Vorticity: $\underline{w} = \nabla \times \underline{u}$

If a flow is irrottational, then:

$$\underline{w} = \nabla \times \underline{u} = 0$$

Thus, can introduce a scalar potential:

$$\underline{u} = \nabla f \qquad \Rightarrow \qquad \nabla \times \nabla f = 0$$

If a flow is incompressible:

$$\nabla \cdot \underline{u} = 0 \qquad \rightarrow \nabla \cdot \nabla f = \nabla^2 f = 0$$

That is, for a 2D, incompressible flow; where $\underline{u} = (u, v)$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{Cartesian}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{1}{r}\frac{\partial v}{\partial q} = 0 \qquad \text{Plane polars}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{\sin q}\frac{\partial}{\partial q}(r \sin q) = 0 \qquad \text{Spherical polars}$$

Now, if we just look at the Cartesian case, if we let:

$$u = \frac{\partial y}{\partial y} \qquad v = -\frac{\partial y}{\partial x}$$

Then:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial y}{\partial x}\right) = 0$$

Thus, the condition for incompressibility is satisfied, with this choice of y … the stream function. The same is true for the following, when the divergence is expressed in the following coordinate systems

In plane polars:

$$u = \frac{1}{r}\frac{\partial y}{\partial q} \qquad v = -\frac{\partial y}{\partial r}$$

In spherical polars:

$$u = \frac{1}{r^2 \sin q}\frac{\partial y}{\partial q} \qquad v = -\frac{1}{r \sin q}\frac{\partial y}{\partial r}$$

To summarise these stream functions… any y will yield a 2D incompressible flow:

$$\underline{u} = \left(\frac{\partial y}{\partial y}, -\frac{\partial y}{\partial x} \right) \qquad \text{Cartesian}$$

$$\underline{u} = \left(\frac{1}{r}\frac{\partial y}{\partial q}, -\frac{\partial y}{\partial r} \right) \qquad \text{Plane polars}$$

$$\underline{u} = \left(\frac{1}{r^2 \sin q}\frac{\partial y}{\partial q}, -\frac{1}{r \sin q}\frac{\partial y}{\partial r} \right) \qquad \text{Spherical polars}$$

Now, if a 2D flow is irrotational, using the Cartesian form of $\underline{u}$:

$$\underline{w} \equiv \nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial y/\partial y & -\partial y/\partial x & 0 \end{vmatrix}$$

$$= -\underline{k}\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \right)$$

Thus, as $\underline{w} = \underline{0}$ in irrotational flow:

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 \qquad \nabla^2 y = 0$$

Thus, Laplace's equation is satisfied for both the stream function & the scalar potential, in an incompressible, irrotational 2D flow:

$$\nabla^2 f = 0 \qquad \nabla^2 y = 0$$

Now, in Cartesian coordinates:

$$u = \frac{\partial f}{\partial x} = \frac{\partial y}{\partial y} \qquad v = \frac{\partial f}{\partial y} = -\frac{\partial y}{\partial x}$$

Or:

$$\frac{\partial f}{\partial x} = \frac{\partial y}{\partial y} \qquad \frac{\partial f}{\partial y} = -\frac{\partial y}{\partial x}$$

Which are the Cauchy-Riemann equations. Thus, an analytic complex function can be described such that:

$$w(z) = f + iy$$ the *complex potential*.

Also, notice:

$$\nabla^2 w = \nabla^2(\phi + i\psi) = \nabla^2\phi + i\nabla^2\psi = 0 + i0 = 0$$
$$\Rightarrow \nabla^2 w = 0$$

Now, we can derive a *complex velocity*, supposing that the complex potential is only a function of *z*, and not its conjugate

$$z = x + iy \qquad \Rightarrow \partial w/\partial z \equiv dw/dz$$
$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz} \qquad \Rightarrow \frac{\partial w}{\partial x} = \frac{dw}{dz}$$
$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = i\frac{dw}{dz}$$

Hence:

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(\phi + i\psi)$$
$$= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$
$$= u - iv$$

Thus, the *complex velocity*:

$$\boxed{\frac{dw}{dz} = u - iv}$$

Lines on which $\psi = const$ are streamlines.