Chapter 7

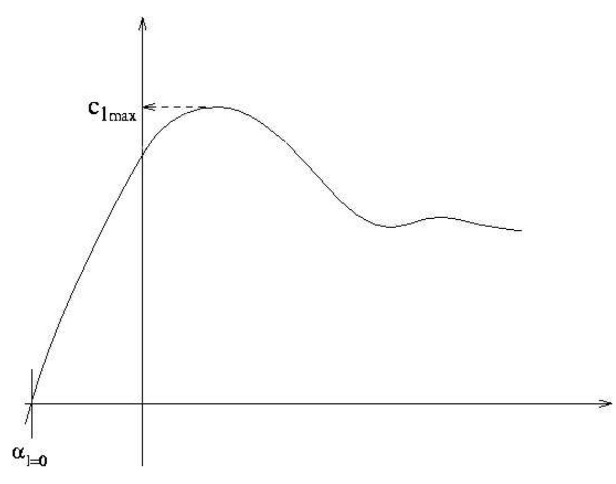
Incompressible Flow Over Airfoils

Aerodynamics of wings:
–2-D sectional characteristics of the airfoil;
–Finite wing characteristics (How to relate 2-D characteristics to 3-D characteristics)

How to obtain 2-D characteristics?
(a)Experimental methods (NACA airfoils)
(b)Analytical methods
(c)Numerical methods

Airfoil characteristics:
(1) $C_l - \alpha$
(2) $C_d - \alpha$
(3) $C_{m,c/4} - \alpha$

Characteristics of $C_l - \alpha$ curve:
(a)$\alpha_{l=0}$
(b)$a_0 = \frac{dC_l}{d\alpha}$
(c)C_{lmax}



AIRFOIL SECTION NOMENCLATURE

Mean camber line is the locus of the points midway between upper and lower surfaces of an airfoil as measured perpendicular to the mean camber line. Normally, the measurement can be made perpendicular to the chord line within acceptable engineering accuracy.

Chord line is the line joining the endpoints of the mean camber line.

Thickness distribution is the height of profile measured normal to the mean camber line.

Thickness ratio generally denoted by t/c is twice the maximum thickness to chord ratio.

Camber is the maximum distance between the mean camber line and the chord measured normal to the chord.

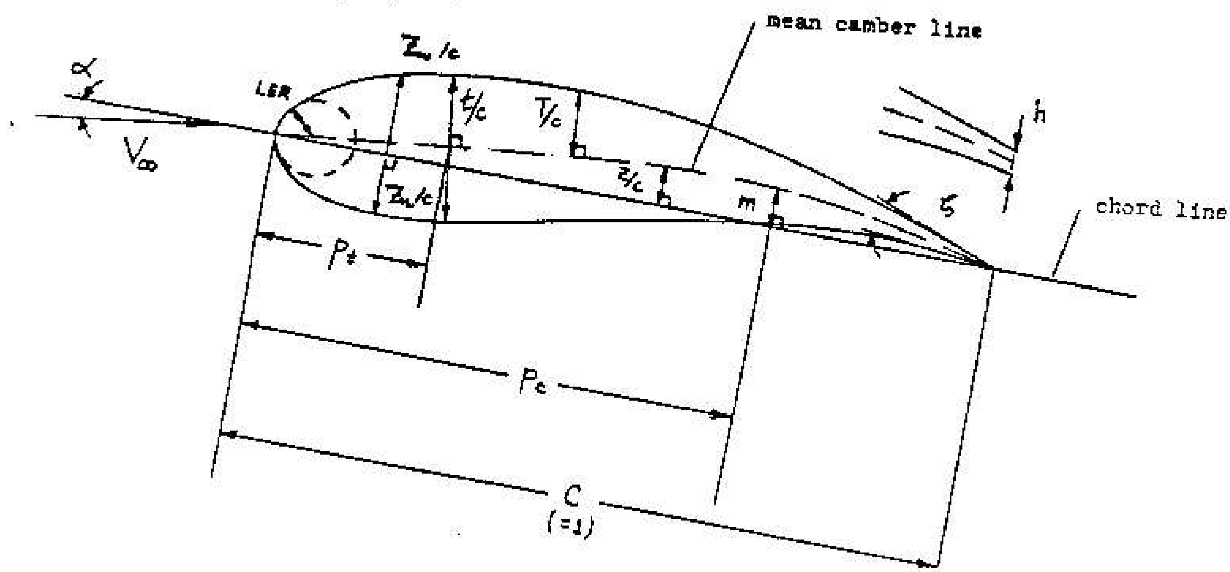
Leading-edge radius is the radius of a circle, tangent to the upper and lower surfaces, with its center located on a tangent to the mean camber line drawn through the leading edge of this line.

Center of pressure: The aerodynamic forces on an airfoil section may be represented by a lift, a drag, and a pitching moment. For every angle of attack there exists a point about which the pitching moment coefficient is zero. This point refered to as the center of pressure moves with change of angle of attack and is not necessarily within the airfoil section.

Aerodynamic center is a point about which the section moment coefficient is independent of the angle of attack. In contrast to the center of pressure the aerodynamic center generally lies within the airfoil section.

AIRFOIL PARAMETER NOMENCLATURE

T/c	= thickness distribution
t/c	= maximum airfoil thickness $[2(T/c)_{max}]$
P_t	= location of maximum thickness referenced to chord
z/c	= camber distribution
m	= maximum airfoil camber
P_c	= location of maximum camber referenced to chord
LED	= airfoil leading-edge radius
h	= trailing-edge thickness referenced to chord $[2(T_{t.e.}/c)]$
α	= airfoil angle of attack referenced to chord line
ζ	= trailing-edge angle



At low to moderate angles of attack $C_l - \alpha$ curve is linear. The flow moves slowly over the airfoil and is attached over most of the surface. At high angles of attack, the flow tends to separate from the top surface.

- $C_{l,max}$ occurs prior to stall
- $C_{l,max}$ is dependent on $R_e = \frac{\rho v c}{\mu}$
- $C_{m,c/4}$ is independent of R_e except for large α
- C_d is dependent on R_e
- The linear portion of the $C_l - \alpha$ curve is independent of R_e and can be predicted using analytical methods.

Theoretical/analytical methods to evaluate 2-D characteristics:
Recall from potential flow that a spinning cylinder produces lift $L = \rho V \Gamma$. For a 2-D Vortex (spinning clockwise):

$$\psi = \frac{\Gamma}{2\pi} ln r$$
$$\phi = -\frac{\Gamma}{2\pi}\theta$$
$$v_\theta = \frac{\Gamma}{2\pi} r$$
$$v_r = 0$$

7.1 Vortex Filament

Consider 2-D/point vortices of same strength duplicated in every plane parallel to the z-x plane along the y-axis from $-\infty$ to ∞. The flow is 2-D and is irrotational everywhere except the y-axis. y-axis is the straight vortex filament and may be defined as a line.

- Definition: A vortex filament is a straight or curved line in a fluid which coincides with the axis of rotation of successive fluid elements.
- Helmholt's vortex theorems:

  1. The strength of a vortex filament is constant along its length.

     Proof: A vortex filament induces a velocity field that is irrotational at every point excluding the filament. Enclose a vortex filament with a sheath from which a slit has been removed. The vorticity at every point on the surface=0. Evaluate the circulation for the sheath.

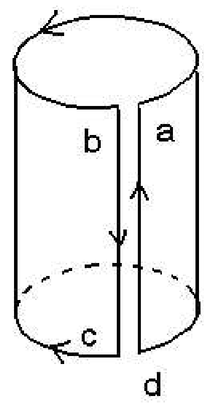
     Circulation $= -\oint_C \vec{V} \cdot d\vec{s} = -\iint_S (\nabla \times \vec{V}) \cdot d\vec{A}$.

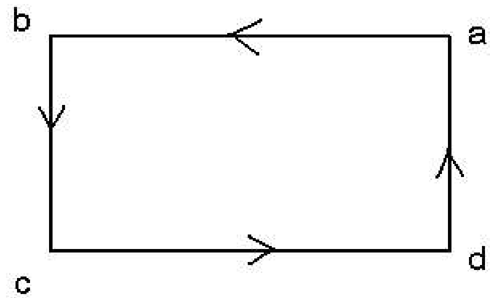
     Sheath is irrotational. Thus $\nabla \times \vec{V} = 0$, $-\oint_C \vec{V} \cdot d\vec{s} = 0$ or $\oint \vec{V} \cdot d\vec{s} = 0$

     $$\int_a^b \vec{V} \cdot d\vec{s} + \int_b^c \vec{V} \cdot d\vec{s} + \int_c^d \vec{V} \cdot d\vec{s} + \int_d^a \vec{V} \cdot d\vec{s} = 0$$

     However,

     $$\int_b^c \vec{V} \cdot d\vec{s} + \int_d^a \vec{V} \cdot d\vec{s} = 0$$





as it constitutes the integral across the slit.

Thus,

$$\int_a^b \vec{V} \cdot d\vec{s} + \int_c^d \vec{V} \cdot d\vec{s} = 0$$

$$\int_a^b \vec{V} \cdot d\vec{s} = -\int_c^d \vec{V} \cdot d\vec{s} = \int_d^c \vec{V} \cdot d\vec{s} = \Gamma$$

2. A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid or form a closed path.

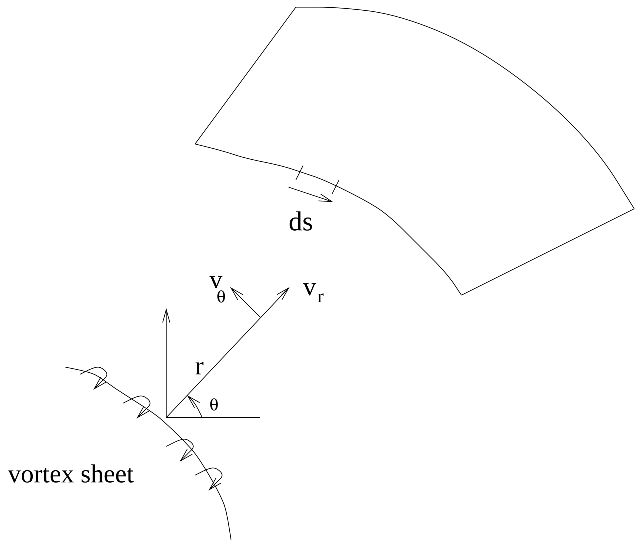
3. In the absence of rotational external flow, a fluid that is irrotational remains irrotational.

4. In the absence of rotational extenal force, if the circulation around a path enclosing a definite group of particles is initially zero, it will remain zero.

5. In the absence of rotational extenal force, the circulation around a path that encloses a tagged group of elements is invariant.

Vortex Sheet or vortex surface

An infinite number of straight vortex filaments placed side by side form a vortex sheet. Each vortex filament has an infinitesimal strength $\gamma(s)$.



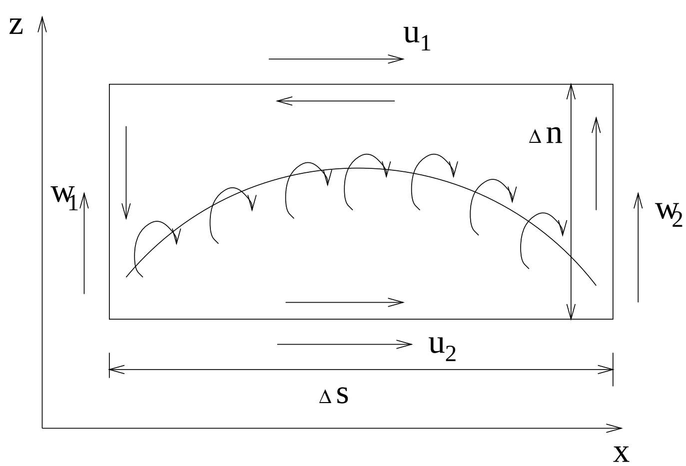
$\gamma(s)$ is the strength of vortex sheet per unit length along s.
$v_\theta = \frac{-\Gamma}{2\pi r}$ for 2-D (point vortex).

A small portion of the vortex sheet of strength γds induces an infinitesimally small velocity dV at a field point $P(r,\theta)$.
So, $v_\theta|vortex filament = \frac{-\gamma ds}{2\pi r}$
$\therefore dv_P = -\frac{\gamma ds}{2\pi r}$.

Circulation Γ around a point vortex is equal to the strength of the vortex. Similarly, the circulation around the vortex sheet is the sum of the strengths of the elemental vortices. Therefore, the circulation Γ for a finite length from point 'a' to point 'b' on the vortex sheet is given by:

$$\Gamma = \int_a^b \gamma(s) ds$$

Across a vortex sheet, there is a discontinuous change in the tangential component of velocity and the normal component of velocity is preserved.



$$\Delta\Gamma = -\int \vec{v} \cdot d\vec{l}$$

$$\Delta\Gamma = -\int_{Box} \vec{v} \cdot d\vec{l} = -[w_2\Delta n - u_1\Delta s - w_1\Delta n + u_2\Delta s]$$

$$\gamma\Delta s = (u_1 - u_2)\Delta s + (w_1 - w_2)\Delta n$$

As $\Delta n \to 0$, we get

$$\gamma\Delta s = (u_1 - u_2)\Delta s \text{ or } \gamma = (u_1 - u_2)$$

$\gamma = (u_1 - u_2)$ states that the local jump in tangential velocity across the vortex sheet is equal to the local sheet strength.

Kutta Condition:

For a given airfoil at a given angle of attack, the value of Γ around the airfoil is such that the flow leaves the trailing edge smoothly.

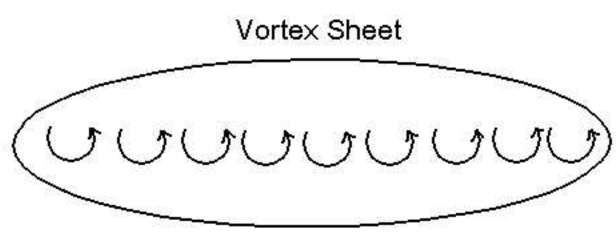
The Kutta condition tells us how to find Γ; it is based on experimental observation. A body with finite angle TE in relative motion through a fluid will create about itself a circulation of sufficient strength to hold the rear stagnation point at the TE. If the TE has a zero angle, the Kutta Condition requires that the velocity of fluid leaving upper and lower surfaces at the TE be equal and non-zero. A body with a finite TE angle will have crossing streamlines at the TE unless the TE is a stagnation point. The Kutta Condition eliminates the crossing streamlines. Consider the TE as a vortex sheet:

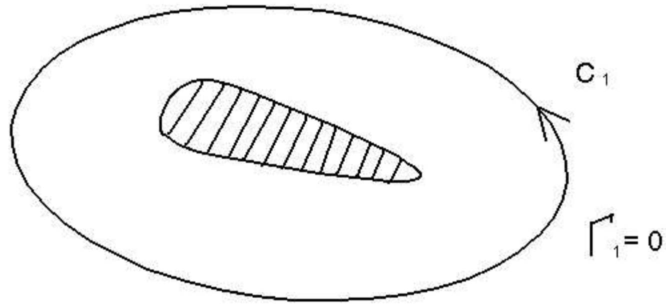
$$\gamma(TE) = V_u - V_l$$

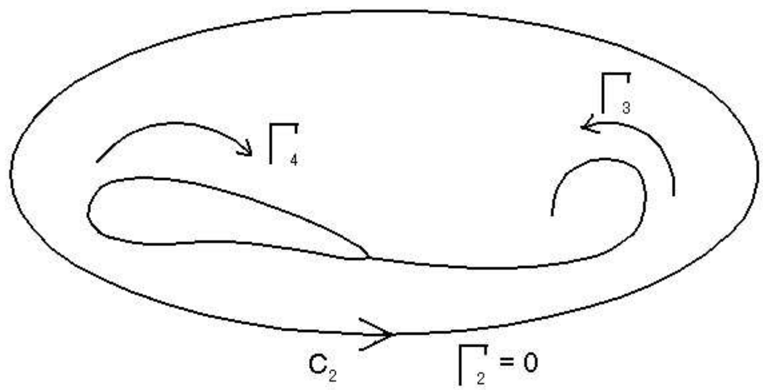
If the TE has a finite angle $V_u = V_l = 0$ because TE is a stagnation point. Or $\gamma(TE) = 0$. If the TE is a cusp (zero angle), $V_u = V_l$ and hence $\gamma(TE) = V_u - V_l = 0$. Thus, Kutta Condition expressed mathematically in terms of vortex strength is $\gamma(TE) = 0$.

7.1.1 Bound Vortex and Starting Vortex

The question might arise: Does a real airfoil flying in a real fluid give rise to a circulation about itself? The answer is yes. When a wing section with a sharp T.E is put into motion, the fluid has a tendency to go around the sharp T.E from the lower to the upper surface. As the airfoil moves along vortices are shed from the T.E which form a vortex sheet.







- Helmholtz's theorem:

  If $\Gamma = 0$ originally in a flow it remains zero.

- Kelvin's theorem:

  Circulation around a closed curve formed by a set of continuous fluid elements remains constant as the fluid elements move through the flow, $\frac{D\Gamma}{Dt} = 0$. Substantial derivative gives the time rate of change following a given fluid element. The circulation about the airfoil is replaced by a vortex of equal strength. This is the bound vortex since it remains bound to the airfoil. A true vortex remains attaced to the same fluid particles and moves with the general flow. Thus, as far as resultant forces are concerned, a bound vortex of proper strength in a uniform flow is equivalent to a body with circulation in a uniform flow.

- Both the theorems are satisfied by the starting vortex and bound vortex system. In the beginning, $\Gamma_1 = 0$ when the flow is started within the contour C_1. When the flow over the airfoil is developed, Γ_2 within C_2 is still zero which includes the starting vortex Γ_3 and the bound vortex Γ_4 which are equal and opposite.

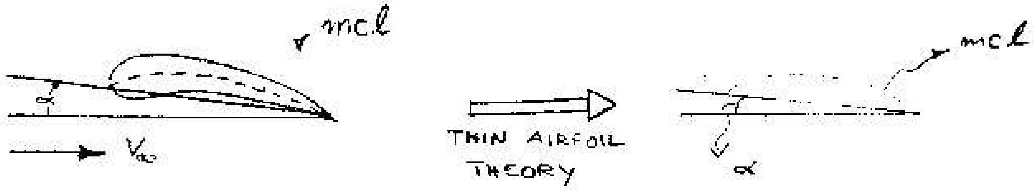
THIN AIRFOIL THEORY

Thin airfoil theory is based on the assumption that under certain conditions an airfoil section may be replaced by its mean camber line (mcl).

Experimental observation: If airfoil sections of the same mcl but different thickness functions are tested experimentally at the same α it is found that the lift L' and the point application of the lift for the different airfoil sections are practically the same provided that

(1) t/c is small;
(2) $(z/c)_{max}$ = m is small;
(3) α is small.

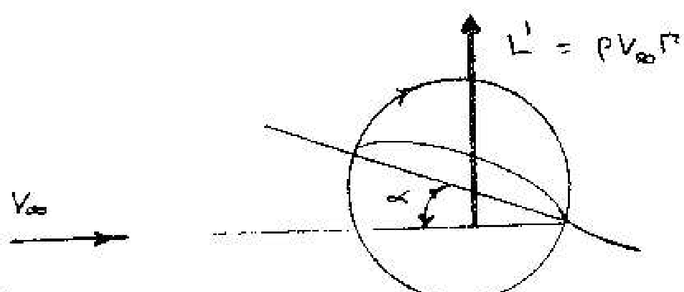
This observation permitted the formulation of thin airfoil theory because it allowed the airfoil to be replaced by the mcl.



The problem now is to find, theoretically the flow of an ideal fluid around this infinitely thin sheet (mcl) flying through the air at a velocity V_∞ at an angle of attack α.

Any solution must satisfy:

(1) Equation of continuity
(2) Irrotaional condition
(3) Outer b.c. – Flow at infinity must be undisturbed.
(4) Inner b.c. – mcl must be a streamline.
(5) In addition, since the thin airfoil is being supported in level flight there must be a lift L' acting on the airfoil.
(6) Since $L' = \rho\ \bar{V}_{rel} \times \bar{\Gamma}$ (Kutta-Joukowski Theorem) any theoretical analysis must introduce a circulation Γ around the airfoil section of sufficient magnitude to satisfy the Kutta condition that the flow leave the TE smoothly.



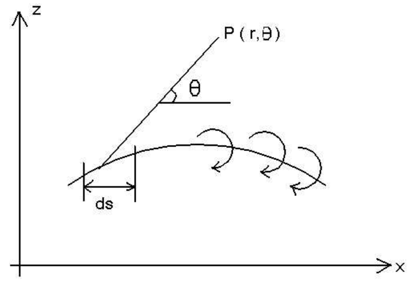
Therefore in thin airfoil theory the mcl is replaced by a vortex sheet of varying strength $\gamma(s)$ such that the above conditions are satisfied and our aim is to determine this 'γ' distribution.

Summary: Thin airfoil theory stated as a problem says for a vortex sheet placed on the mcl in a uniform flow of V_∞ determine $\gamma(s)$ such that the mcl is a streamline subject to the condition $(\gamma)_{TE} = 0$.

7.2 Fundamental Equation of Thin Airfoil Theory

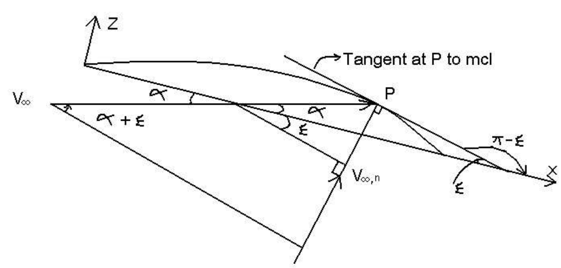
Principle: Mean camber line is a streamline of the flow.
Velocity induced by a 2-D vortex is $\vec{V} = v_\theta \hat{e}_\theta = -\frac{\Gamma}{2\pi r}\hat{e}_\theta$ where Γ is the strength of the 2-D vortex. Similarly the velocity induced by the vortex sheet of infinitesimal length ds is given by

$$d\vec{v}_P = -\frac{\gamma(s)ds}{2\pi r}\hat{e}_\theta$$



To force the mean camber line to be a streamline, the sum of all velocity components normal to the mcl must be equal to zero. Consider the flow induced by an elemental vortex sheet ds at a point P on the vortex sheet. It is perpendicular to the line connecting the center of ds to the point P given by

$$d\vec{v}_P = -\frac{\gamma(s)ds}{2\pi r}\hat{e}_\theta$$



Thus dw'_P the velocity normal to the mcl is:

$$dw'_P = dv_P \cos\beta = -\frac{\gamma(s)\cos\beta ds}{2\pi r}$$

where β is the angle made by dv_p to the normal at P, and r is the distance from the center of ds to the point P.
The induced velocity due to the vortex sheet representing the entire mcl is given by;

$$w'_P(s) = -\frac{1}{2\pi}\int_{LE}^{TE} \frac{\gamma(s)\cos\beta}{r} ds$$

Now determine the component of the freestream velocity normal to the mcl.

$$V_{\infty,n} = V_\infty \sin(\alpha + \epsilon)$$

where α is the angle of attack and ϵ is the angle made by the tangent at point P to the x-axis.
The slope of the tangent line at point P is given by:

$$\frac{dz}{dx} = \tan(\pi - \epsilon) = -\tan\epsilon$$

or

$$\epsilon = \tan^{-1}(-\frac{dz}{dx})$$

$$V_{\infty,n} = V_\infty(\sin\alpha + \tan^{-1}(-\frac{dz}{dx}))$$

In order that the mcl is a streamline. $w'_P(s) + V_{\infty,n} = 0$, or

$$-\frac{1}{2\pi}\int_{LE}^{TE}\frac{\gamma(s)\cos\beta}{r}ds + V_\infty(\sin\alpha + \tan^{-1}(-\frac{dz}{dx})) = 0$$

within thin airfoil theory approximation $s \to x$, $ds \to dx$, $\cos\beta = 1$ and $r \to (x_0 - x)$, where x varies from 0 to c, and x_0 refers to the point P.

After changing these variables and making the small angle approximation for sin and tan, and upon rearrangement we get:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x)}{x_0 - x}dx = V_\infty(\alpha - \frac{dz}{dx})$$

7.3 Flat Plate at an Angle of Attack

The following analysis is an exact solution to the flat plate or an approximate solution to the symmetric airfoil. The mean camber line becomes the chord and hence:

$$\frac{dz}{dx} = 0$$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x)}{(x_0 - x)}dx = V_\infty\alpha$$

In order to facilitate analytic solution, we do a variable transformation such that:

$$x = \frac{c}{2}(1 - \cos\theta)$$

$$x_0 = \frac{c}{2}(1 - \cos\theta_0)$$

$\theta = 0$ at LE and $\theta = \pi$ at TE and θ increases in CW, $dx = \frac{c}{2}\sin\theta d\theta$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\frac{c}{2}\sin\theta}{\frac{c}{2}[(1-\cos\theta_0) - (1-\cos\theta)]}d\theta = V_\infty\alpha$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta = V_\infty\alpha$$

Here we simply state a rigorous solution for $\gamma(\theta)$ as:

$$\gamma(\theta) = 2\alpha V_\infty \frac{1+cos\theta}{\sin\theta}$$

We can verify this solution by substitution as follows:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta = \frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{(1+\cos\theta)}{(\cos\theta-\cos\theta_0)}d\theta$$

We now use the following result to evaluate the above integral.

$$\int_0^\pi \frac{(\cos n\theta)}{(\cos\theta-\cos\theta_0)}d\theta = \frac{\pi\sin n\theta_0}{\sin\theta_0}$$

$$\frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{(1+\cos\theta)}{(\cos\theta-\cos\theta_0)}d\theta = \frac{V_\infty\alpha}{\pi}\left(\int_0^\pi \frac{1}{\cos\theta-\cos\theta_0}d\theta + \int_0^\pi \frac{\cos\theta}{\cos\theta-\cos\theta_0}d\theta\right) \tag{7.1}$$

$$= \frac{V_\infty\alpha}{\pi}(0+\pi) = V_\infty\alpha \tag{7.2}$$

Thus, it satisfies the equation:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta = V_\infty\alpha$$

In addition, the solution for γ also satisfies the Kutta condition.
When $\theta = \pi$,

$$\gamma(\pi) = 2V_\infty\alpha\frac{1-1}{0}$$

By using L'Hospital' rule, we get

$$\gamma(\pi) = 2V_\infty\alpha\frac{-\sin\pi}{\cos\pi} = 0$$

Thus it satifies the Kutta condition.

7.4 2-D lift coefficient for a thin/symmetrical airfoil

$$L^{'} = \rho_\infty V_\infty \Gamma = \rho V_\infty \int_{LE}^{TE} \gamma(s)ds$$

Where s is along the mcl.
By using thin airfoil approximation:

$$L^{'} \simeq \rho V_\infty \int_{LE}^{TE} \gamma(x)dx = \rho V_\infty \int_0^c \gamma(x)dx$$

Using the transformation $x = \frac{c}{2}(1 - cos\theta)$

$$L^{'} \simeq \rho_\infty V_\infty \int_0^c \gamma(x)dx = \frac{1}{2}\rho_\infty V_\infty c \int_0^\pi \gamma(\theta)\sin\theta d\theta$$

Substituting the solution:

$$\gamma(\theta) = \frac{2\alpha V_\infty(1+\cos\theta)}{\sin\theta}$$

$$L^{'} \simeq \frac{1}{2}\rho_\infty V_\infty c \int_0^\pi 2V_\infty \alpha(1+\cos\theta)d\theta$$

$$L^{'} \simeq \alpha\rho_\infty V_\infty^2 c \int_0^\pi (1+\cos\theta)d\theta$$

$$L^{'} \simeq c\pi\alpha\rho_\infty V_\infty^2$$

$$L^{'} \simeq 2\pi\alpha(\frac{\rho_\infty V_\infty^2}{2})(c \times 1)$$

$$L^{'} \simeq 2\pi\alpha q_\infty S$$

or $C_l = \frac{L^{'}}{q_\infty S} = 2\pi\alpha$, and $\frac{dC_l}{d\alpha} = 2\pi$

$\frac{dC_l}{d\alpha} = 2\pi$ shows that lift curve is linearly proportional to the angle of attack.

7.4.1 Calculation of Moment Coefficient

$$M^{'}_{LE} = -\int_0^c x(dL^{'}) \tag{7.3}$$

$$= -\int_0^c x(\rho_\infty V_\infty d\Gamma) \tag{7.4}$$

$$= -\int_0^c x(\rho_\infty V_\infty \gamma(x)dx) \tag{7.5}$$

$$= -\rho_\infty V_\infty \int_0^c \gamma(x)xdx \tag{7.6}$$

$$= -\rho_\infty V_\infty \int_0^\pi \frac{2\alpha V_\infty(1+\cos\theta)}{\sin\theta} \cdot \frac{c}{2}(1-\cos\theta) \cdot \frac{c}{2}\sin\theta d\theta \tag{7.7}$$

$$= -2\alpha\rho_\infty V_\infty^2 \frac{c^2}{4}\int_0^\pi (1-\cos^2\theta)d\theta \tag{7.8}$$

$$= -\alpha\rho_\infty V_\infty^2 \frac{c^2}{2}\left(\frac{\pi}{2}\right) \tag{7.9}$$

$$= -q_\infty c^2 \left(\frac{\alpha\pi}{2}\right) \tag{7.10}$$

$$C_{m,LE} = \frac{M'_{LE}}{q_\infty Sc} = \frac{M'_{LE}}{q_\infty c^2} = -\frac{\pi\alpha}{2}$$

$$C_l = 2\pi\alpha$$

$$C_{m,LE} = -\frac{C_l}{4}$$

$$M'_{LE} = M'_{c/4} - L'_{c/4}$$

$$C_{m,LE} = C_{m,c/4} - C_l/4$$

$$C_{m,LE} = -C_l/4$$

$$C_{m,c/4} = 0$$

$C_{m,c/4}$ is equal to zero for all values of α.

$c/4$ is the aerodynamic center. Aerodynamic center is that point on an airfoil where moments are independant of angle of attack.

7.5 Thin Airfoil Theory for Cambered Airfoil

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x)dx}{x_0 - x} = V_\infty(\alpha - \frac{dz}{dx}) \qquad (A)$$

where $\frac{dz}{dx}$ is the slope of mcl at x_0.
For symmetric airfoil, mcl is a straight line and hence $\frac{dz}{dx} = 0$ everywhere. On the other hand, for a cambered airfoil $\frac{dz}{dx}$ varies from point to point.
As before, we do a variable transformation given by:

$$x = \frac{c}{2}(1 - \cos\theta)$$
$$dx = \frac{c}{2}\sin\theta d\theta$$

Equation (A) becomes:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\theta)\sin\theta d\theta}{(\cos\theta - \cos\theta_0)} = V_\infty(\alpha - \frac{dz}{dx}) \qquad (B)$$

has a solution of the form:

$$\gamma(\theta) = 2V_\infty[A_0(\frac{1+\cos\theta}{\sin\theta}) + \sum_{n=1}^{\infty} A_n \sin n\theta]$$

Sub. solution in equation (B)

$$\frac{1}{\pi}\int_0^\pi \frac{A_0(1+\cos\theta)}{(\cos\theta - \cos\theta_0)}d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^\pi \frac{A_n \sin n\theta \sin\theta}{(\cos\theta - \cos\theta_0)}d\theta = (\alpha - \frac{dz}{dx}_{x_0})$$

Using the integral

$$\int_0^\pi \frac{\cos n\theta}{(\cos\theta - \cos\theta_0)}d\theta = \pi\frac{\sin n\theta_0}{\sin\theta_0}$$

The first term becomes:

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi \frac{A_0(1+\cos\theta)}{(\cos\theta-\cos\theta_0)}d\theta \\
&= \frac{1}{\pi}\int_0^\pi \frac{A_0}{(\cos\theta-\cos\theta_0)}d\theta + \frac{1}{\pi}\int_0^\pi \frac{A_0\cos\theta}{(\cos\theta-\cos\theta_0)}d\theta \\
&= A_0
\end{aligned}$$

Using the integral

$$\int_0^\pi \frac{\sin n\theta\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta = -\pi\cos n\theta_0$$

The second term becomes:

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^\pi \frac{A_n\sin n\theta\sin\theta}{(\cos\theta-\cos\theta_0)}d\theta \cong -\sum_{n=1}^{\infty}A_n\cos n\theta_0$$

Therefore Equation (B) becomes:

$$A_0 - \sum_{n=1}^{\infty}A_n\cos n\theta_0 = \alpha - \frac{dz}{dx}_{x_0}$$

Upon rearrangement the slope at a point P on the mcl is given by:

$$\frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty}A_n\cos n\theta_0$$

From Fourier serious

$$f(\theta) = B_0 + \sum_{n=1}^{\infty}B_n\cos n\theta$$

Where,

$$\begin{aligned}
B_0 &= \frac{1}{\pi}\int_0^\pi f(\theta)d\theta \\
B_n &= \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta d\theta \\
n &= 1, 2, ..., \infty
\end{aligned}$$

$$\begin{aligned}
(\alpha - A_0) &= B_0 = \frac{1}{\pi}\int_0^\pi (\frac{dz}{dx})d\theta \\
A_n &= B_n
\end{aligned}$$

Evaluation of Γ

$$\Gamma = \int_0^c \gamma(x)dx = \frac{c}{2}\int_0^c \gamma(\theta)\sin\theta d\theta$$

From thin airfoil theroy

$$\begin{aligned}
&\gamma(\theta) = 2V_\infty[\frac{A_0(1+\cos\theta)}{\sin\theta} + \sum_{n=1}^{\infty}A_n\sin n\theta] \\
&\Gamma = \frac{c}{2}\int_0^c \gamma(\theta)\sin\theta d\theta = \frac{c}{2}\int_0^\pi 2V_\infty A_0(1+\cos\theta)d\theta + \frac{c}{2}\int_0^\pi 2V_\infty\sum_{n=1}^{\infty}A_n\sin n\theta\sin\theta d\theta \\
&= cV_\infty[A_0(\theta+\sin\theta)|_0^\pi + \sum_{n=1}^{\infty}A_n\int_0^\pi \sin n\theta\sin\theta d\theta]
\end{aligned}$$

Using

$$\sum_{n=1}^{\infty} A_n \int_0^{\pi} \sin n\theta \sin\theta d\theta$$
$$= \frac{\pi}{2} \quad for \quad n = 1$$
$$= 0 \quad for \quad n \neq 1$$

$$\Gamma = cV_\infty[A_0\pi + A_1\frac{\pi}{2}]$$
$$L' = \rho_\infty V_\infty \Gamma = \rho V_\infty^2 c[\pi A_0 + \frac{\pi}{2}A_1]$$
$$C_l = \frac{L'}{q_\infty s} = \frac{L'}{q_\infty c} = 2[\pi A_0 + \frac{\pi}{2}A_1]$$

C_l is normalized by the α as seen by the chord connecting the LE and TE of the mcl.
c is the chord connecting the LE and TE of the mcl.

$$\begin{aligned} C_l &= 2\pi A_0 + \pi A_1 \\ &= 2\pi\left(\alpha - \frac{1}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)d\theta\right) + \pi\left(\frac{2}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)\cos\theta d\theta\right) \\ &= 2\pi\left[\alpha + \frac{1}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)(\cos\theta - 1)d\theta\right] \end{aligned}$$

$$\frac{dC_l}{d\alpha} = 2\pi$$

as is the case for symmetric airfoil.
Also,

$$C_l = \frac{dC_l}{d\alpha}(\alpha - \alpha_{L=0}) = 2\pi(\alpha - \alpha_{L=0})$$

$$\therefore \alpha_{L=0} = -\frac{1}{\pi}\int_0^{\pi}(\cos\theta - 1)\frac{dz}{dx}d\theta$$

Determination of moment coefficient

$$M'_{LE} = -\rho_\infty V_\infty \int_0^c x\gamma(x)dx$$
$$C_{m,LE} = \frac{M'_{LE}}{q_\infty sc} = \frac{-2}{V_\infty c^2}\int_0^c x\gamma(x)dx$$

As before we do a variable transformation from x to θ. Thus,

$$x = \frac{c}{2}(1 - \cos\theta)$$
$$dx = \frac{c}{2}\sin\theta d\theta$$
$$\gamma(\theta) = 2V_\infty[\frac{A_0(1+\cos\theta)}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta]$$
$$C_{m,LE} = -\int_0^{\pi} A_0(1 - \cos^2\theta)d\theta - \sum_{n=1}^{\infty} A_n \int_0^{\pi}(1 - \cos\theta)\sin\theta \sin n\theta d\theta$$

Using the following definite integrals:

$$\int_0^{\pi} \cos^2\theta d\theta = \frac{\pi}{2}$$
$$\int_0^{\pi} \sin^2\theta d\theta = \frac{\pi}{2}$$
$$\int_0^{\pi} \sin\theta \sin n\theta d\theta$$
$$= \frac{\pi}{2} \quad for \quad n = 1$$
$$= 0 \quad for \quad n = 2, ..., \infty$$
$$\int_0^{\pi} \sin\theta\cos\theta\sin n\theta d\theta$$
$$= 0 \quad for \quad n = 1$$
$$= \frac{\pi}{4} \quad for \quad n = 2$$
$$= 0 \quad for \quad n = 3, ..., \infty$$

$$\begin{aligned} C_{m,LE} &= -\int_0^{\pi} A_0 d\theta + \int_0^{\pi} A_0 \cos^2\theta d\theta \\ &\quad -\sum_{n=1}^{\infty}\left(A_n\int_0^{\pi}\sin\theta\sin n\theta d\theta - A_n\int_0^{\pi}\sin\theta\cos\theta\sin n\theta d\theta\right) \\ &= -\pi A_0 + A_0\frac{\pi}{2} - A_1\frac{\pi}{2} + A_2\frac{\pi}{4} \\ &= -A_0\frac{\pi}{2} - A_1\frac{\pi}{2} + A_2\frac{\pi}{4} \end{aligned}$$

$$C_l = 2\pi A_0 + A_1\pi$$
$$C_{m,LE} = -\frac{\pi}{2}\left[\frac{2A_0 + 2A_1 - A_2}{2}\right] = -\left[\frac{C_l}{4} + \frac{\pi}{4}(A_1 - A_2)\right]$$
$$M'_{LE} = M'_{\frac{c}{4}} - L'\frac{c}{4} \qquad (1)$$
$$C_{m,LE} = c_{m,\frac{c}{4}} - \frac{C_l}{4} = -\left[\frac{C_l}{4} + \frac{\pi}{4}(A_1 - A_2)\right] \qquad (2)$$
$$C_{m,\frac{c}{4}} = -\frac{\pi}{4}(A_1 - A_2)$$
$$C_{m,LE} = -x_{cp}\frac{C_l}{c} \qquad (3)$$
$$x_{cp} = [\frac{c}{4} + \frac{\pi c}{4C_l}(A_1 - A_2)] = \frac{1}{4}[c + \frac{\pi c}{C_l}(A_1 - A_2)]$$

Relationship between pressure on mcl and γ

$$dL' = (p_l - p_u)\cos\eta(ds \cdot 1)$$
$$L' = \int_{LE}^{TE}(p_l - p_u)\cos\eta ds \qquad (A)$$
$$L' = \int_{LE}^{TE}\rho V_\infty \gamma(s) ds \qquad (B)$$

Equating (A) and (B) and setting $\cos\eta \cong 1$, we get

$$\int_{LE}^{TE}(p_l - p_u)ds = \int_{LE}^{TE}\rho V_\infty\gamma(s)ds$$

or

$$(p_l - p_u) = \rho V_\infty \gamma(s) \qquad (1)$$

Using Bernoullis equation:

$$p_l + \frac{1}{2}\rho(u_l^2 + w^2) = p_u + \frac{1}{2}\rho(u_u^2 + w^2)$$
$$p_l - p_u = \frac{\rho}{2}(u_u + u_l)(u_u - u_l) \qquad (2)$$

From vortex sheet theory:

$$u_u - u_l = \gamma(s) \qquad (3)$$

From (1), (2) and (3)

$$V_\infty = \frac{u_u + u_l}{2}$$

i.e., within thin airfoil approximation, the average of top and bottom surface velocities at any point on the mcl is equal to the freestream velocity.

$$\begin{aligned}
c_{p,l} - c_{p,u} &= \frac{(p_l - p_\infty)}{q_\infty} - \frac{(p_u - p_\infty)}{q_\infty} \\
&= \frac{p_l - p_u}{q_\infty} \\
&= \frac{\frac{1}{2}\rho(u_u + u_l)(u_u - u_l)}{q_\infty} \\
&= \frac{\gamma(s)(2V_\infty)}{v_\infty^2} \\
&= \frac{2\gamma(s)}{V_\infty}
\end{aligned}$$