GAUSSIAN ELIMINATION & LU DECOMPOSITION

1. Gaussian Elimination

It is easiest to illustrate this method with an example. Let's consider the system of equations

To solve for x, y, and z, we must eliminate some of the unknowns from some of the equations. Consider adding -2 times the first equation to the second equation and also adding 6 times the first equation to the third equation:

$$\begin{array}{rcrr}
 x - 3y + z &= 4 \\
 0x + 2y + 6z &= -10 \\
 0x + 15y - 9z &= 33
 \end{array}$$

We have now eliminated the x term form the last two equations, Now simplify the last two equations by 2 and 3, respectively:

$$x - 3y + z = 4$$

$$0x + y + 3z = -5$$

$$0x + 5y - 3z = 11$$

To eliminate the y term in the last equation, multiply the second equation by -5 and add it to the third equation:

$$x - 3y + z = 4
 0x + y + 3z = -5
 0x + 0y - 18z = 36$$

From the third equation, we can get z = -2, substituting this into the second equation yields y = -1. Using both of these results in the first equation gives x = 3. This process of progressively solving for the unknowns is back-substitution.

Now, let's see this example in matrix:

First, convert the system of equations into an augmented matrix:

2. LU Decomposition

If A is a square matrix and it can be factored as A = LU where L is a lower triangular matrix and U is an upper triangular matrix, then we say that A has an **LU-Decomposition** of LU.

If A is a square matrix and it can be reduced to a row-echelon form, U, without interchanging any rows, then A can be factored as A = LU where L is a lower triangular matrix.

LU decomposition of a matrix is not unique.

There are three factorization methods:

Crout Method: diag (U) = 1; $u_{ii} = 1$

Doolittle Method: diag (L) = 1; $l_{ii} = 1$

Choleski Method: diag (U) = diag (L) ; $u_{ii} = l_{ii}$

To solve several linear systems Ax = b with the same A, and A is big, we would like to avoid repeating the steps of Gaussian elimination on A for every different B. The most efficient and accurate way is LU-decomposition, which in effect records the steps of Gaussian elimination. This is Doolittle Method.

Without pivoting:

Ax = b

LUx = b

To solve this, first we solve Ly = b for y by forward-substitution method,

then solve Ux = y for x by backward-substitution method.

With pivoting:

Ax = b

PAx = Pb, where P is permutation matrix.

$$LUx = Pb$$

To solve this, first we solve Ly = Pb for y by forward-substitution method,

then solve Ux = y for x by backward-substitution method.

The main idea of the LU decomposition is to record the steps used in Gaussian elimination on A in the places where the zero is produced.

Let's see an example of LU-Decomposition without pivoting:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix}$$

The first step of Gaussian elimination is to subtract 2 times the first row form the second row. In order to record what was done, the multiplier, 2, into the place it was used to make a zero.

$$\xrightarrow{R2-2R1} \begin{bmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ 0 & 2 & -10 \end{bmatrix}$$

There is already a zero in the lower left corner, so we don't need to eliminate anything there. We record this fact with a (0). To eliminate a_{32} , we need to subtract -2 times the second row from the third row. Recording the -2:

$$\xrightarrow{R_{3}-(-2)R_{2}} \begin{bmatrix} 1 & -2 & 3\\ (2) & -1 & 6\\ (0) & (-2) & 2 \end{bmatrix}$$

Let U be the upper triangular matrix produced, and let L be the lower triangular matrix with the records and ones on the diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -10 \end{bmatrix}$$

Then,

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix} = A$$

Go back to the first example, rewrite the system of equation into matrix equation:

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix} \qquad Ax = b$$

$$A: \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 1 \\ (2) & -2 & 6 \\ (-6) & -15 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 1 \\ (2) & -2 & 6 \\ (-6) & \left(-\frac{15}{2} \right) & -54 \end{bmatrix}$$