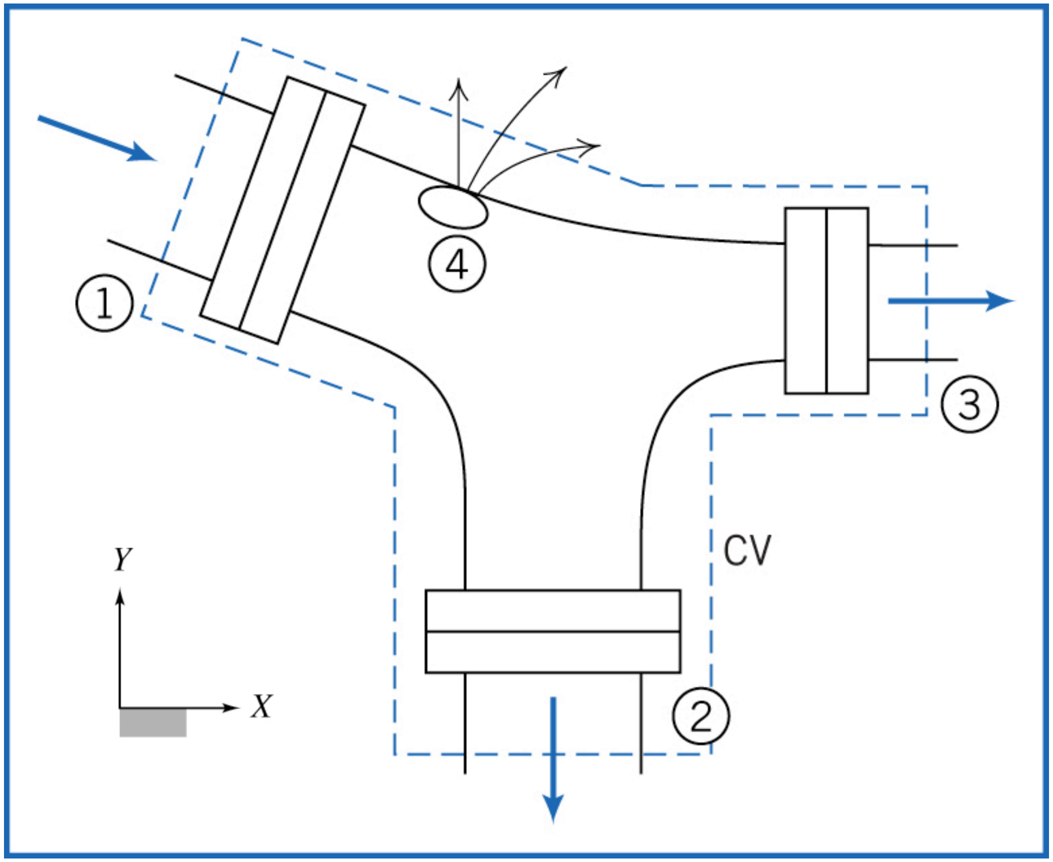
Control Volume Analysis (and Bernoulli's Equation)

Control Volume Analysis

- Consider the control volume in more detail for both mass, energy, and momentum:
  - open and closed systems
  - steady and transient analysis
- Control Volume
  - encloses the system or region of interest
  - can have multiple inlets/exits or none at all if it is a closed system (as we have seen)
  - is important much like the free body diagram

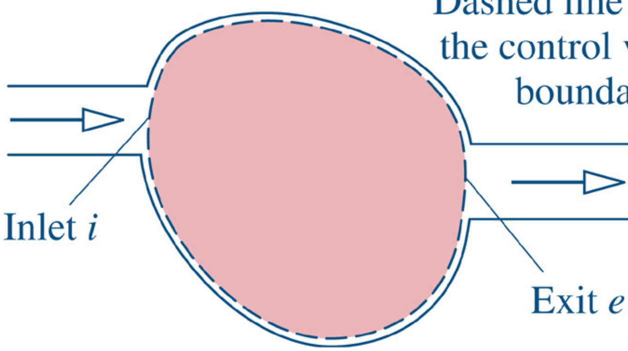
Control Volume Analysis

- Consider the following system:



Control Volume Analysis

- Simple Two Port System (Mass Balance)



- Rate Balance

$$\begin{bmatrix} \text{Time rate of change} \\ \text{of mass contained within} \\ \text{the control volume at time } t \end{bmatrix}_{CV} = \begin{bmatrix} \text{time rate of flow} \\ \text{of mass in across} \\ \text{inlet at time } t \end{bmatrix}_i - \begin{bmatrix} \text{time rate of flow} \\ \text{of mass out across} \\ \text{the exit at time } t \end{bmatrix}_e$$

$$\frac{dm_{CV}}{dt} = \dot{m}_i - \dot{m}_e$$

Conservation of Mass

- Extended to multiple inlet/exit control volumes, we may write the balance as:

$$\frac{dm_{CV}}{dt} = \sum_{inlets} \dot{m}_i - \sum_{exits} \dot{m}_e$$

- If there are **no** inlets/exits, the the system is closed and we have:

$$\frac{dm_{CV}}{dt} = 0 \quad or \quad m_{CV} = \text{Constant}$$

Conservation of Mass

- If the system is under steady state then the left hand side of the mass balance becomes:

$$\frac{dm_{CV}}{dt} = 0$$

or

$$\sum_{inlets} \dot{m}_i = \sum_{exits} \dot{m}_e$$

- We will examine many problems involving steady and unsteady flow.

Conservation of Mass

- The mass flow rate entering a port is defined as:

$$\dot{m} = \int_A \rho V_n dA$$

- If we have a stream with normal uniform velocity passing through an area A, the mass flow rate for this one-dimensional flow is:

$$\dot{m} = \rho \overline{V} A$$

Conservation of Mass

- The ***Conservation of Mass*** equation for a multiport control volume can then be taken as:

$$\frac{dm_{CV}}{dt} = \sum_{inlets}\left(\rho \overline{V} A\right)_i - \sum_{exits}\left(\rho \overline{V} A\right)_e$$

- This will be the starting point for all future analyses. We will then make assumptions as to *steady/unsteady* or *open/closed*.

Conservation of Mass

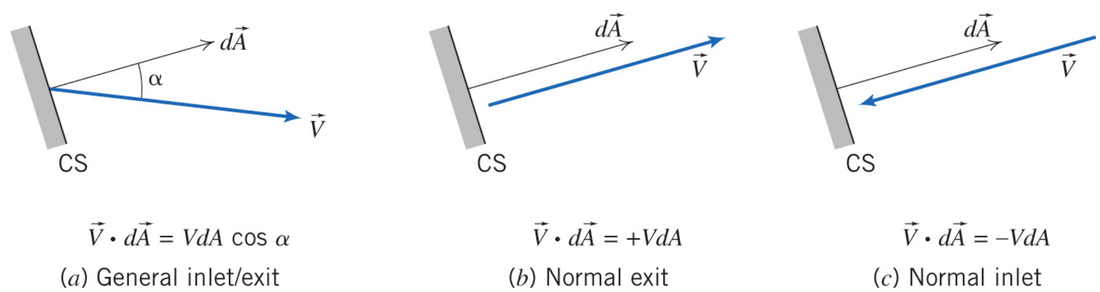
- The integral form of conservation of mass is frequently written as:

$$\frac{d}{dt}\int_V \rho dV + \int_A \vec{\rho} V \cdot \hat{n} dA = 0$$

- The unit vector dot product sign convention is:

$$\left(\vec{V}\cdot\hat{n}\right) \quad "-" = \text{inflow}$$

$$\left(\vec{V}\cdot\hat{n}\right) \quad "+" = \text{outflow}$$

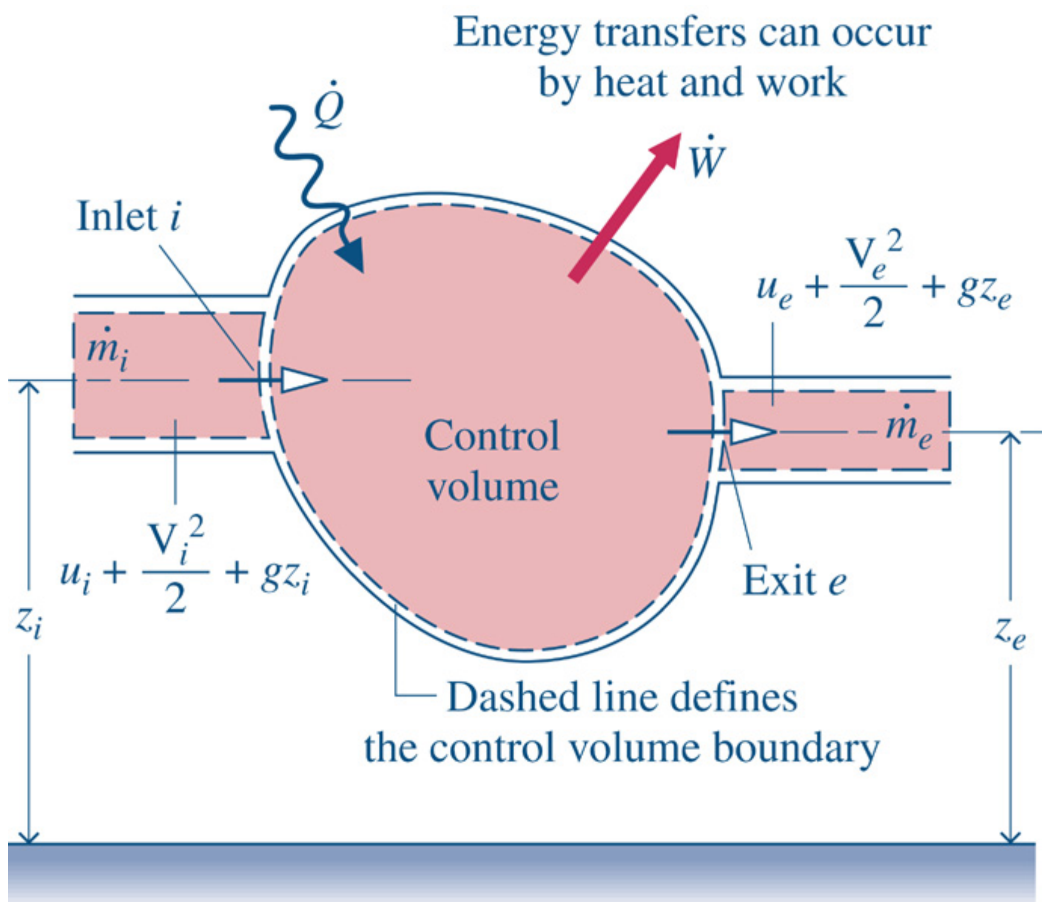


Conservation of Energy

- The *First Law of Thermodynamics* states that energy must be conserved, i.e. it can not be created or destroyed.
- The energy balance for a control volume follows a similar approach to that for *Conservation of Mass*, but has additional considerations.
- As before we will consider open and closed systems and steady/transient flows

Conservation of Energy

- Energy Balances



Conservation of Energy

- Rate Balance

$$\begin{bmatrix} \text{time rate of change} \\ \text{of energy contained} \\ \text{within the control} \\ \text{volume at time } t \end{bmatrix}_{CV} = \begin{bmatrix} \text{net rate of energy} \\ \text{transferred in} \\ \text{as heat transfer} \\ \text{at time } t \end{bmatrix}_{\dot{Q}} - \begin{bmatrix} \text{net rate of energy} \\ \text{transferred out} \\ \text{as work at time } t \end{bmatrix}_{\dot{W}} + \begin{bmatrix} \text{net rate of energy} \\ \text{transfer into the} \\ \text{control volume} \\ \text{accompanying mass} \\ \text{flow through ports} \end{bmatrix}$$

- Referring to the figure we see that in equation form this becomes:

$$\frac{dE_{CV}}{dt} = \dot{Q} - \dot{W} + \dot{m}_i\left(u_i + \frac{V_i^2}{2} + gz_i\right) - \dot{m}_e\left(u_e + \frac{V_e^2}{2} + gz_e\right)$$

Conservation of Energy

- If the control volume contains multiple inlets/ exits then we may write:

$$\frac{dE_{CV}}{dt} = \dot{Q} - \dot{W} + \sum_{inlets} \dot{m}_i \left(u_i + \frac{V_i^2}{2} + gz_i \right) - \sum_{exits} \dot{m}_e \left(u_e + \frac{V_e^2}{2} + gz_e \right)$$

- We must now consider what the work term really represents. At this point it is the *net* work transfer.
- It is more useful to account for this in more detail as there are different types of work.

Conservation of Energy

- Since work is always done on or by a control volume when matter flows across inlets/exits, it is convenient to separate work into two contributions:
  – work associated with fluid pressure
  – work associated with rotating shafts, boundary displacement, etc
- The former is referred to frequently as "*flow work*"

Conservation of Energy

- Work due to fluid flow can be considered if we consider that pressure times area is a force, p*A, and force times velocity, F*V, is power (or work), thus:

$$\begin{bmatrix} time\ rate\ of\ energy \\ transfer\ by\ work \\ due\ to\ flow \end{bmatrix} = pAV$$

- We will define the work term as:

$$\dot{W} = \dot{W}_{CV} + \sum_{exits} p_e A_e V_e - \sum_{inlets} p_i A_i V_i$$

Conservation of Energy

- Redefining the above in terms of mass flow rate and density (or specific volume) leads to:

$$\dot{W} = \dot{W}_{CV} + \sum_{exits} \frac{p_e \dot{m}_e}{\rho_e} - \sum_{inlets} \frac{p_i \dot{m}_i}{\rho_i}$$

$$\dot{W} = \dot{W}_{CV} + \sum_{exits} p_e v_e \dot{m}_e - \sum_{inlets} p_i v_i \dot{m}_i$$

- Combining with the energy balance gives:

$$\frac{dE_{CV}}{dt} = \dot{Q}_{CV} - \dot{W}_{CV} + \sum_{inlets} \dot{m}_i \left(u_i + p_i v_i + \frac{V_i^2}{2} + g z_i \right) - \sum_{exits} \dot{m}_e \left(u_e + p_e v_e + \frac{V_e^2}{2} + g z_e \right)$$

Conservation of Energy

- Conservation of energy is also frequently used in the following form using enthalpy:

$$\frac{dE_{CV}}{dt} = \dot{Q}_{CV} - \dot{W}_{CV} + \sum_{inlets} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i \right) - \sum_{exits} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e \right)$$

- Steady State

$$\dot{Q}_{CV} - \dot{W}_{CV} + \sum_{inlets} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i \right) - \sum_{exits} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e \right) = 0$$

- Closed System

$$\frac{dE_{CV}}{dt} = \dot{Q}_{CV} - \dot{W}_{CV} \qquad \text{or} \qquad \Delta E_{CV} = Q_{CV} - W_{CV}$$

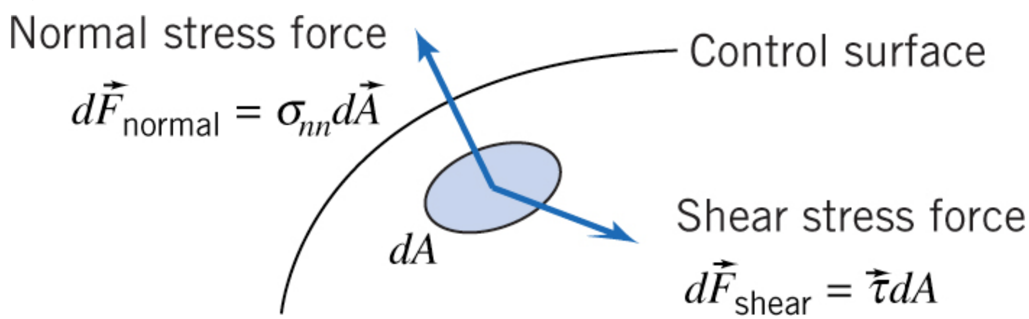
$$E = KE + PE + U$$

Conservation of Energy

- In integral form, we frequently write:

$$\frac{d}{dt}\int_V e\rho dV + \int_A e\rho\left(\vec{V}\cdot \hat{n}\right)dA = \dot{Q}_{CV} - \dot{W}_{CV}$$

- We will leave the basic equation in the above form and deal with work as required in problems.
- We may split up the work term to include several types of work, such as shaft work, viscous normal and shear work, and other forms such as electrical work.



Conservation of Momentum

- Rate Balance

$$\begin{bmatrix}\text{time rate of change}\\ \text{of momentum}\\ \text{within the CV}\end{bmatrix} = \begin{bmatrix}\text{net rate of momentum}\\ \text{flow into the CV}\end{bmatrix} - \begin{bmatrix}\text{net rate of momentum}\\ \text{flow out of the CV}\end{bmatrix} + \begin{bmatrix}\text{sum of external}\\ \text{forces acting on the CV}\end{bmatrix}$$

- External forces include:
  - Gravitational
  - Pressure
  - Viscous
- NOTE: This is a vector equation with three components for x, y, and z, directions

Conservation of Momentum

- Using the basic concepts of conservation principles and recognizing that we are considering a vector quantity, we write:

$$\frac{d\vec{M}}{dt} = \vec{M}_{in} - \vec{M}_{out} + \sum_{external} \vec{F}$$

- Using integral formulation for the momentum change and flow, we have (with same convention on unit vector product):

$$\frac{d}{dt}\int_V \vec{V}\rho dV + \int_A \rho\vec{V}\left(\vec{V}\cdot\hat{n}\right)dA = \sum_{external}\vec{F}$$

Conservation of Momentum

- For systems with one-dimensional inlets/exits, we can write:

$$\frac{d}{dt}\int_V \vec{V}\rho dV = \sum_{in}\dot{m}_i\vec{V}_i - \sum_{out}\dot{m}_e\vec{V}_e + \sum_{external}\vec{F}$$

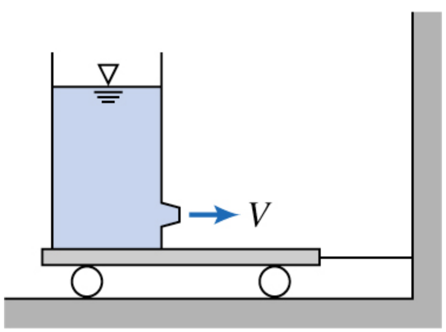
- For steady flow systems we have the simpler:

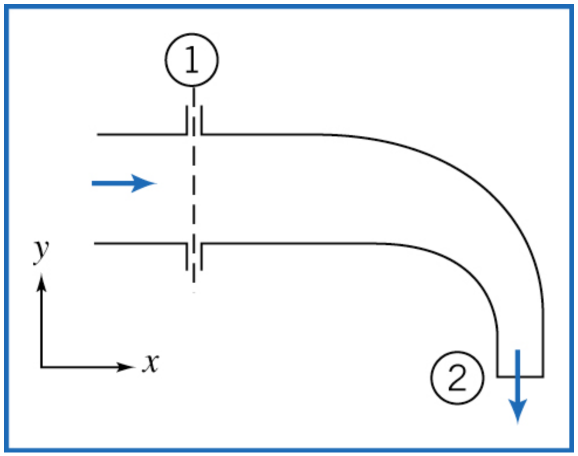
$$\sum_{out}\dot{m}_e\vec{V}_e - \sum_{in}\dot{m}_i\vec{V}_i = \sum_{external}\vec{F}$$

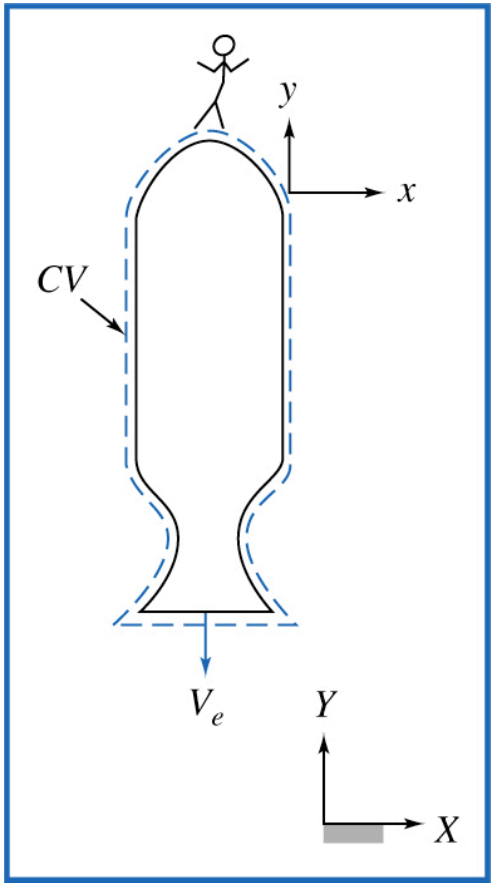
- This has 3 components. The sign of the vector velocity, must be considered (+ or -), accordingly.

Conservation of Momentum

- The momentum equation for a control volume can be used to determine reaction forces and thrust forces, among other things.
- It is used frequently in fluid mechanics in the same manner as conservation of momentum in rigid body dynamics.







Summary of Equations

$$\frac{d}{dt}\int_V \rho dV + \int_A \vec{\rho} V \cdot \hat{n} dA = 0$$

$$\frac{d}{dt}\int_V \rho dV = \sum_{inlets} \dot{m}_i - \sum_{exits} \dot{m}_e$$

$$\frac{d}{dt}\int_V e\rho dV + \int_A e\rho\left(\vec{V}\cdot \hat{n}\right)dA = \dot{Q}_{CV} - \dot{W}_{CV}$$

$$\frac{d}{dt}\int_V e\rho dV = \dot{Q}_{CV} - \dot{W}_{CV} + \sum_{inlets} \dot{m}_i\left(h_i + \frac{V_i^2}{2} + gz_i\right) - \sum_{exits} \dot{m}_e\left(h_e + \frac{V_e^2}{2} + gz_e\right)$$

$$\frac{d}{dt}\int_V \vec{V}\rho dV + \int_A \rho\vec{V}\left(\vec{V}\cdot \hat{n}\right)dA = \sum_{external} \vec{F}$$

$$\frac{d}{dt}\int_V \vec{V}\rho dV = \sum_{in} \dot{m}_i\vec{V}_i - \sum_{out} \dot{m}_e\vec{V}_e + \sum_{external} \vec{F}$$

Example - 14

- Consider the hydro dam as sketched. How much power can be converted assuming that a 30 m^3/s flow is obtained in a 3.5 m circular pipe which passes through a turbine. Assume all losses are negligible.

Example - 15

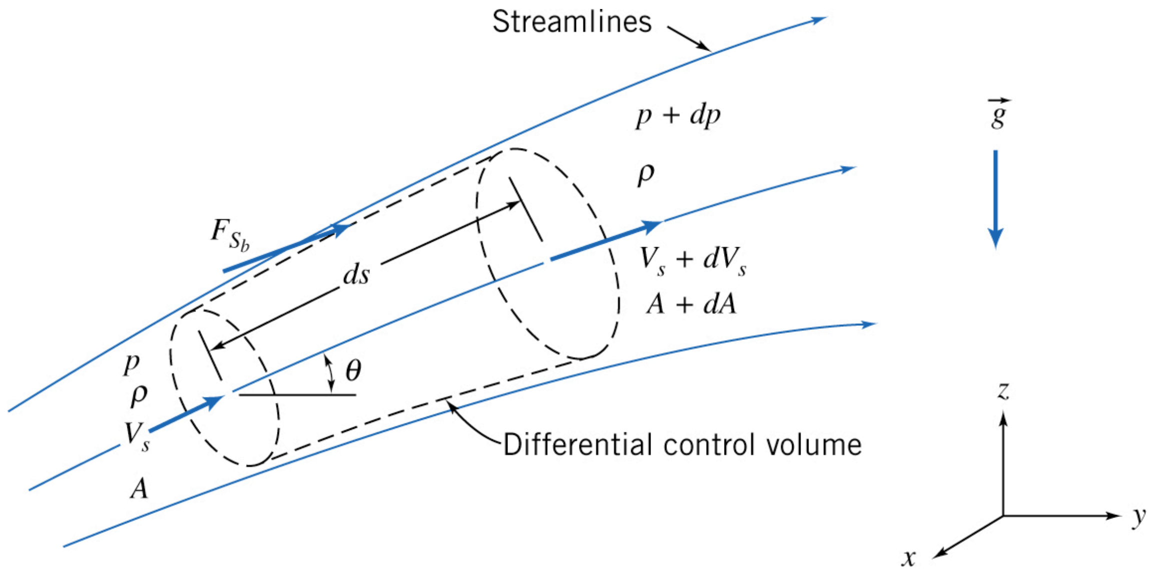
- Consider the piping element sketched in class. Find the force on the T-section, which is required to restrain the pipe. Assume the flow is frictionless.

Bernoulli's Equation

- We will now spend some time on Bernoulli's equation.
- It is one of the most famous equations in Fluid Mechanics, and also one of the most mis-used equations.
- We will consider its applications, and also examine two points of view from which it may be obtained.
- It has many useful applications both quantitatively and qualitatively.

Bernoulli's Equation

- Stream Tube Analysis:
  - Frictionless, incompressible, steady flow, along a stream line:



Bernoulli's Equation

- Mass Balance:

$$-\rho V_s A + \rho(V_s + dV_s)(A + dA) = 0$$

$$-\rho V_s A + \rho V_s A + \rho V_s dA + \rho A dV_s + \rho \underbrace{dA dV_s}_{small \sim 0} = 0$$

$$V_s dA + A dV_s = 0$$

- Momentum Balance:

$$\underbrace{\rho(V_s + dV_s)(A + dA)(V_s + dV_s)}_{out} - \underbrace{\rho V_s A V_s}_{in} = \underbrace{pA - (p + dp)(A + dA) + \left(p + \frac{dp}{2}\right) dA}_{net-pressure} - \underbrace{\rho g \sin\theta \left(A + \frac{dA}{2}\right) ds}_{gravity}$$

$$\underbrace{(\rho V_s A)(V_s + dV_s)}_{out} - \underbrace{\rho V_s A V_s}_{in} = \underbrace{-A dp - \left(\frac{dp}{2}\right) dA}_{net-pressure} - \underbrace{\rho g \sin\theta \left(A + \frac{dA}{2}\right) ds}_{gravity}$$

Bernoulli's Equation

- Simplifying (using *dpdA~0* and *dAds~0)*:

$$\rho V_s A dV_s = -A dp - \rho g A dz$$

$$\sin\theta = \frac{dz}{ds}$$

- Finally, we may write the above as:

$$d\left(\frac{V_s^2}{2}\right) + \frac{dp}{\rho} + g dz = 0$$

$$d\left(\frac{V_s^2}{2} + \frac{p}{\rho} + gz\right) = 0$$

or

$$\left(\frac{V_s^2}{2} + \frac{p}{\rho} + gz\right) = C$$

Bernoulli's Equation

- We may now write for any two points along the stream tube:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

- The three terms are referred to as: pressure head, velocity head, and elevation head. We can also write the above as:

$$\underbrace{(p_1 - p_2)}_{static-pressure} + \underbrace{\rho\left(\frac{V_1^2}{2} - \frac{V_2^2}{2}\right)}_{dynamic-pressure} + \underbrace{\rho g(z_1 - z_2)}_{elevation} = 0$$

Bernoulli's Equation

- The inherent assumptions are:
  - Incompressible flow
  - Frictionless flow
  - Steady flow
  - Flow along a streamline or through a stream tube.
- If there is friction along the stream tube, the balance will **not** be maintained.

Example - 16

- Consider flow discharging from a tank with a fluid level maintained at a height H above the center of discharge orifice. What is the exit velocity of the stream. The solution to this problem is also known as Torricelli’s theorem.

Example - 17

- Derive an ideal expression for the measurement of flow through a venturi meter and an orifice plate flow meter. A venturi is a section of pipe with a gradual restriction, while an orifice plate is a disk with a smaller flow area (orifice) in the center. More on this later in Chapter 8. For now, we will consider the ideal cases.

Example - 18

- Consider the Pitot-Static tube system sketched in class. This system is used to measure local velocity in a flow stream. You will need to use some basic manometry theory to complete the solution.

Bernoulli's Equation

- When there are losses, we usually write the Bernoulli equation as:

$$\underbrace{\left(\frac{p_1}{\rho g}+\frac{V_1^2}{2g}\right)}_{\text{Total Pressure at 1}}-\underbrace{\left(\frac{p_2}{\rho g}+\frac{V_2^2}{2g}\right)}_{\text{Total Pressure at 2}}=\underbrace{\left(z_2-z_1\right)}_{Elevation-Difference}+\underbrace{h_L}_{Losses}$$

- Where h_L is termed a head loss or friction loss.
- We can also introduce the idea of an Energy Grade Line (EGL) and Hydraulic Grade Line (HGL) for a flow.

Energy/Hydraulic Grade Lines

- Energy and Hydraulic Grade lines are defined as follows:

$$\frac{p}{\rho g} + z = HGL$$

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = EGL$$

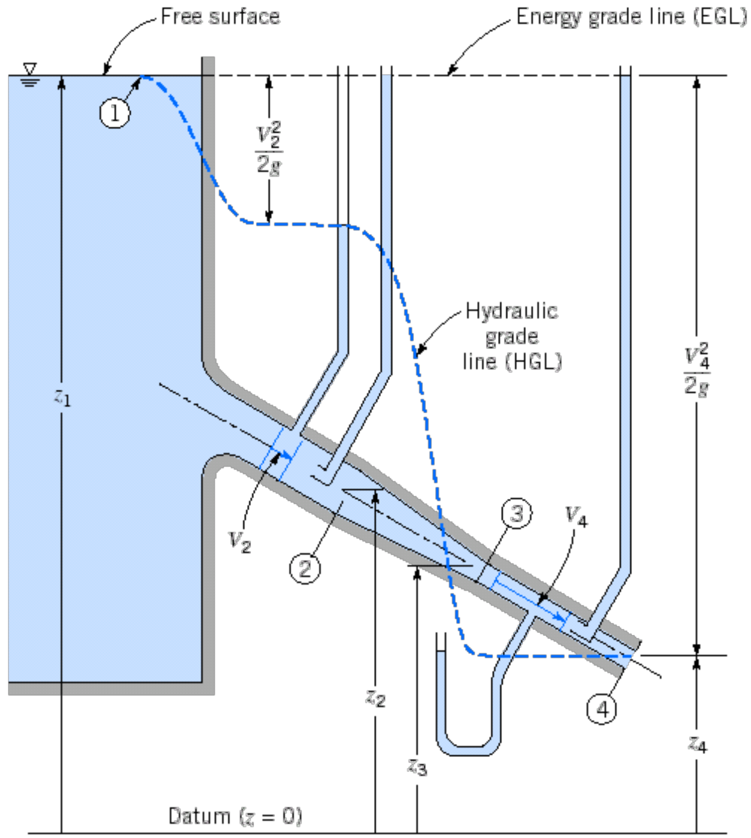


Fig. 6.6 Energy and hydraulic grade lines for frictionless flow.

Bernoulli's Equation

- We can also obtain Bernoulli's equation from the steady Conservation of Energy Equation:

$$\dot{Q}_{CV} - \dot{W}_{CV} + \sum_{inlets} \dot{m}_i \left(h_i + \frac{V_i^2}{2} + g z_i \right) - \sum_{exits} \dot{m}_e \left(h_e + \frac{V_e^2}{2} + g z_e \right) = 0$$

- If the Q and W terms are zero and we recognize that $h = u + pv$, where $v = 1/\rho$
- and for an isothermal flow $u_i = u_e$ for one inlet/ exit system, we obtain:

$$\left(\frac{p_i}{\rho} + \frac{V_i^2}{2} + g z_i \right) - \left(\frac{p_e}{\rho} + \frac{V_e^2}{2} + g z_e \right) = 0$$

Bernoulli's Equation

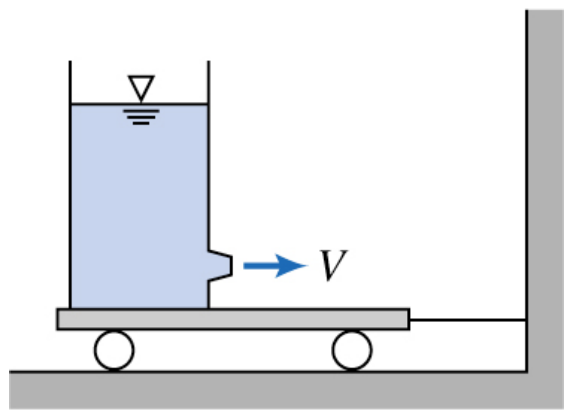
- We have now obtained the Bernoulli equation, but this time from energy balance considerations.
- Earlier we derived the Bernoulli equation from the momentum balance on a stream tube.
- How? The Bernoulli equation is a form of mechanical energy balance (forces integrated over distance).
- Work is F*ds, thus the equivalence between the two.

Example - 19

- Consider the sudden expansion in the pipe/duct section sketched in class. Using the conservation of mass, momentum, and energy, determine the loss of energy due to the flow expansion (deceleration).

Example - 20

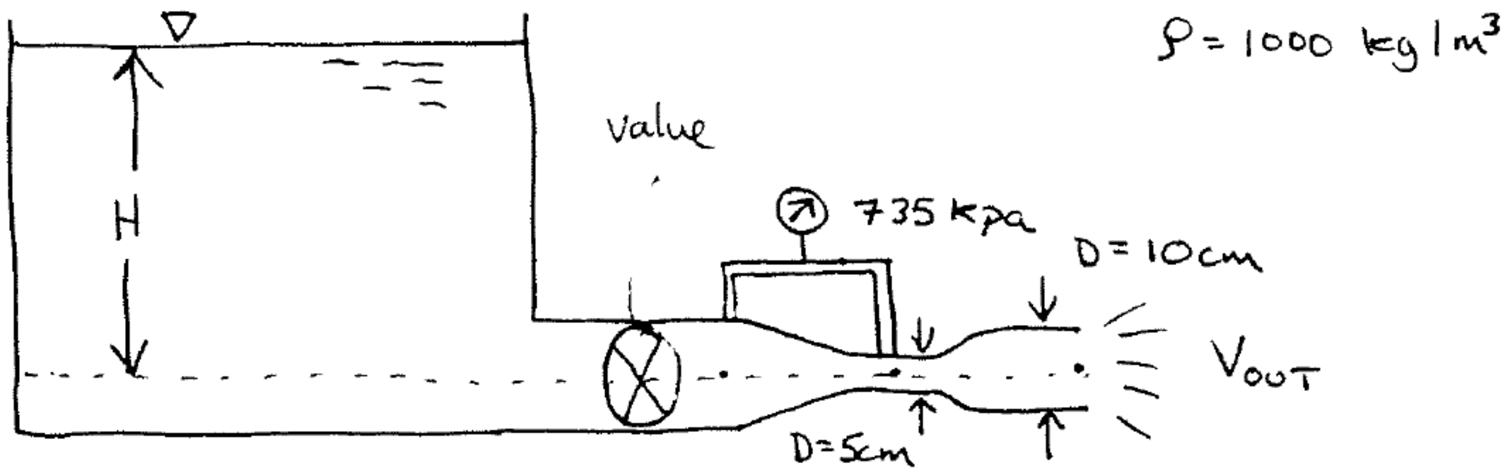
- Develop an expression for the tension in the cable assuming that the level of the tank is maintained at fixed height of H.



Example - 21

- 2001 Midterm Question #1

1) At the instant the valve is opened, the Venturi meter indicates a pressure difference of 735 kPa. What is the level of fluid in the tank and the velocity at the exit of the pipe. Assume an ideal Venturi meter and frictionless flow through the pipe and valve.

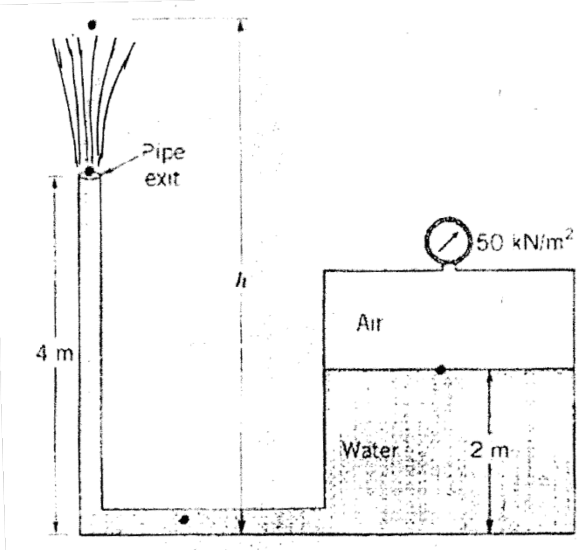


Example - 22

- 2005 Quiz #1, Question #1

1) Water flows steadily from a large tank and exits through a vertical, constant diameter pipe as shown below. The air in the tank is pressurized to 50 kPa (gage). Assuming frictionless flow determine:
   a. The height, h, to which the water rises. **(3 marks)**
   b. The velocity of the water in the pipe. **(2 marks)**
   c. The pressure in the horizontal section of pipe. **(2 marks)**
   d. If the air pressure is doubled, what is the new height? **(2 marks)**
   e. How is h affected by headlosses. **(1 mark)**

The density of water may be taken as $\rho = 1000\ kg/m^3$.



Example - 23

- 2005 Quiz #2, Question #2

2) Water with a flow rate of 50 L/min, flows steadily through a 3.5 cm diameter pipe as shown below. Two holes placed 10 m apart of diameter 5 mm allow water to escape from the pipe. Each water jet rises to the specified height shown, $h_1 = 50$ cm and $h_2 = 23.8$ cm. Using the principles of conservation of mass, momentum, and energy determine:
   a. The discharge Q_{out} **(4 marks)**
   b. Assuming *frictionless* flow, what is the pressure gain/loss due to changes in momentum only, in the direction of the main flow **(2 marks)**
   c. Is the flow frictionless? If not what is the total pressure (head) loss due to friction? **(2 marks)**
   d. If piezo-tubes of 1 m length are attached to the locations of each jet, what is the discharge Q_{out} and head loss in the section of pipe **(2 mark)**

The density of water may be taken as $\rho = 1000$ kg/m^3.