Conformal Mapping in Wing Aerodynamics

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Contents

1 Introduction

The fundamental problem of wing theory is to determine the flow around a geometrically determined wing or wing system[1]. Assuming that a wing is moving uniformly with constant velocity through a fluid, one must calculate the presure distribution along the boundary of the wing.

The use of conformal mappings in fluid mechanics can be traced back to the work of Gauss, Riemann, Weierstrass, C. Neumann, H.A. Schwarz, and Hilbert. Lord Rayleigh, a British physicist and mathematician, is attributed to give the first complete treatment of conformal mapping in aerodynamics. Near the beginning of the twentieth century Martin Kutta, a German mathematician, and Nikolay Zhukovski (Joukowsky in the modern literature), a Russian scientist, published a series of papers on airfoil theory that began a new era in fluid and aerodynamics.

The purpose of this exposition is to give the reader an elementary introduction to the use of conformal mapping in two-dimensional airfoil theory with ideal fluids. Sections 2 and 3 will provide the reader with the prerequisite backround knowledge of basic airfoil theory and two-dimensional fluid dynamics respectively. Section 4 will show how conformal mappings are used to reconcile the complicated geometries of airfoils, resulting in a simplifiction of the problem[2]. For a more complete treatment of wing theory, the reader is advised to consult Robinson[3]. For fluid dynamics, White[4] is the classical reference.

2 Basic Airfoil Theory and Terminology

As mentioned in the introduction, the object of wing theory is to investigate the aerodynamic action on a wing, or system of wings, given the embedding of the wing in a fluid with given velocity. This aerodynamic action acting on an airfoil can be generally described by Bernoulli's principle.

2.1 Airfoils

An *airfoil* in our context is the shape of a wing as seen in cross-section, see Figure 1. The *chord line* is the straight line connecting the leading edge of the airfoil to the trailing edge. The *angle of attack*, commonly denoted by α, is the angle between the chord line and the relative wind. The *mean camber line* is the locus of points between the top and bottom surfaces of the airfoil. The resulting curvature of the mean camber line gives the airfoil its *profile*. Ordinary airfoils tend to have to have a slightly cambered (arched) body.

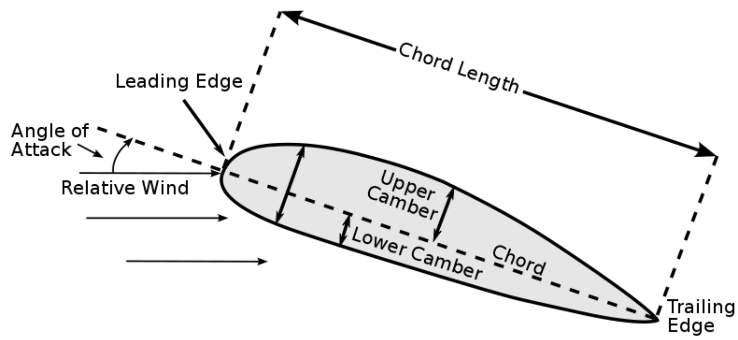

Figure 1: Airfoil schematic

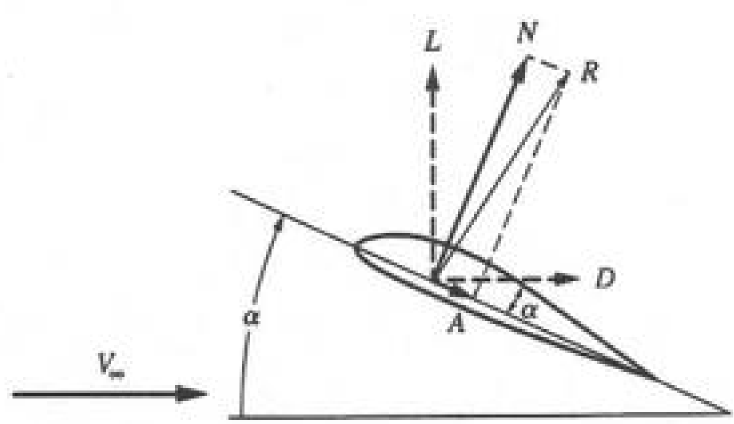

Figure 2: Forces acting on an airfoil

Figure 3: Kutta condition occurs in Flow # 1

2.2 Bernoulli's Principle

We seek to understand the forces acting on an airfoil that moves horizontally through the air, see Figure 2. The airfoil will experience a force R, which can be decomposed into two components, namely the *drag* D in the direction of the airflow and the *lift* L in the direction perpendicular to the flow. Generally, the lift is important for keeping the airplane flying, but the drag is undesired and must be compensated for by the thrust of the propeller.

In order to understand the source of the lift force L, we need to understand Bernoulli's principle. The lift force experienced by the airfoil is primarily do to a difference in pressure at the upper and lower surfaces of the airfoil. This difference in pressure at the upper and lower surface of the airfoil result from the particular shape of the airfoil, and by assuming that the *Kutta condition* holds.

Definition The *Kutta condition* states that the fluid flowing over the upper and lower surfaces of the airfoil meets at the trailing edge of the airfoil, see Figure 3.

In reality, the Kutta condition holds because of friction between the boundary of the airfoil and the fluid. Also, the angle of attack α of the airfoil must not be so large that the flow around the airfoil is no longer smooth or continuous. When the angle of attack is too large, the airplane will stall.

Thus, by the Kutta condition, since the length of the upper surface of a typical airfoil is greater than the length of the lower surface of the airfoil, the fluid velocity at the upper surface of the airfoil must be greater than the fluid velocity at the lower surface of the airfoil. Bernoulli's principle states that an increase in the velocity of the flow of a fluid coincides with a decrease in the pressure of the fluid. Thus the fluid pressure above the airfoil will be less than the fluid pressure below the airfoil, generating lift.

For steady motion of a fluid of density ρ along a body at rest, the pressure p at any point of the fluid of velocity U can be calculated from *Bernoulli's equation*, which we write as

$$p + \frac{1}{2}\rho U^2 = p_0 + \frac{1}{2}\rho V^2,$$

where p_0 is the pressure and V is the velocity at "infinity."

To simplify our problem, we will take our take our airfoil to extend to infinity on both sides in a direction normal to the plane. Thus, we will not have to worry about wing tips, reducing the study of flow to the two-dimensional case.

3 2-Dimensional Fluid Dynamics

In this section we will provide some basic facts and definitions of fluid dynamics, and will derive the result known as the *Kutta-Joukowski theorem*, which relates the lift force acting on a cylinder to the velocity of the cylinder relative to the surrounding fluid. To keep the mathematics simple, we will need to make a few key assumptions about the nature of the surrounding fluid.

3.1 Assumption of an Ideal Fluid

There are two ways to proceed when studying the fluid interactions around an airfoil. To make the distinction, we will need the following definitions.

Definition *Sheer stress* is a measure of the force of friction from a fluid acting on a body in the path of that fluid.

Definition The *viscosity* of a fluid is the measure of its resistance to gradual deformation by sheer stress. For example, honey is more viscous than water.

The first way to proceed when studying fluids is to assume that the fluid is a real, viscous fluid. This way of proceeding is clearly the most accurate, but it involves mathematics and physics that are beyond the scope of this paper. Thus, we will assume that we are working with an *ideal fluid*, that is a fluid with zero viscosity and that is *incompressible*, meaning its density remains constant. In the real world, zero viscosity is observed only at very low temperatures, in superfluids such as liquid helium. However, the assumption of an ideal fluid can still yield an accurate model provided certain conditions are met.

This first of these conditions is that the fluid must satisfy the Kutta condition described above. The second is that we require the airfoil's velocity relative to its surrounding fluid to be subsonic. At the speed of sound, shock waves occur in the fluid that make the fluid flow discontinuous, which ruins our assumption of ideal fluid flow.

3.2 Potential Flow

Consider an ideal fluid flowing over the ζ-plane with vector velocity U which is everywhere horizontal and independent of depth. Let C be a curve in the ζ-plane passing through the points $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, and let S be the surface through C with elements perpendicular to the ζ-plane. Let V_s denote the component of U along C at the point $\zeta = \xi + i\eta$, and let the ξ and η components of U at ζ be u and v, respectively. Then, the flux accross C through S is

$$H = \int_{\alpha}^{\beta} -v\, d\xi + u\, d\eta.$$

Now H is independent of the path from α to β if and only if

$$\frac{\partial(-v)}{\partial\eta} = \frac{\partial u}{\partial\xi}. \tag{1}$$

If (1) holds, the integral

$$\psi(\xi, \eta) := \int_{\alpha}^{\zeta} -v\, d\xi + u d\eta$$

defines a function ψ known as the *stream function*. If (1) holds, the fluid is incompressible. Notice that, in mathematical terms, equality of (1) implies that the divergence of of the fluid is zero. Thus, when we assume we have an ideal fluid that is incompressible, we are really assuming that the divergence of the fluid at every point is zero. Intuitively, this means that the fluid is not reduced in volume by an increase in pressure, and so the density of the fluid must remain constant.

We define the *circulation* along C from α to β to be

$$G = \int_{\alpha}^{\beta} V_s \, ds = \int_{\alpha}^{\beta} u \, d\xi + v \, d\eta.$$

Now, G is independent of the path from α to β if and only if

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \xi}. \tag{2}$$

If (2) holds, the integral

$$\phi(\xi, \eta) := \int_{\alpha}^{\zeta} u \, d\xi + v \, d\eta$$

defines a funtion ϕ known as the *velocity potential*. Notice here that equality of (2) implies that the curl of the fluid is zero. Thus, a fluid is said to be *irrotational* if (2) holds at every point.

The complex function

$$F = \phi + i\psi = f(\zeta)$$

is called the *generalized potential function*. Since

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \psi}{\partial \eta} = u,$$

and

$$\frac{\partial \phi}{\partial \eta} = -\frac{\partial \psi}{\partial \xi} = v,$$

F is an analytic function of $\zeta = \xi + i\eta$. A flow that can be represented by such a function F is called a *potential flow*. Since F is analytic, the functions ϕ and ψ are harmonic functions.

We may obtain the velocity components of F from $\frac{dF}{d\zeta}$. Thus,

$$\frac{dF}{d\zeta} = \frac{\partial \phi}{\partial \xi} + i\frac{\partial \psi}{\partial \xi} = u - iv. \tag{3}$$

Definition The family of curves satisfying $\psi = c_1$, for some constant c_1, are called *stream lines*, and the family of curves satisfying $\phi = c_2$, for some consant c_2, are called *potential lines*.

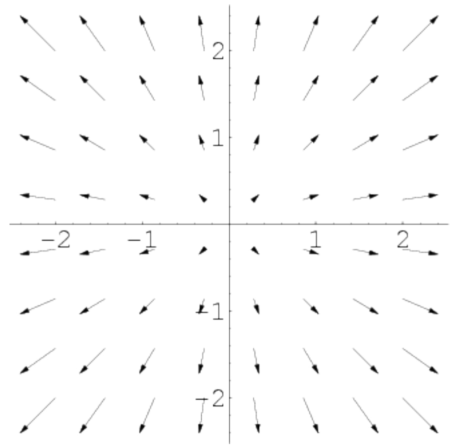

Figure 4: Vector field for a source flow ($m > 0$ and $\zeta_0 = 0$)

Theorem 3.1 *The stream lines and the potential lines are mutually orthogonal.*

Proof We prove that the gradients of ϕ and ψ are orthogonal. By the Cauchy-Riemann equations, we have

$$\begin{aligned}\nabla\phi\cdot\nabla\psi &= \frac{\partial\phi}{\partial\xi}\frac{\partial\psi}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial\psi}{\partial\eta}\\ &= \frac{\partial\psi}{\partial\eta}\frac{\partial\psi}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta}\\ &= 0,\end{aligned}$$

as desired. ∎

3.3 Sources, Sinks, Vortexes, and Doublets

We now state some important and well known generalized potential functions. First, for a fluid flowing with uniform velocity $Ue^{i\alpha}$ at the point ζ_0, where U and α are real, the potential function is

$$F_R = \phi_R + i\psi_R = Ue^{-i\alpha}(\zeta - \zeta_0).$$

For a *source/sink* of strength $m \in \mathbb{R}$ at ζ_0, see Figure 4 and Figure 5, the potential function is

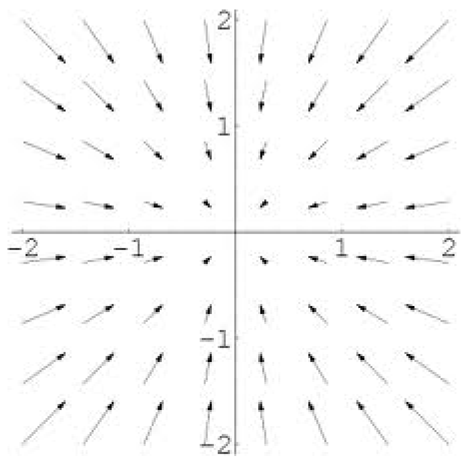

Figure 5: Vector field for a sink flow ($m < 0$ and $\zeta_0 = 0$)

$$F_s = \frac{m\log(\zeta - \zeta_0)}{2\pi}.$$

For a *vortex* of circulation $\Gamma \in \mathbb{R}$ centered at ζ_0, see Figure 6, the potential outside of a circle of radius a centered at z_0 is

$$F_V = \frac{-i\Gamma\log(\frac{\zeta-\zeta_0}{a})}{2\pi}.$$

Finally, for a *doublet* whose axis is the line of angle α with the ξ-axis and of moment (strength) M at ζ_0, the potential is

$$F_D = -\frac{Me^{i\alpha}}{\zeta - \zeta_0}.$$

3.4 Flow Around a Cylinder and the Kutta-Joukowski Theorem

Due to the fact that Laplace's equation is a linear partial differential equation, linear combinations of harmonic functions are harmonic. Thus, if $F_1, \ldots, F_n$ are n generalized potentials, then any linear combination of these functions is also a generalized potential. Thus, by *superposition* of potential flows, that is, by taking linear combinations of potential flows, we may construct flow fields of various complexity.

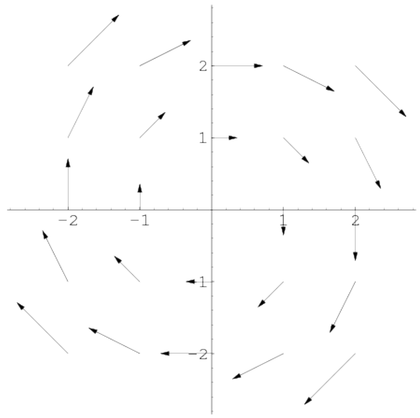

Figure 6: Vector field for a vortex

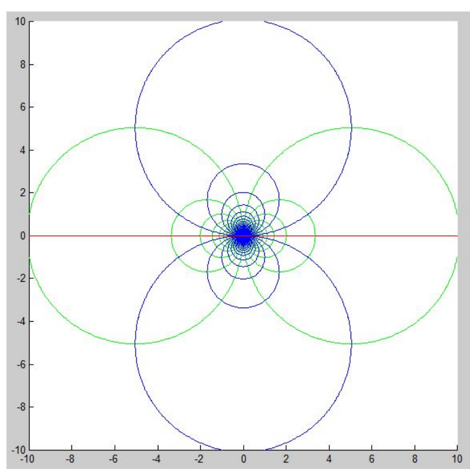

Figure 7: Doublet flow pattern

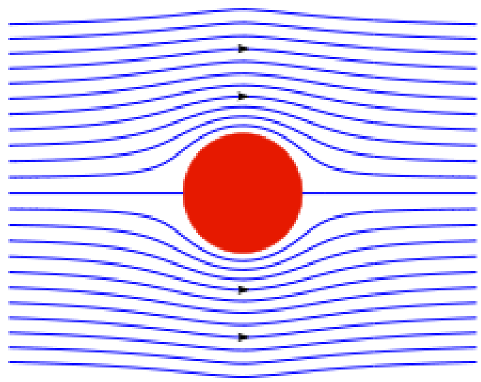

Figure 8: Flow around a cylinder

In this section we study the superposition of a rectilinear flow with uniform velocity with a doublet flow and a vortex flow, each having common center ζ_0. This will model the flow around a cylinder, see Figure 8. The potential $F_c = \phi_c + i\psi_c$ for such a flow around a cylinder of radius a is

$$F_c = F_R - F_D - F_V = Ue^{-i\alpha}(\zeta - \zeta_0) + \frac{Me^{i\alpha}}{\zeta - \zeta_0} + \frac{i\Gamma}{2\pi}\log\left(\frac{\zeta - \zeta_0}{a}\right). \tag{4}$$

Now, in order to properly model the flow around the cylinder we must require that the flow across the boundary of the circle $\zeta - \zeta_0 = ae^{i\theta}$ be zero. This means that for ζ on the circle, the imaginary part ψ_c of F_c must be zero, and from (4) it follows that $M = Ua^2$. The velocity function for this flow is given by

$$\frac{dF}{d\zeta} = Ue^{-i\alpha} - \frac{Ua^2e^{i\alpha}}{(\zeta - \zeta_0)^2} + \frac{i\Gamma}{2\pi}\frac{1}{\zeta - \zeta_0} = U_\xi - iV_\eta. \tag{5}$$

By Bernoulli's equation, we have

$$\frac{\rho i}{2}\int_{|\zeta|=r}(U_\xi - iV_\eta)^2\,d\zeta = p_\xi + ip_\eta = R,$$

where R is the total resultant force of the drag p_ξ and lift p_η, resulting from the pressure on the circular boundary $\zeta = re^{i\theta}$. If $\alpha = 0$, it follows that

$$p_\eta = 0, \qquad p_n = \frac{\rho\Gamma U}{2}\left(1 + \frac{a^2}{r^2}\right).$$

Thus, if $a = r$, $p_\eta = \rho\Gamma U$. We have shown that if our circle of radius a represents the boundary of a cross-section of an infinily long, solid, cylindrical body placed in an inviscous airstream, *then the body will experience a per unit length lift force of of measure $\rho\Gamma U$ at right angles to the stream.* This result is known as the *Kutta-Joukowski theorem.*

It has been shown in the literature that the Kutta-Joukowski theorem holds for any body with simply connected and uniform cross-section, and thus we may use this result for airfoils. For a more in-depth discussion of the Kutta-Joukowski theorem, see White[4], pages 85-91.

4 Application of Conformal Mappings

Do to the fact that airfoils have complicated geometries, it is difficult to solve for the fluid flow around airfloils using Laplace's equation and potential flow theory. Kutta (1902) calculated the stream lines around an airfoil using conformal mappings, independent of the earlier work by Lord Rayleigh. The conformal mapping technique will help us to understand and calculate the flow around airfoils used in aeronautic practice.

4.1 Methodology

Suppose that, in the ζ-plane, there is a region G, which is the cross-section of a uniform, infinitely long body for which the potential function for a plane flow is known and is given by

$$F(\zeta) = \phi(\xi, \eta) + \psi(\xi, \eta), \tag{6}$$

where $\psi = 0$ on G since the flow across ∂G must be zero. For example, G could be the cylindrical body used above, along with the superimposed potential function F_c.

Write $z = x + iy$. We need a transformation T that will suitably map points from the z-plane to the ζ-plane. Denote the region in the z-plane that is the pre-image of G under T by P (Later on, we will think of P as an airfoil, see Figure 9). We want T to satisfy the following four conditions.

(i) T maps unique points of ∂P to unique points of ∂G.

(ii) $\frac{dt}{dz}$ is finite and non-zero on the entire ζ-plane outside of G, for then, because of (i) T maps points outside of P to points ouside of G.

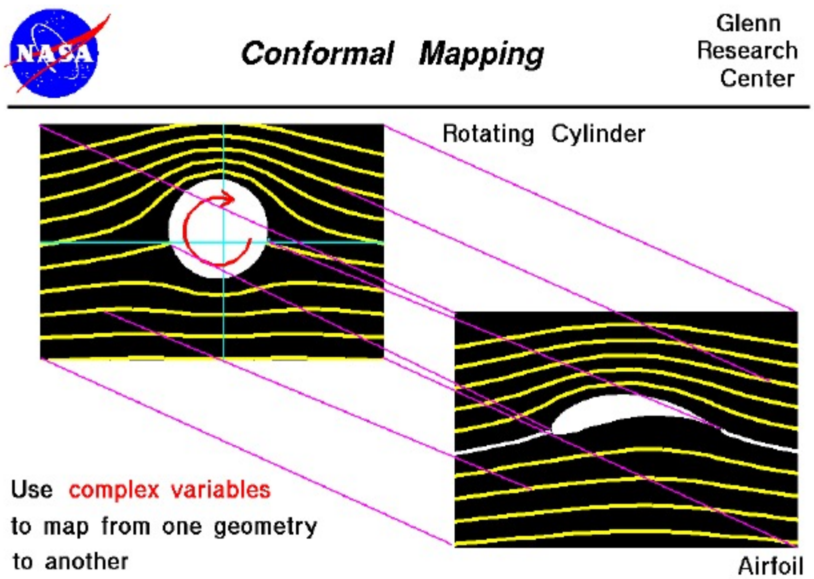

Figure 9: Mapping cylinder cross-section to and from an airfoil cross-section

(iii) T is analytic at infinity and as $z \to \infty$, $\zeta \to \infty$ and $\frac{dt}{dz} \to K$, where K is a real, finite, non-zero constant.

(iv) T has an inverse ouside of G.

We can obtain a new potential function $\mathscr{F}(z)$ for the flow around P from (6) and T, given by

$$\mathscr{F}(z) = F(T(z)) = \Phi(x, y) + i\Psi(x, y). \tag{7}$$

Note that for each ζ on ∂G, $\phi(\xi, \eta) = 0$, and so $F(\zeta)$ is purely real on ∂G. Thus, by (7) and condition (i) of T, it follows that $\Phi(x, y) = 0$ for points on ∂P, and thus we have that the flow accross ∂P is zero.

We derived earlier that the velocity function for flow in the ζ-plane is $\frac{dF}{d\zeta} = u - iv$, see (3). Write $u - iv = w_\zeta$, for clarity. Then we can derive the velocity function for flow in the z-plane, w_z. By the chain rule, we have

$$\frac{d\mathscr{F}}{dz} = \frac{dF}{d\zeta} \cdot \frac{d\zeta}{dz} = w_z \cdot \frac{dt}{dz} = u_z - iv_z = w_z. \tag{8}$$

Observe that condition (iii) on T ensures that the fluid velocity w_z at infinity in the z-plane is proportional to the fluid velocity w_ζ at infinity in the ζ-plane.

We have $\mathscr{F}(z) = F(\zeta)$ when $\zeta = t(z)$, and thus it follows that for corresponding z and ζ, $\phi(\xi, \eta) = \Phi(x, y)$ and $\psi(\xi, \eta) = \Psi(x, y)$. Let $c_1, c_2 \in \mathbb{R}$. Then to each potential line $\phi = c_1$ in the ζ-plane there is a corresponding potential line $\Phi = c_1$ in the z-plane, and to each stream line $\psi = c_2$ in the ζ-plane there is a corresponding stream line $\Psi = c_2$ in the z-plane.

Now, condition (ii) on T that $\frac{dT}{dz}$ is finite and non-zero on the entire ζ-plane outside of G is made such that the family of curves $\phi = c_1$ and $\psi = c_2$ in the ζ-plane outside of G are mapped conformally, preserving angles, into the z-plane outside of P, forming the families $\Phi = c_1$ and $\Psi = c_2$. Since the curves $\phi = c_1$ and $\psi = c_2$ are orthogonal, the curves $\Phi = c_1$ and $\Psi = c_2$ must also be orthogonal.

Thus, given a potential $F(\zeta)$ and and a region G representing the cross-section of a uniform, infinitely long body, we have shown how to find the potential $\mathscr{F}(z)$ for the flow of an ideal fluid around a body with any profile and cross-section given by P, which is formed from G by a conformal mapping.

4.2 Application

In this section we will think of G as being the cross-section of a cylinder of radius a and we will think of P as an airfoil. We derived the potential function for flow around a cylinder in section 3.4. It is given by

$$F(\zeta) = Ue^{-i\alpha}(\zeta - \zeta_0) + \frac{Ua^2 e^{i\alpha}}{\zeta - \zeta_0} + \frac{i\Gamma}{2\pi}\log\left(\frac{\zeta - \zeta_0}{a}\right). \tag{9}$$

Now, we want our airfoil P to satisfy the Kutta condition described in section 2.2. Thus, we will assume that P has a "sharp trailing edge" and we will assume that the angle of attack α is small. In this case we can calculate the circulation Γ around the airfoil and we can use that to calculate the lift. Denote the trailing edge of the airfoil P by z_t and the corresponding point on the cylinder G by ζ_t.

We will orient ourselves such that the airfoil P is facing left (like in the figures above). Then the trailing edge z_t of P occurs on the lower right hand side of P. It will turn out that the corresponding point ζ_t of G will be on the lower right hand side of the cylinder G and thus there is an angle $0 < \beta < \frac{\pi}{2}$ such that $\zeta_t = \zeta_0 + ae^{-i\beta}$.

The Kutta condition is modeled mathematically by requiring $\frac{dF}{d\zeta}\Big|_{\zeta_t} = 0$. From (5) and the Kutta condition, it follows that

$$\frac{dF}{d\zeta}\bigg|_{\zeta_t} = Ue^{-i\alpha} - Ue^{i(\alpha+2\beta)} + \frac{i\Gamma}{2\pi a}e^{i\beta} = 0.$$

Thus,

$$\Gamma = \frac{2\pi aU}{i}\left(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right) = 4\pi aU\sin(\alpha+\beta). \tag{10}$$

By the Kutta-Joukowski Theorem, the lift force acting on the airfloil can be computed with the formula $\rho\Gamma U$. Thus, from (10) the lift force is given by $L = 4\pi\rho aU^2\sin(\alpha+\beta)$.

Now that we have developed all of the theoretical backround necessary to understand conformal transformations between cylinders and airfoils, we present a couple of examples. We just need a transformation T, satisfying the conditions in section 4.1, mapping points from the z-plane to the ζ-plane.

The first set of airfoils are represented by the *Joukowski transformation*, which is given by

$$z = \zeta + \frac{b^2}{\zeta}. \tag{11}$$

Given a properly defined inverse, this transformation satisfies the conditions in section 4.1. The non-conformal points of this transformation are the points $z = \pm 2b$ corresponding to the cusp at the end of a Joukowski airfoil. Changing the value of b changes the shape of the resulting airfoil. A small b value produces a thicker, cylindrical airfoil. A larger value of b will create a thinner airfoil. If $b = a$ (the radius of the cylinder in the ζ-plane), then the corresponding airfoil will be a flat line, in particular, it is just the chord of the airfoil, see figure 1.

The Joukowski family of airfoils resulting from (11) have two important restrictions. One is that these airfoils have zero tail angle at the trailing edge, and the other is that they are relatively thin.

The second transformation is a generalization of the Joukowski transformation. The *Kármán-Trefftz transformation* is given by

$$z = \lambda \frac{\left(1+\frac{b^2}{\zeta}\right)^\lambda + \left(1-\frac{b^2}{\zeta}\right)^\lambda}{\left(1+\frac{b^2}{\zeta}\right)^\lambda - \left(1-\frac{b^2}{\zeta}\right)^\lambda},$$

where $\lambda = 2 - \frac{\theta}{\pi}$ and θ is the tail angle at the trailing edge of the airfoil.

Notice that if $\theta = 0$, a straight-forward, but messy calculation shows that the Kármán-Trefftz transformation reduces to the Joukowski transformation. The Kármán-Trefftz transformation yeilds airfoils having improved fluid velocity and pressure distributions as compared to those of the Joukowski transformation.

5 Conclusion

Solving the problem of fluid flow around a wing is a highly complex task. However, reducing the problem to the two-dimensional study of airfoils allows one to employ techniques of complex variables, in particular utilizing the geometric properties of conformal mappings. To obtain intuition for the Joukowski and Kármán transformations, there are a multitude of Matlab scripts and Java applets available online.

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