Initial Value Problems

Euler’s Method
Taylor’s Method

Ordinary differential equation

- Differential equations are commonly used for mathematical modeling in science and engineering.
- Often there is no known analytic solution and numerical approximations are required.

Example

- First order ordinary differential equation

$$\frac{du}{dt} = -0.27(u - 60)^{3/4}$$

$$\frac{dy}{dx} = x + y$$

Example

- Second order ordinary differential equation

$$\frac{d^2\theta}{dx^2} = -\frac{PL^2}{EI}\sin\theta$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2e^x - 3kosx$$

prepared by Razana Alwee

Initial Value Problem

- Given the first order ordinary differential equation,

$$\frac{dy}{dx} = f(x, y) \qquad a \le x \le b$$

with

$$y(a) = \alpha$$

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Initial Value Problem

- We want to approximate,

$$y(x_1),\ y(x_2),\ \ldots,\ y(x_N)$$

for $\quad a=x_0,\ x_1,\ \ldots.\ x_N=b$

where $\quad x_i = x_0+ih$

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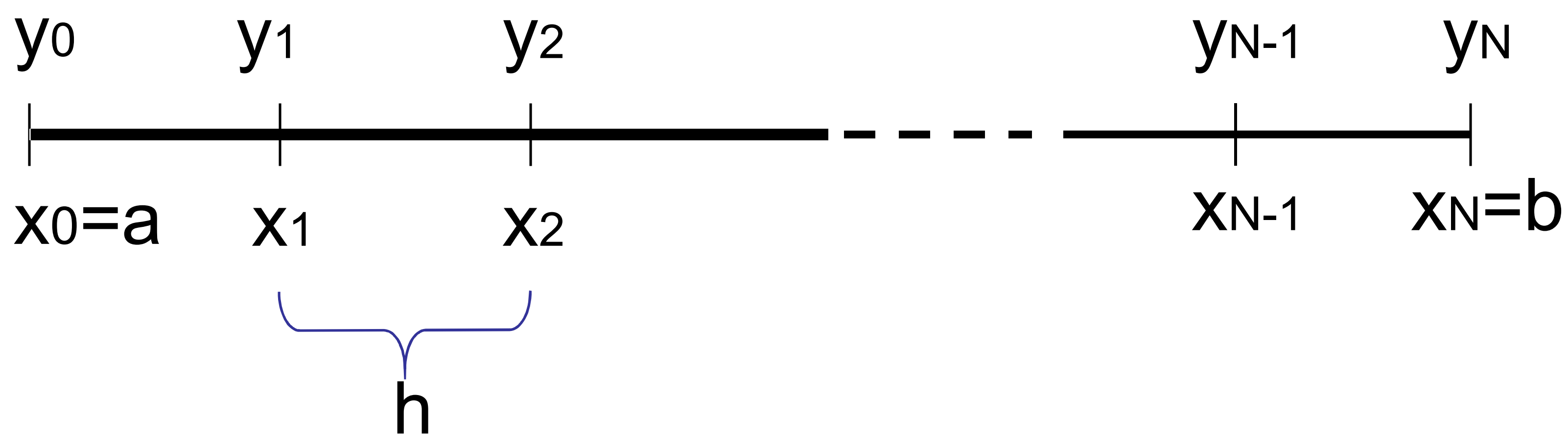
Initial Value Problem



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Euler's Method

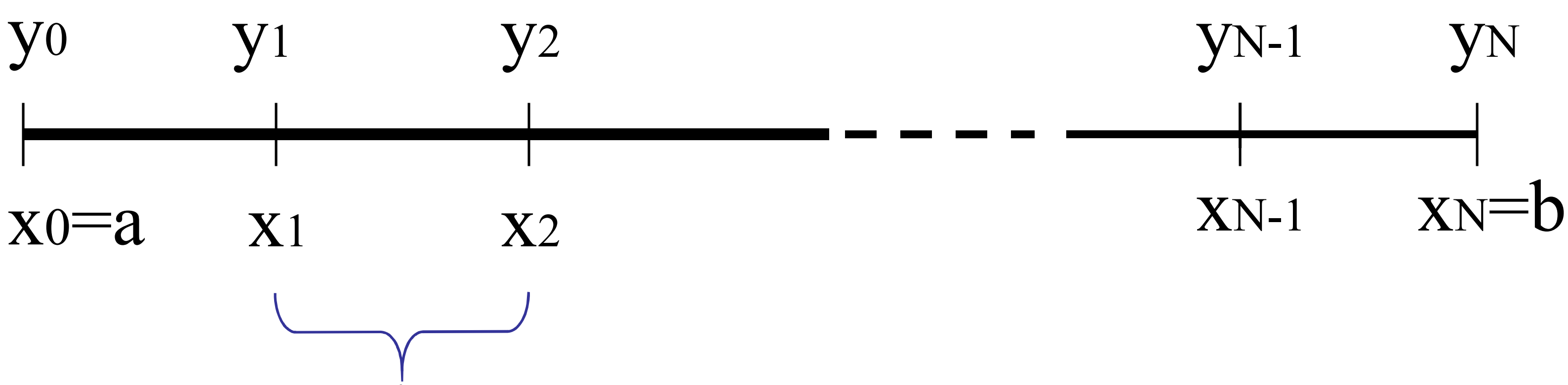
- Let [a,b] be the interval over which we want to find the solution to the initial value problem,

$$\frac{dy}{dx} = f(x, y)$$

$$y(a) = \alpha$$

Euler's Method

- We subdivide the interval [a,b] into N equal subinterval.



$$h = \frac{b-a}{N}$$

$$x_i = a + ih, \qquad i = 0, 1, 2, \ldots, N$$

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Euler's Method

- Taylor's theorem,

$$y(x_{i+1}) = y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(\xi_i)$$

- If the step size h is chosen small enough, then we may neglect the second-order term and get,

$$y(x_{i+1}) = y(x_i) + hy'(x_i) \qquad i = 0, 1, 2, \ldots, N-1$$

Euler's Method

$$y'(x_i) = f(x_i, y(x_i))$$

$$y(x_{i+1}) \approx y(x_i) + hf(x_i, y(x_i)) \qquad i = 0, 1, 2, \ldots, N-1$$

- Which is Euler's approximation.

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Euler's Method

- Let,

$$y_i \approx y(x_i),$$

- Thus, Euler's method is,

$$y_0 = \alpha$$

$$y_{i+1} \approx y_i + hf(x_i, y_i), \qquad i = 0,1,\ 2,\ ....,\ N-1$$

Example

- Use Euler's method to approximate the solution for the following initial value problem.

$$\frac{dy}{dx} = x + y, \qquad 0.0 \le x \le 0.5$$

$$y(0) = 1$$

with N=5.

Example

$$y(0) = 1 \longrightarrow x_0 = 0 \qquad y_0 = 1$$

$$\frac{dy}{dx} = x + y, \longrightarrow f(x, y) = x + y$$

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Example

$$y_{i+1} \approx y_i + hf(x_i, y_i) \quad = y_i + h(x_i + y_i)$$

- N=5

$$h = \frac{0.5 - 0.0}{5} = 0.1$$

Example

$$y(0.1) = y_1 \approx y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0)$$

$$= 1 + (0.1) f(0,1)$$
$$= 1 + (0.1)(0 + 1)$$
$$= 1.1,$$

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Example

$$y(0.2) = y_2 \approx y_1 + hf(x_1, y_1) = y_1 + h(x_1 + y_1)$$

$$= 1.1 + (0.1)f(0.1, 1.1)$$
$$= 1.1 + (0.1)(0.1 + 1.1)$$
$$= 1.22,$$

Example

$$y(0.3) = y_3 \approx y_2 + hf(x_2, y_2) = y_2 + h(x_2 + y_2)$$

$$= 1.22 + (0.1) f(0.2, 1.22)$$

$$= 1.22 + (0.1)(0.2 + 1.22)$$

$$= 1.362,$$

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Example

$$y(0.4) = y_4 \approx y_3 + hf(x_3, y_3) = y_3 + h(x_3 + y_3)$$

$$= 1.362 + (0.1)f(0.3, 1.362)$$

$$= 1.362 + (0.1)(0.3 + 1.362)$$

$$= 1.5282,$$

Example

$$y(0.5) = y_5 \approx y_4 + hf(x_4, y_4) = y_4 + h(x_4 + y_4)$$

$$= 1.5282 + (0.1)f(0.4, 1.5282)$$
$$= 1.5282 + (0.1)(0.4 + 1.5282)$$
$$= 1.7210.$$

Exercise

- Use Euler's method to approximate the solution for the following initial value problem.

$$y' = xy + x \qquad 0.0 \le x \le 1.0$$

$$y(0) = 0$$

with h=0.2.

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Taylor's Method

- Taylor's Theorem,

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \ldots$$

$$+ \frac{h^n}{n!} y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

Taylor's Method

- The general step for Taylor's method of order N,

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \ldots + \frac{h^n}{n!} y^{(n)}(x_i)$$

Example

- Given the initial value problem,

$$y' = x^2 - y \qquad 0.0 \le x \le 1.0$$

$$y(0) = 1$$

- Approximate y(0.2) and y(0.4) using Taylor's method of order three.

Example

- The Taylor's method of order three,

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{3!} y'''(x_i)$$

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Example

- Given,

$$y(0) = 1 \longrightarrow \quad x_0 = 0 \qquad y_0 = 1$$

- h=0.2

Example

- Given,

$$y' = x^2 - y$$

$$\begin{aligned} y'' &= (x^2 - y)' \\ &= 2x - y' \\ &= 2x - (x^2 - y) \\ &= 2x - x^2 + y \end{aligned}$$

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Example

$$\begin{aligned} y''' &= (2x - x^2 + y)' \\ &= 2 - 2x + y' \\ &= 2 - 2x + x^2 - y. \end{aligned}$$

Example

$$y_{i+1} = y_i + h(x_i^2 - y_i) + \frac{h^2}{2}\left(2x_i - x_i^2 + y_i\right)$$

$$+ \frac{h^3}{3!}(2 - 2x_i + x_i^2 - y_i)$$

Example

$$y_1 = y_0 + h(x_0^2 - y_0) + \frac{h^2}{2}\left(2x_0 - x_0^2 + y_0\right) + \frac{h^3}{3!}(2 - 2x_0 + x_0^2 - y_0)$$

$$y_1 = 1 + (0.2)(0^2 - 1) + \frac{(0.2)^2}{2}\left(2(0) - 0^2 + 1\right) + \frac{(0.2)^3}{6}(2 - 2(0) + 0^2 - 1)$$

$$y_1 = 1 - 0.2 + 0.02 + 0.0013 = 0.8213\ .$$

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Example

$$y_2 = y_1 + h(x_1^2 - y_1) + \frac{h^2}{2}\left(2x_1 - x_1^2 + y_1\right) + \frac{h^3}{3!}(2 - 2x_1 + x_1^2 - y_1)$$

$$y_2 = 0.8213 + (0.2)(0.2^2 - 0.8213) + \frac{(0.2)^2}{2}\left(2(0.2) - 0.2^2 + 0.8213\right) + \frac{(0.2)^3}{6}(2 - 2(0.2) + 0.2^2 - 0.8213)$$

$$y_2 = 0.8213 - 0.15626 + 0.023626 + 0.0010916 = 0.6898$$

Example

- The solution,

$$y(0.2) \approx y_1 = 0.8213$$

$$y(0.4) \approx y_2 = 0.6898$$

- Exercise: Approximate y(0.6), y(0.8), and y(1.0).

Exercise

- Use Taylor's method of order two and three to approximate the solution for the following initial value problem.

$$y' = 3y + 3x \qquad 0 \le x \le 1$$
$$y(0) = 1$$

with h=0.5.

Do calculations in 4 decimal points.

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