“JUST THE MATHS”

UNIT NUMBER

15.9

ORDINARY DIFFERENTIAL EQUATIONS 9 (Simultaneous equations (B))

by

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UNIT 15.9 - ORDINARY DIFFERENTIAL EQUATIONS 9

SIMULTANEOUS EQUATIONS (B)

15.9.1 INTRODUCTION

For students who have studied the principles of eigenvalues and eigenvectors (see Unit 9.6), a second method of solving two simultaneous linear differential equations is to interpret them as a single equation using matrix notation. The discussion will be limited to the simpler kinds of example, and we shall find it convenient to use t, x_1 and x_2 rather than x, y and z.

15.9.2 MATRIX METHODS FOR HOMOGENEOUS SYSTEMS

To introduce the technique, we begin by considering two simultaneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2.\end{aligned}$$

which are of the "homogeneous" type, since no functions of t, other than x_1 and x_2, appear on the right hand sides.

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(ii) Secondly, in a similar way to the method appropriate to a single differential equation, we make a trial solution of the form

$$\mathrm{X} = \mathrm{K}e^{\lambda t},$$

where

$$\text{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

is a constant matrix of order 2×1.

This requires that

$$\lambda \text{K} e^{\lambda t} = \text{MK} e^{\lambda t} \quad \text{or} \quad \lambda \text{K} = \text{MK},$$

which we may recognise as the condition that λ is an eigenvalue of the matrix M, and K is an eigenvector of M.

The solutions for λ are obtained from the "characteristic equation"

$$|\text{M} - \lambda \text{I}| = 0.$$

In other words,

$$\begin{vmatrix} a - \lambda & b \\ c & b - \lambda \end{vmatrix} = 0,$$

leading to a quadratic equation having real and distinct solutions ($\lambda = \lambda_1$ and $\lambda = \lambda_2$), real and coincident solutions (λ only) or conjugate complex solutions ($\lambda = l \pm jm$).

(iii) The possibilities for the matrix K are obtained by solving the homogeneous linear equations

$$\begin{aligned} (a - \lambda_1 k_1 + b k_2 &= 0, \\ c k_1 + (d - \lambda_1) k_2 &= 0, \end{aligned}$$

giving $k_1 : k_2 = 1 : \alpha$ (say)

and

$$
\begin{aligned}
(a-\lambda_2)k_1 + bk_2 &= 0,\\
ck_1 + (d-\lambda_2)k_2 &= 0,
\end{aligned}
$$

giving $k_1 : k_2 = 1 : \beta$ (say).

Finally, it may be shown that, according to the types of solution to the auxiliary equation, the solution of the differential equation will take one of the following three forms, in which A and B are arbitrary constants:

(a)

$$
A\begin{bmatrix}1\\ \alpha\end{bmatrix}e^{\lambda_1 t} + B\begin{bmatrix}1\\ \beta\end{bmatrix}e^{\lambda_2 t};
$$

(b)

$$
\left\{(At+B)\begin{bmatrix}1\\ \alpha\end{bmatrix} + \frac{A}{b}\begin{bmatrix}0\\ 1\end{bmatrix}\right\}e^{\lambda t};
$$

or

(c)

$$
e^{lt}\left\{\begin{bmatrix}A\\ pA+qB\end{bmatrix}\cos mt + \begin{bmatrix}B\\ pB-qA\end{bmatrix}\sin mt\right\},
$$

where, in (c), $1 : \alpha = 1 : p + jq$ and $1 : \beta = 1 : p - jq$.

EXAMPLES

1. Determine the general solution of the simultaneous differential equations

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1 + 5x_2,\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -x_1 + 2x_2.
\end{aligned}
$$

Solution

The characteristic equation is

$$\begin{vmatrix} -4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 + 2\lambda - 3 = 0 \;\text{ or }\; (\lambda-1)(\lambda+3) = 0.$$

When $\lambda = 1$, we need to solve the homogeneous equations

$$\begin{aligned} -5k_1 + 5k_2 &= 0, \\ -k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : 1$.

When $\lambda = -3$, we need to solve the homogeneous equations

$$\begin{aligned} -k_1 + 5k_2 &= 0, \\ -k_1 + 5k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{1}{5}$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} e^{-3t}$$

or, alternatively,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-3t},$$

where A and B are arbitrary constants.

2. Determine the general solution of the simultaneous differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - x_2,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 + 3x_2.$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 4 = 0 \text{ or } (\lambda - 2)^2 = 0.$$

When $\lambda = 2$, we need to solve the homogeneous equations

$$-k_1 - k_2 = 0,$$
$$k_1 + k_2 = 0,$$

both of which give $k_1 : k_2 = 1 : -1$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ (At + B) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} e^{2t},$$

where A and B are arbitrary constants.

3. Determine the general solution of the simultaneous differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - 5x_2,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 2x_1 + 3x_2.$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 13 = 0,$$

which gives $\lambda = 2 \pm j3$.

When $\lambda = 2 + j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1-j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1-j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1-j3}{5}$.

When $\lambda = 2 - j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1+j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1+j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1+j3}{5}$.

The general solution is therefore

$$\frac{e^{2t}}{5}\left\{\begin{bmatrix} A \\ -A+3B \end{bmatrix}\cos 3t + \begin{bmatrix} B \\ -B-3A \end{bmatrix}\sin 3t\right\},$$

where A and B are arbitrary constants.

Note:

From any set of simultaneous differential equations of the form

$$\begin{aligned} a\frac{\mathrm{d}x_1}{\mathrm{d}t} + b\frac{\mathrm{d}x_2}{\mathrm{d}t} + cx_1 + dx_2 &= 0, \\ a'\frac{\mathrm{d}x_1}{\mathrm{d}t} + b'\frac{\mathrm{d}x_2}{\mathrm{d}t} + b'x_1 + c'x_2 &= 0, \end{aligned}$$

it is possible to eliminate $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in turn, in order to obtain two equivalent equations of the form discussed in the above examples.

15.9.3 EXERCISES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + 3x_2, \\ \frac{dx_2}{dt} &= 3x_1 + x_2. \end{aligned}$$

2. Solve completely, the simultaneous differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + 2x_2, \\ \frac{dx_2}{dt} &= 4x_1 + x_2, \end{aligned}$$

given that $x_1 = 3$ and $x_2 = -3$ when $t = 0$.

3. Solve completely, the simultaneous differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 + 9x_2, \\ \frac{dx_2}{dt} &= 11x_1 + x_2, \end{aligned}$$

given that $x_1 = 20$ and $x_2 = 20$ when $t = 0$.

4. Solve completely, the simultaneous differential equations

$$\begin{aligned} \frac{dx_1}{dt} - \frac{dx_2}{dt} + 2x_1 - 2x_2 &= 0, \\ \frac{dx_1}{dt} + 2\frac{dx_2}{dt} - 7x_1 - 5x_2 &= 0, \end{aligned}$$

given that $x_1 = 2$ and $x_2 = 0$ when $t = 0$.

5. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2.\end{aligned}$$

6. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2.\end{aligned}$$

15.9.4 ANSWERS TO EXERCISES

1.

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

2.

$$\begin{bmatrix}1\\1\end{bmatrix}e^{5t} + 2\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}.$$

3.

$$2\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t} + \begin{bmatrix}18\\22\end{bmatrix}e^{10t}.$$

4.

$$\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

5.

$$\left\{(At+B)\begin{bmatrix}1\\-1\end{bmatrix} + \frac{A}{2}\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}.$$

6.

$$e^{7t}\left\{\begin{bmatrix}A\\-A+2B\end{bmatrix}\cos 2t + \begin{bmatrix}B\\-B-2A\end{bmatrix}\sin 2t\right\}.$$