Solution of Nonlinear Equations: Graphical and Incremental Search Methods

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Solution of Nonlinear Equations

Part I

Solution of Nonlinear Equations

General Form of the Problem

Many engineering problems involve finding one or more values of x that satisfy one of the following forms of equations:

1. Form 1:
$$f(x) = 0$$
2. Form 2:
$$g(x) = C$$
$$f(x) = g(x) - C = 0$$
3. Form 3:
$$g(x) = h(x)$$
$$f(x) = g(x) - h(x) = 0$$

Types of Nonlinear Equations

- Polynomial equations
- Transcendental equations
  - Exponential equations
  - Logarithmic equations
  - Trigonometric equations
  - Hyperbolic equations

Graphical Interpretation

Solutions to equations of the form $f(x) = 0$ can be seen as places where the graph of $f(x)$ crosses or touches the x axis.

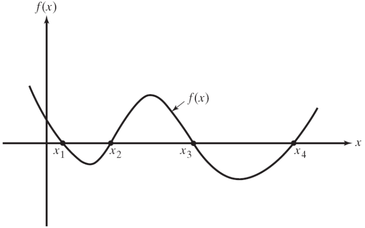
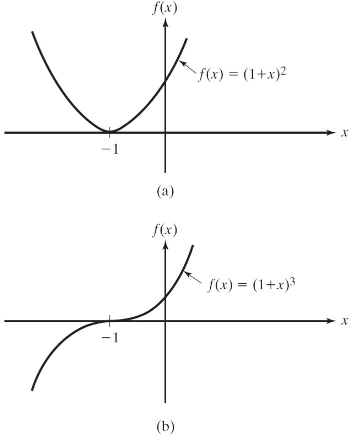
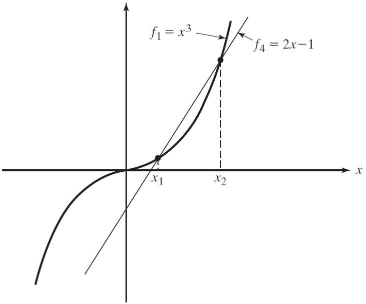


Figure 2.1
x_1, x_2, x_3, x_4 - roots of the equation.

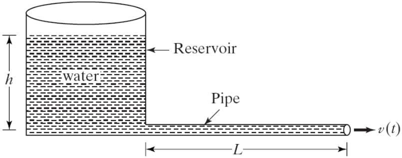


Graphical Interpretation



Solutions to equations of the form $f(x) = g(x)$ can be seen as places where the graphs of $f(x)$ and $g(x)$ intersect.

Mathematical Model

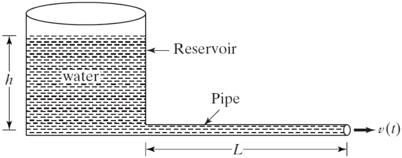


Water is discharged from a reservoir through a long pipe as shown. By neglecting the change in the level of the reservoir, the transient velocity of the water flowing from the pipe, $v(t)$, can be expressed as

$$\frac{v(t)}{\sqrt{2gh}} = \tanh\left(\frac{t}{2L}\sqrt{2gh}\right),$$

where h is the height of the fluid in the reservoir, L is the length of the pipe, g is the acceleration due to gravity, and t is the time elapsed from the beginning of the flow.

Governing Equations



$$\frac{v(t)}{\sqrt{2gh}} = \tanh\left(\frac{t}{2L}\sqrt{2gh}\right)$$

Find the value of h necessary for achieving a velocity of $v = 5$ m/s at time $t = 3$ s when $L = 5$ m and $g = 9.81$ m/s^2.

Solution of Equation

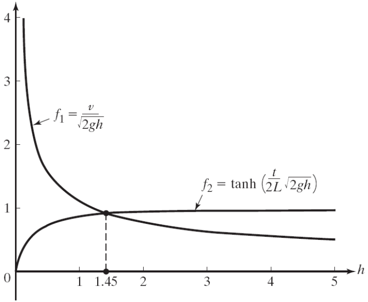
Substitute the values for v, t, L, and g into the previous equation on the left side

$$\frac{v(t)}{\sqrt{2gh}} = \frac{5}{\sqrt{2(9.81)h}} = \frac{1.1288}{\sqrt{h}}$$

and the right side

$$\tanh\left(\frac{t}{2L}\sqrt{2gh}\right) = \tanh\left(\frac{3}{2(5)}\sqrt{2(9.81)h}\right) = \tanh\left(1.3288\sqrt{h}\right)$$

Solution of Equation



Plot the two sides of the equation as separate functions of h, then find their intersections. In this case, the two graphs intersect around $h = 1.45$ m, so the original equation is satisfied with $h = 1.45$ m.

Incremental Search Method

Incremental search is the most basic automated numerical method for solving nonlinear equations.

The method:

1. Pick a starting point x_0 and a step size Δx. Use a positive Δx if you want to search to the right, and a negative Δx if you want to search to the left.
2. Let $x_1 = x_0 + \Delta x$ and calculate $f(x_0)$ and $f(x_1)$.
3. If the sign of $f(x)$ changes between x_0 and x_1, it is assumed that a root of $f(x)$ exists on the interval (x_0, x_1).
4. If the sign of $f(x)$ does not change between x_0 and x_1, let $x_2 = x_1 + \Delta x$ and repeat the process.

Incremental Search Example

Find the root of the equation

$$f(x) = \frac{1.1288}{\sqrt{h}} - \tanh\left(1.3288\sqrt{h}\right) = 0$$

using the incremental search method with $x_0 = 1.0$ and $\Delta x = 0.1$. Evaluate the function $f(x)$ at $x = 1.0, 1.1, 1.2, \cdots$:

$$\begin{aligned}
x_0 &= 1.0 & f(x_0) &= 0.2598 \\
x_1 &= 1.1 & f(x_1) &= 0.1923 \\
x_2 &= 1.2 & f(x_2) &= 0.1336 \\
x_3 &= 1.3 & f(x_3) &= 0.0822 \\
x_4 &= 1.4 & f(x_4) &= 0.0366 \\
x_5 &= 1.5 & f(x_5) &= -0.0040
\end{aligned}$$

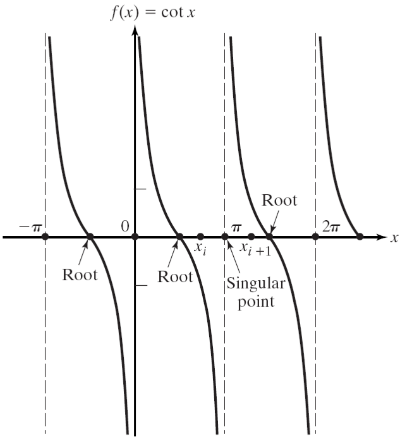
Incremental Search Example

Since the sign of $f(x)$ changed between $x = 1.4$ and $x = 1.5$, we assume there is a root of $f(x)$ between 1.4 and 1.5. Repeating this method with $x_0 = 1.4$ and $\Delta x = 0.01$ would allow us to make a more accurate estimate of the root.

Incremental Search Limitations

- Only finds real-valued roots of $f(x)$. It cannot find complex roots of polynomials.
- Only finds roots where $f(x)$ crosses the x axis. It cannot find roots where $f(x)$ is tangent to the x axis.
- May be fooled by singularities in $f(x)$, such as in the tangent and cotangent functions.
- If the step size Δx is too large, you may miss closely-spaced roots by skipping over them.

Example of Singularities



Homework

- Read articles on course homepage labeled "Lec. 01 Reading".
- Read and work along with examples given in Chapters 1–2 of *Getting Started with MATLAB 7* (link on course homepage).
- Be prepared to receive homework assignments on bisection and Newton-Raphson methods Monday.