	1.75		0.7298
4	2.00		0.8165
5	2.25		0.9000
6	2.50		0.9806
7	2.75		1.0585
8	3.00		1.1339
9	3.25		1.2070
10	3.50		1.2780
11	3.75		1.3470
12	4.00	1.4142	
	Total	1.8614	10.6364

Example

$$\int_1^4 \frac{x}{\sqrt{x+4}}\ dx = \int_1^4 f(x)\ dx$$

$$= \frac{h}{2}\left[f_0 + f_{12} + 2\sum_{i=1}^{11} f_i \right]$$

$$= \frac{0.25}{2}\left[1.8614 + 2(0.6364 \right.$$

$$= 2.8918$$

(Note: The exact value of the integral is 2.8925)

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Example

- For h=0.5, N=6, the error is 0.0029
- For h=0.25, N=12, the error is 0.0007

- The error decreases as the number of segments (N) increases.
- When h is reduced by a factor of ½ the successive errors are diminished by approximately ¼.

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Exercise

- Approximate the following integrals using the Trapezoidal rule with N=10.

(a) $\int_0^2 e^x \, dx$

(b) $\int_1^2 \sqrt{x^3 - 1} \, dx$

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Simpson’s Rule

- The trapezoidal rule usually requires a large number of function evaluations to achieve an accurate answer.
- Another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points.

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Simpson's Rule

- Let

$$p_2(x) = f_k + r\Delta f_k + \frac{1}{2} r(r-1)\Delta^2 f_k$$

Simpson's Rule

$$\int_{x_k}^{x_{k+2}} f(x)\ \ dx = \int_{x_k}^{x_k+2h}\left(f_k + r\Delta f_k + \frac{1}{2}r(r-1)\Delta^2 f_k \right)\ dx$$

$$= h\int_0^2 \left(f_k + r\Delta f_k + \frac{1}{2}r(r-1)\Delta^2 f_k \right)\ dr$$

$$= h\left(rf_k + \frac{r^2}{2}\Delta f_k + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 f_k \right)\Bigg|_0^2$$

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Simpson's Rule

$$= h\left(2f_k + 2\Delta f_k + \frac{1}{3}\Delta^2 f_k\right)$$

$$= h\left(2f_k + 2\left(f_{k+1} - f_k\right) + \frac{1}{3}\left(f_{k+2} - 2f_{k+1} + f_k\right)\right)$$

$$= \frac{h}{3}\left(f_k + 4f_{k+1} + f_{k+2}\right)$$

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Simpson's Rule

- Subdivide the interval [a,b] into N subintervals (N is even).

$$\int_a^b f(x)\ dx = \int_{x_0}^{x_2} f(x)\ dx + \int_{x_2}^{x_4} f(x)\ dx$$

$$+ \ldots + \int_{x_{N-2}}^{x_N} f(x)\ dx$$

$$= \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 4f_{N-1} + f_N\right)$$

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Simpson's Rule

- This equation is known as Simpson's 1/3 rule.

$$\int_a^b f(x)\ dx = \frac{h}{3}\left[\left(f_0 + f_N \right) + 4\sum_{i=1}^{N/2} f_{2i-1} + 2\sum_{i=1}^{N/2-1} f_{2i} \right]$$

- Where N is even.

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Simpson's 3/8 Rule

- Let,

$$p_3(x) = f_k + r\Delta f_k + \frac{1}{2}r(r-1)\Delta^2 f_k + \frac{1}{6}r(r-1)(r-2)\Delta^3 f_k$$

$$\int_{x_k}^{x_{k+3}} f(x) \;\; dx = \int_{x_k}^{x_{k+3}} p_3(x) \;\; dx$$

Simpson's 3/8 Rule

$$\int_a^b f(x)\ dx = \frac{3h}{8}\left[\left(f_0 + f_N\right) + 3\sum_{i=1}^{N/3}\left(f_{3i-2} + f_{3i-1}\right) + 2\sum_{i=1}^{N/3-1} f_{3i}\right]$$

- where N= 3, 6, 9, 12, …..
- This equation is called Simpson's 3/8 rule because h is multiplied by 3/8.

Example

- Approximate the following integrals using the Simpson's 1/3 rule with h=0.5 and h=0.25.

$$\int_1^4 \frac{x}{\sqrt{x+4}} \, dx$$

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Example

- For h=0.5, a=1 and b=4

$$N = \frac{b-a}{h} = \frac{4-1}{0.5} = 6$$

(N is even)

prepared by Razana Alwee

Example

i	x_i	$f_i = f(x_i) = \frac{x_i}{\sqrt{x_i + 4}}$		
0	1.0	0.4472		
1	1.5		0.6396	
2	2.0			0.8165
3	2.5		0.9806	
4	3.0			1.1339
5	3.5		1.2780	
6	4.0	1.4142		
	Total	1.8614	2.8982	1.9504

Example

$$\int_1^4 \frac{x}{\sqrt{x+4}}\,dx = \frac{h}{3}\left[\left(f_0 + f_6\right) + 4\sum_{i=1}^{3} f_{2i-1} + 2\sum_{i=1}^{2} f_{2i}\right]$$

$$= \frac{0.5}{3}\left[1.8614 + 4(2.8982) + 2(1.9504)\right]$$

$$= 2.8925$$

Example

- For h=0.25, a=1 and b=4

$$N = \frac{b-a}{h} = \frac{4-1}{0.25} = 12$$ (N is even)

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i	x_i	$f_i = f(x_i) = \dfrac{x_i}{\sqrt{x_i + 4}}$		
0	1.00	0.4472		
1	1.25		0.5455	
2	1.50			0.6396
3	1.75		0.7298	
4	2.00			0.8165
5	2.25		0.9000	
6	2.50			0.9806
7	2.75		1.0585	
8	3.00			1.1339
9	3.25		1.2070	
10	3.50			1.2780
11	3.75		1.3470	
12	4.00	1.4142		
	Total	1.8614	5.7878	4.8486

Example

$$\int_1^4 \frac{x}{\sqrt{x+4}}\,dx = \frac{h}{3}\left[\left(f_0 + f_{12} \right) + 4\sum_{i=1}^{6} f_{2i-1} + 2\sum_{i=1}^{5} f_{2i} \right]$$

$$= \frac{0.25}{3}\left[1.8614 + 4(5.7878) + 2(4.8486) \right]$$

$$= 2.8925$$

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Example

- Approximate the following integrals using the Simpson's 3/8 rule with h=0.25.

$$\int_1^4 \frac{x}{\sqrt{x+4}}\ dx$$

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Example

- For h=0.25, *a*=1 and *b*=4

$$N = \frac{b-a}{h} = \frac{4-1}{0.25} = 12$$

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i	x_i	$f_i = f(x_i) = \dfrac{x_i}{\sqrt{x_i + 4}}$		
0	1.00	0.4472		
1	1.25		0.5455	
2	1.50		0.6396	
3	1.75			0.7298
4	2.00		0.8165	
5	2.25		0.9000	
6	2.50			0.9806
7	2.75		1.0586	
8	3.00		1.1339	
9	3.25			1.2070
10	3.50		1.2780	
11	3.75		1.3470	
12	4.00	1.4142		
	Total	1.8614	7.7191	2.9174

Example

$$\int_1^4 \frac{x}{\sqrt{x+4}}\,dx = \frac{3h}{8}\left[\left(f_0 + f_{12}\right) + 3\sum_{i=1}^{4}\left(f_{3i-2} + f_{3i-1}\right) + 2\sum_{i=1}^{3} f_{3i}\right]$$

$$= \frac{3(0.25)}{8}\left[1.8614 + 3(7.7191) + 2(2.9174)\right]$$

$$= 2.8925$$

Simpson’s Rule

- Simpson’s 1/3 rule is usually the method of preference because it attains third-order accuracy with three points rather than the four points required for the 3/8 version.

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Exercise

- Approximate the following integral

$$\int_0^3 \frac{1}{\sqrt{x^3+1}} \, dx$$

using

a) The Simpson's 1/3 rule

b) The Simpson's 3/8 rule

with N=12.

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