Lecture 27

- Continuity Equation
- Bernoulli's Equation

Cutnell+Johnson: 11.7-11.10

Incompressibility

There is one big difference between liquids and gases. The density of a gas is easy to change. However, fluids are usually *incompressible*. Incompressiblility means that the density of a fluid is independent of the pressure. This is not perfectly true: fluids do contract and expand a little, but not much at all: this expansion and contraction can easily be neglected. We've already used fluid incompressibility. For example, the formula for how the pressure depends on the depth of the fluid assumed that the density remained constant, even though the pressure increases. Pascal's Principle also depends on it. Pascal's principle says that if you push at one end of the fluid, the pressure increases everywhere. If the fluid were compressible, what would just happen is that part of the fluid would become more dense. This is what happens to a solid. A gas, on the other hand, will compress uniformly.

Continuity Equation

So far, we've discussed static fluids. Now we'll discuss fluids in motion. If something moves, it has to go somewhere. This simple fact is just the conservation of mass (this holds unless you're in a nuclear reactor, which converts mass to energy). The conservation of mass results in what's called the continuity equation. Let's consider a small chunk of moving fluid moving at some speed v. During some small time interval Δt, this chunk moves a distance $d = v\Delta t$. The volume of fluid that has flowed past this point is therefore $V = dA = Av\Delta t$, where A is the cross-sectional area. The mass of this chunk of fluid is therefore

$$\Delta m = \rho V = \rho A v \Delta t$$

Thus this gives us the rate of mass flow:

$$\frac{\Delta m}{\Delta t} = \rho A v$$

If the fluid is in a steady state, this means mass isn't accumulating or diminishing. In other words, you're not doing something like adding a water to a jar. In a steady state, the rate of mass flow must be the same everywhere.

$$\rho A v = \text{constant}$$

Another way of writing this equation is to say that the mass flow must be the same at any two points in the same fluid. Thus

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

.

The continuity equation is familiar if you've ever handled a garden hose. If you shut off say half of the end with your thumb, you've decreased the cross-sectional area A in half. This means that the velocity of the water must double. The water needs to hurry up to get out. In general, this means as a tube gets wider or narrower, the fluid flow gets slower or faster. The continuity equation also applies to gases: the wind gets faster in between large buildings. Again, the point is that the mass must go somewhere. However, it is simpler for a liquid, because of the incompressibility. The density must the same everywhere, so $\rho_1 = \rho_2$. This means that

$$A_1 v_1 = A_2 v_2$$

in a liquid.

Bernoulli's Equation

In order to make the discussion a little easier, we'll make a few assumptions. We'll neglect viscosity. You can think of viscosity as being like fluid friction. Honey is extremely viscous: when you try to stir it, most of the energy is spent on friction: the interactions between the honey molecules prevent the fluid from sliding freely past each other. On the hand, water is much less viscous, and we will neglect the viscosity altogether. This means that all the force one puts into pushing a fluid goes into fluid motion or pressure. It is simpler to row a boat in water than it is in honey. Another thing we'll assume for simplicity is the the fluid is not rotating as it flows. In other words, the whole fluid has the same velocity.

With these assumptions, we can derive Bernoulli's equation. This equation is a generalization of our earlier $P_2 = P_1 + \rho g h$ to the case where the fluid is moving. To derive it, let's first consider a pipe which goes horizontally. To make the fluid go faster, you have to do work. Consider a chunk of fluid of mass m. This has kinetic energy

$$KE = \frac{1}{2} m v^2$$

To make this chunk of fluid go faster, you have to apply a force. As we saw with Archimedes' Principle, the way you apply a force in a fluid is to have a difference in pressure. Thus if there is a difference in pressure between one point and another, there is a net force on the fluid in between. This force is $P_2 - P_1$ times the cross-sectional area A. To move the fluid a distance d requires doing the work

$$W = Fd = (P_1 - P_2)Ad = (P_1 - P_2)V$$

Recall in such a situation, the work is the change in kinetic energy. The change in energy of this chunk is

$$W = \Delta KE = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Equating the two says that that

$$(P_1 - P_2)V = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Rearranging the terms and using the fact that the density $\rho = m/V$, we have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

This equation is useful for a Venturi meter, as we'll see in this

Problem Consider two pipes with $r_1 = 2r_2$. Join them together and let water flow. The pressure difference between the two parts is $P_1 - P_2 = 100\,Pa$. What is the velocity in the larger half?

Answer First, we use the continuity equation to relate the two velocities. The two areas are related by $A_1 = 4A_2$ (note that it's 4, not 2: if the radius doubles, the area quadruples.) Thus the continuity equation says that

$$v_1 A_1 = v_2 A_2$$

so that $v_2 = 4v_1$ (the fluid in the larger pipe flows slower, and with greater pressure). Now using the above equation gives

$$\begin{aligned} P_1 - P_2 &= \frac{1}{2}\rho(v_2^2 - v_1^2) \\ 100\,Pa &= \frac{1}{2}(1000\,kg/m^3)(16v_1^2 - v_1^2) \end{aligned}$$

Solving for v_1 gives

$$v_1 = \sqrt{\frac{100}{(500)(15)}}m/s = .12m/s$$

To get the full Bernoulli's equation, we just need to include gravity like we did before. Now let's consider a pipe which goes up and downhill. Then we need to do work to get the water to go uphill as well, from y_1 to y_2. The potential energy is as always mgh. We then have

$$(P_1 - P_2)V = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 + mg(y_2 - y_1)$$

Rearraging again gives Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

If you set the velocities equal to zero like we were doing last time, then this equation reduces to our earlier one.

There are many consequences of Bernoulli's equation. In fact, Bernoulli's equation also applies to air. It's why planes fly, and curve balls curve. The basic idea is that when air is moving faster, its pressure is lower. Thus if you manage to have an object like a baseball or a plane wing with different pressures on different sides, the object will feel a net force. A spinning baseball drags some of the air with it. This means on opposite sides, the air is moving at different speeds. This results in a change in pressure, and makes the ball move sideways. The shape of a plane wing causes the air rushing past to go faster on top. This lowers the pressure and hence causes the wing to lift up.