# *CHAPTER 11*

# *TWO-DIMENSIONAL AIRFOIL THEORY*

## 11.1 THE CREATION OF CIRCULATION OVER AN AIRFOIL

In Chapter 10 we worked out the force that acts on a solid body moving in an inviscid fluid. In two dimensions the force is

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi \hat{n} dl - \bar{U}_{\infty}(t) \times \Gamma(t)\hat{k} \tag{11.1}$$

where $\Gamma(t) = \oint_{C_w} \bar{U} \bullet \hat{c}\, dC$ is the circulation about the body integrated counter-clockwise about a path that encloses the body. It should be stated at the outset that the creation of circulation about a body is fundamentally a viscous process. The figure below from Van Dyke's Album of Fluid Motion depicts what happens when a viscous fluid is moved impulsively past a wedge.

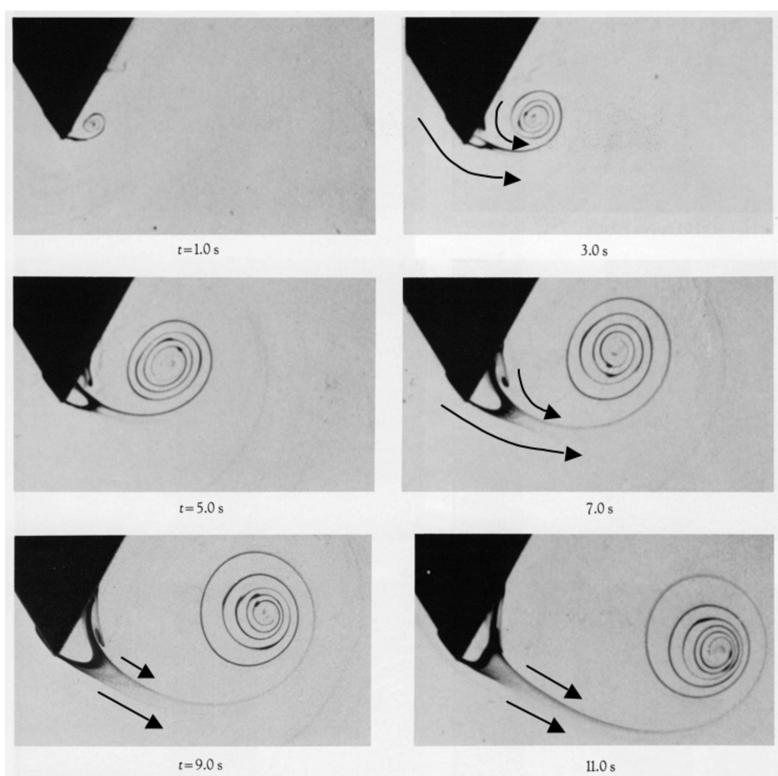



*Figure 11.1 Starting vortex formation about a corner in an impulsively started incompressible fluid.*

A purely potential flow solution would negotiate the corner giving rise to an infinite velocity at the corner. In a viscous fluid the no slip condition prevails at the wall and viscous dissipation of kinetic energy prevents the singularity from occurring. Instead two layers of fluid with oppositely signed vorticity separate from the two faces of the corner to form a starting vortex. The vorticity from the upstream facing side is considerably stronger than that from the downstream side and this determines the net sense of rotation of the rolled up vortex sheet. During the vortex formation process, there is a large difference in flow velocity across the sheet as indicated by the arrows in Figure 11.1. Vortex formation is complete when the velocity difference becomes small and the starting vortex drifts away carried by the free stream.

Figure11.2 shows a sequence of events in the development of lift on a wing in a free stream impulsively started from rest to velocity $U_\infty$ to the right. The wedge flow depicted above accurately characterizes the events that occur near the wing trailing edge during the initiation of lift.

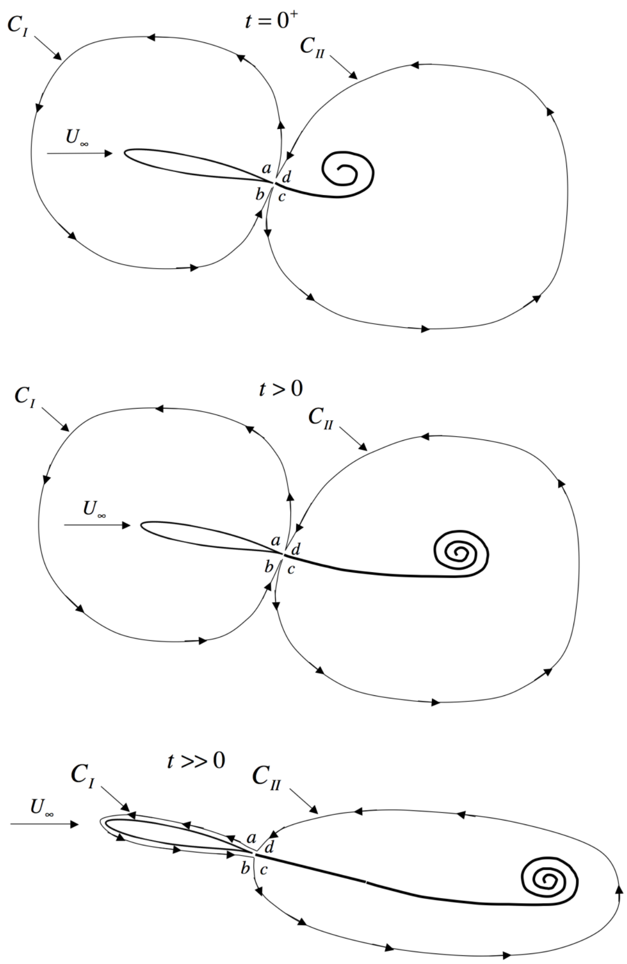



*Figure 11.2 Starting vortex formation from an impulsively started wing in an incompressible flow. Contours distinguish the circulation about the wing and the opposite signed circulation about the starting vortex.*

The contour about the system never penetrates the vortex sheet separating from the wing trailing edge. Since the contour lies entirely in potential flow without crossing any discontinuities the velocity potential is a continuous single valued function along the contour. Therefore

$$\int_a^b U \bullet \hat{c}\, dC + \int_c^d U \bullet \hat{c}\, dC = \oint_{C_1+C_2} \nabla\Phi \bullet \hat{c}\, dC = 0 \tag{11.2}$$

and

$$\Gamma_{Wing} = -\Gamma_{Vortex} \ . \tag{11.3}$$

A note on the sign of the circulation about the wing: Based on the orientation of the $(x,y)$ axes in figure 11.3 with the free stream velocity in the positive $x$ direction and lift on the wing in the positive $y$ direction, the circulation about the counterclockwise rotating starting vortex is positive while the circulation about the wing is negative.

During the starting process the shed vortex induces a down wash on the wing producing drag. The downwash caused by the bound vorticity on the wing is negligible. The figure below depicts this effect

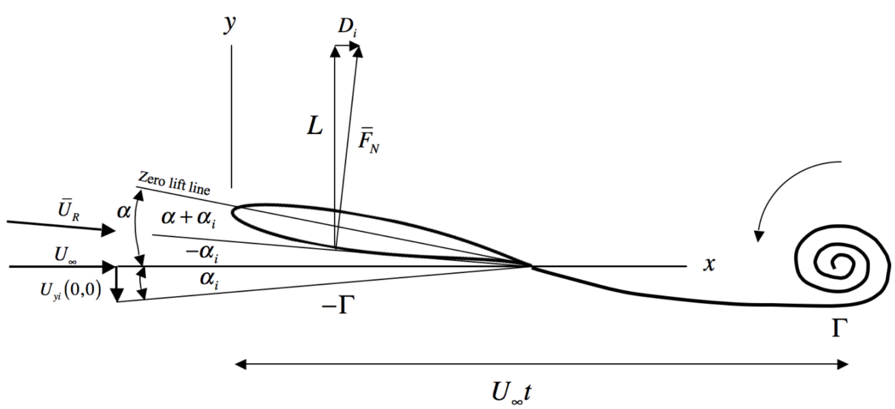



*Fig 11.3 Effect of starting vortex downwash on lift and drag of an airfoil.*

Assume the geometric angle of attack of the wing with respect to the oncoming flow aligned with the x-axis is $\alpha$. Assume the angle is relatively small and the lift curve slope is

$$\frac{dC_L}{d\alpha} = a_0 \tag{11.4}$$

The tangential velocity of a line vortex is

$$U_\theta = \frac{\Gamma_{Vortex}}{2\pi r} \tag{11.5}$$

where $r$ is the distance from the vortex center. After a period of time $t$ has passed, the starting vortex has been carried a distance $U_\infty t$ from the wing. During this period, the starting vortex induces a downwash velocity on the wing that we can estimate as

$$U_{yi}(0,0,t) = \frac{\Gamma_{Wing}}{2\pi U_\infty t} = \frac{-\Gamma_{Vortex}}{2\pi U_\infty t} \quad . \tag{11.6}$$

Note again that $\Gamma_{Wing} < 0$. This downwash rotates the velocity vector of the flow coming at the wing by the angle

$$\alpha_i = ArcTan\left(\frac{U_{yi}(0,0,t)}{U_\infty}\right) \cong \frac{U_{yi}(0,0,t)}{U_\infty} < 0 \quad . \tag{11.7}$$

The velocity vector relative to the wing is

$$\bar{U}_R = \left(U_\infty, U_{yi}(0,0,t)\right) \tag{11.8}$$

with magnitude

$$U_R = \left(U_\infty^2 + U_{yi}(0,0,t)^2\right)^{1/2} \quad . \tag{11.9}$$

The lift and induced drag at any instant are

$$L = F_N Cos(\alpha_i) = F_N \frac{U_\infty}{\left(U_\infty^2 + U_{yi}(0,0,t)^2\right)^{1/2}} \tag{11.10}$$

$$D_i = -F_N Sin(\alpha_i) = -F_N \frac{U_{yi}(0,0,t)}{\left(U_\infty^2 + U_{yi}(0,0,t)^2\right)^{1/2}} \quad . \tag{11.11}$$

The normal force on the wing is related to the relative velocity vector and the circulation by

$$F_N = -\rho U_R \Gamma_{Wing} \quad . \tag{11.12}$$

The lift and induced drag are

$$L = F_N Cos(\alpha_i) = -\rho U_\infty \Gamma_{Wing} \tag{11.13}$$

and

$$D_i = \rho U_{yi}(0,0,t)\Gamma_{Wing} \quad . \tag{11.14}$$

Using (11.6), (11.14) can be expressed as

$$D_i = \rho \frac{\Gamma_{wing}^2}{2\pi U_\infty t} \quad . \tag{11.15}$$

The lift coefficient is proportional to the angle of attack with respect to the relative velocity vector.

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 C} = a_0\left(\alpha + \alpha_i\right) \tag{11.16}$$

where $L$ is the lift per unit span. Use (11.7) and (11.13) to express (11.16) in terms of the as yet unknown circulation.

$$-\rho U_\infty \Gamma_{Wing} = a_0 \frac{1}{2}\rho U_\infty^2 C\left(\alpha + \frac{\Gamma_{Wing}}{2\pi U_\infty^2 t}\right) \quad . \tag{11.17}$$

Solve for the circulation (essentially the time dependent lift coefficient).

$$\frac{-2\Gamma_{Wing}(t)}{U_\infty C} = \left(\frac{\dfrac{4\pi U_\infty t}{a_0 C}}{\dfrac{4\pi U_\infty t}{a_0 C} + 1}\right) a_0 \alpha = C_L(t) \tag{11.18}$$

The induced drag per unit span is $D_i = \rho\left(\Gamma_{wing}\right)^2 / \left(2\pi U_\infty t\right)$. The time dependent coefficient of induced drag is

$$C_{D_i}(t) = \left(\frac{\dfrac{4\pi U_\infty t}{a_0 C}}{\left(\dfrac{4\pi U_\infty t}{a_0 C} + 1\right)^2}\right) a_0 \alpha^2 \quad . \tag{11.19}$$

The characteristic time that comes out of this analysis is

$$\tau_{Starting\ Airfoil} = \frac{a_0 C}{4\pi U_\infty} \quad . \tag{11.20}$$

In terms of the characteristic time,

$$\frac{-2\Gamma_{Wing}(t)}{a_0 U_\infty C\alpha} = \frac{C_L}{a_0\alpha} = \left(\frac{t/\tau_{Starting\ Airfoil}}{t/\tau_{Starting\ Airfoil}+1}\right), \quad \frac{C_{D_i}}{a_0\alpha^2} = \left(\frac{t/\tau_{Starting\ Airfoil}}{\left(t/\tau_{Starting\ Airfoil}+1\right)^2}\right). \tag{11.21}$$

The evolution of lift and drag over the wing is shown below.

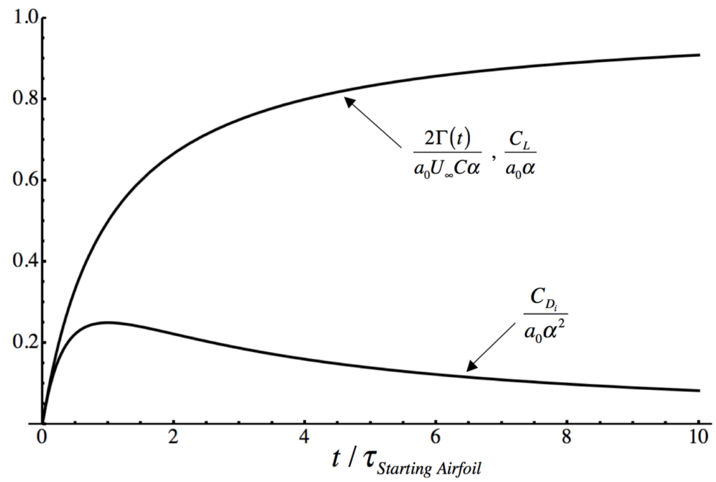



*Figure 11.4 Onset of circulation and lift, growth and decay of induced drag on an impulsively started airfoil.*

Classical theory gives $a_0 = 2\pi$ for the lift curve slope of an airfoil at small angles of attack. In this instance the starting time is about the time it takes the flow to travel one-half of a chord length.

At $t = 0$ all three are zero. As $t \to \infty$ the lift and circulation approach a constant value while the induced drag, after reaching a maximum at about one characteristic time, decays to zero as the lift approaches a constant value.

A more sophisticated analysis of this problem that takes into account the dynamics of the vortex sheet can be found in Saffman (*Vortex Dynamics*, page 111).

## 11.2 THE JOUKOWSKY AIRFOIL

Recall the complex stream function introduced in Chapter 10.

$$W(z) = \Phi(x,y) + i\Psi(x,y) \tag{11.22}$$

where

$$z = x + iy = r\left(Cos(\theta) + iSin(\theta)\right) = re^{i\theta} \ . \tag{11.23}$$

The radius and angle are

$$r = \left(x^2 + y^2\right)^{1/2}$$
$$Tan(\theta) = \frac{y}{x} \qquad (11.24)$$

Both the velocity potential and stream function satisfy the Cauchy-Riemann equations.

$$U = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}$$
$$V = \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \qquad (11.25)$$

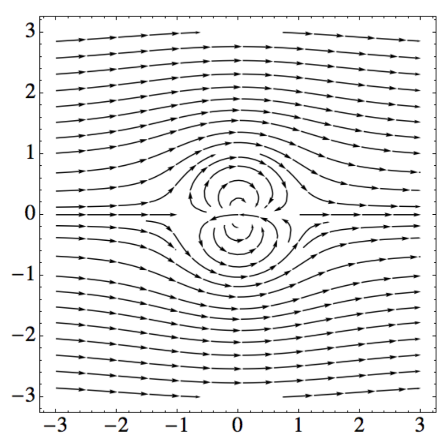
Cross differentiating (11.25) it is easy to show that the velocity potential and stream function both satisfy Laplace's equation. Moreover two-dimensional potential flows can be constructed from any analytic function of a complex variable, $W(z)$.

The complex velocity is independent of the path along which the derivative is of the complex potential is taken.

$$\frac{dW}{dz} = \frac{\partial \Phi}{\partial x}\frac{dx}{dz} + i\frac{\partial \Psi}{\partial x}\frac{dx}{dz} = U - iV$$
$$\frac{dW}{dz} = \frac{\partial \Phi}{\partial y}\frac{dy}{dz} + i\frac{\partial \Psi}{\partial y}\frac{dy}{dz} = \frac{V}{i} + i\frac{U}{i} = U - iV \qquad (11.26)$$

In Chapter 10 we constructed flow past a circular cylinder from the superposition of a uniform stream and a dipole.

$$W = U_\infty z + \frac{\kappa}{2\pi}\left(\frac{1}{z}\right) \quad \Phi = U_\infty x + \frac{\kappa}{2\pi}\left(\frac{x}{x^2 + y^2}\right) \quad \Psi = U_\infty y + \frac{\kappa}{2\pi}\left(\frac{y}{x^2 + y^2}\right) \qquad (11.27)$$




The radius of the cylinder is

$$R = \left( \frac{\kappa}{2\pi U_\infty} \right)^{1/2} . \tag{11.28}$$

With the velocity on the surface of the cylinder known, the Bernoulli constant is used to determine the pressure coefficient on the cylinder.

$$\begin{aligned} \frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^2 &= \left( \frac{P}{\rho} + \frac{1}{2}U^2 \right)_{R=\left(\frac{\kappa}{2\pi U_\infty}\right)^{1/2}} \\ C_p = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} &= \left( 1 - \left( \frac{U}{U_\infty} \right)^2 \right) = 1 - 4Sin^2(\theta) \end{aligned} \tag{11.29}$$

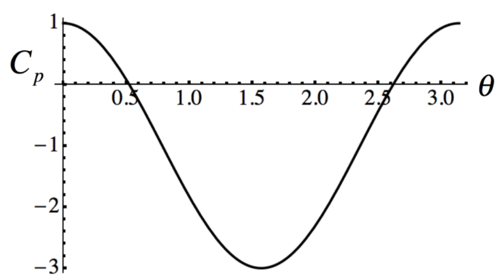
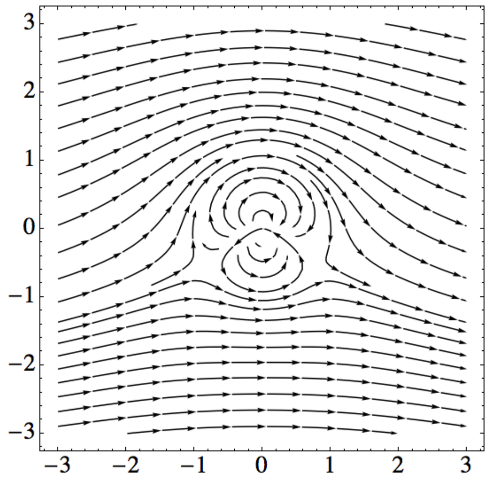



*Fig 11.5 Pressure coefficient for irrotational flow past a circle.*

In Chapter 10 we added circulation to the problem by adding a vortex to the flow. Take the circulation of the vortex to be in the clockwise direction with the positive angle measured in the clockwise direction from the left stagnation point.

$$\begin{aligned} W &= U_\infty z + \frac{\kappa}{2\pi}\left(\frac{1}{z}\right) + \frac{i\Gamma}{2\pi} Ln(z) \\ \Phi &= U_\infty x + \frac{\kappa}{2\pi}\left(\frac{x}{x^2 + y^2}\right) - \frac{\Gamma}{2\pi}\theta \\ \Psi &= U_\infty y + \frac{\kappa}{2\pi}\left(\frac{y}{x^2 + y^2}\right) + \frac{\Gamma}{2\pi} Ln(r) \end{aligned} \tag{11.30}$$




Note that the radius of the circle is unchanged by the vortex but the forward and rearward stagnation points are moved symmetrically below the centerline producing lift on the cylinder in the amount $L = -\rho U_\infty \Gamma$.

Now let the oncoming flow be at an angle of attack $\alpha$ below the horizontal. Take $\alpha$ to be positive when the flow is below the horizontal and negative when it is above. Superpose this uniform flow with a dipole and a vortex. Take the circulation of the vortex to be in the clockwise direction.

$$W(z_1) = U_\infty z_1 e^{-i\alpha} + \frac{\kappa}{2\pi}\left(\frac{1}{z_1}\right) + \frac{i\Gamma}{2\pi} Ln(z_1)$$

$$\Phi(x_1, y_1) = U_\infty\left(x_1 Cos(\alpha) + y_1 Sin(\alpha)\right) + \frac{\kappa}{2\pi}\left(\frac{x_1}{x_1^2 + y_1^2}\right) - \frac{\Gamma}{2\pi} ArcTan(y_1 / x_1) \quad (11.31)$$

$$\Psi = U_\infty\left(y_1 Cos(\alpha) - x_1 Sin(\alpha)\right) + \frac{\kappa}{2\pi}\left(\frac{y_1}{x_1^2 + y_1^2}\right) + \frac{\Gamma}{2\pi} Ln\left(\left(x_1^2 + y_1^2\right)^{1/2}\right)$$

where $z_1 = x_1 + iy_1$ is the complex variable with dimensions $[z] = Length$. The radius of the circle is still

$$R = \sqrt{\frac{\kappa}{2\pi R U_\infty}} \ . \quad (11.32)$$

Using (11.32) the potentials (11.31) can be non-dimensionalized as follows.

$$
\begin{aligned}
w(z_1) &= \left(\frac{z_1}{R}\right)e^{-i\alpha} + \left(\frac{R}{z_1}\right) + i\gamma Ln\left(\frac{z_1}{R}\right) \\
\phi(x_1, y_1) &= \left(\frac{x_1}{R}\right)Cos(\alpha) + \left(\frac{y_1}{R}\right)Sin(\alpha) + \frac{RCos(\theta_1)}{r_1} - \gamma\theta_1 \\
\psi(x_1, y_1) &= y_1 Cos(\alpha) - x_1 Sin(\alpha) - \frac{RSin(\theta_1)}{r_1} + \gamma Ln\left(\frac{r_1}{R}\right)
\end{aligned}
\qquad (11.33)
$$

where

$$
w = \frac{W}{U_\infty R} \qquad \phi = \frac{\Phi}{U_\infty R} \qquad \psi = \frac{\Psi}{U_\infty R} \ . \qquad (11.34)
$$

The dimensionless circulation of the vortex is

$$
\gamma = \frac{\Gamma}{2\pi R U_\infty} \ . \qquad (11.35)
$$

The flow is shown below for $\gamma = 1$ , $R = 1$ and $\alpha = \pi / 18$ .

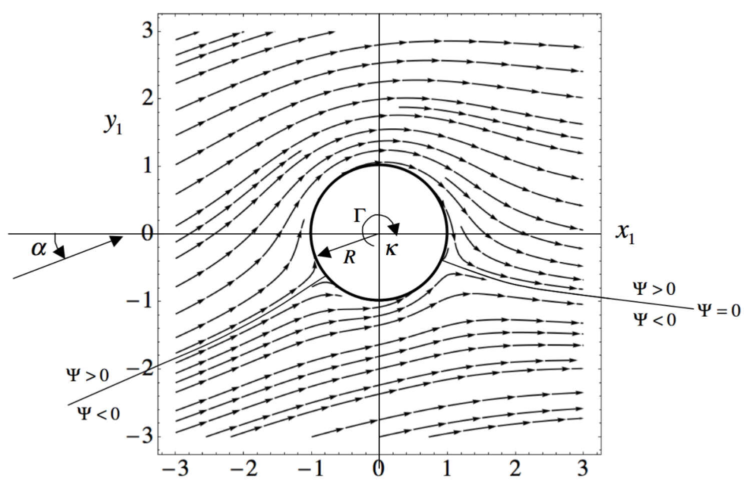



*Figure 11.6 Flow at angle of attack $\alpha$ past a circular cylinder with lift.*

The dimensionless potentials expressed in $(r_1, \theta_1)$ coordinates are

$$
\begin{aligned}
\phi &= \frac{r_1}{R}Cos(\alpha)Cos(\theta_1) + \frac{r_1}{R}Sin(\alpha)Sin(\theta_1) + \frac{R}{r_1}Cos(\theta_1) - \gamma\theta_1 \\
\psi &= \frac{r_1}{R}Cos(\alpha)Sin(\theta_1) - \frac{r_1}{R}Sin(\alpha)Cos(\theta_1) - \frac{R}{r_1}Sin(\theta_1) + \gamma Ln\left(\frac{r_1}{R}\right)
\end{aligned}
\ . \qquad (11.36)
$$

The complex velocity is

$$\frac{dw(z_1)}{dz_1} = \frac{e^{-i\alpha}}{R} - \left(\frac{R}{z_1^2}\right) + i\frac{\gamma}{z_1} \quad . \tag{11.37}$$

The velocities in polar coordinates are

$$\begin{aligned} U_{r_1} &= \frac{1}{R}Cos(\alpha)Cos(\theta_1) + \frac{1}{R}Sin(\alpha)Sin(\theta_1) - \frac{R}{r_1^2}Cos(\theta_1) \\ U_{\theta_1} &= -\frac{1}{R}Cos(\alpha)Sin(\theta_1) + \frac{1}{R}Sin(\alpha)Cos(\theta_1) - \frac{R}{r_1^2}Sin(\theta_1) - \frac{\gamma}{r_1} \end{aligned} \quad . \tag{11.38}$$

The flow is determined by $\alpha$ which is specified and $\gamma$ and $R$ which remain to be determined. On the cylinder $U_{r_1} = 0$ and $r_1 = R$. The tangential velocity in the cylinder is

$$U_{\theta_1}\Big|_{r_1=R} = \frac{1}{R}\left(-Cos(\alpha)Sin(\theta_1) + Sin(\alpha)Cos(\theta_1) - Sin(\theta_1) - \gamma\right) \quad . \tag{11.39}$$

Stagnation points occur where $U_{\theta_1}\Big|_{r_1=R} = 0$. The two roots of (11.39) , denoted $\theta_{10}$, satisfy

$$Sin(\alpha)Cos(\theta_{10}) = Cos(\alpha)Sin(\theta_{10}) + Sin(\theta_{10}) + \gamma \quad . \tag{11.40}$$

Solve (11.40) for the stagnation points $(r,\theta) = (R,\theta_{10})$ given $\gamma$ and $\alpha$. The roots of (11.40) are

$$Sin(\theta_{10}) = -\frac{\gamma}{2} \pm \frac{Sin(\alpha)}{2(1+Cos(\alpha))}\sqrt{2(1+Cos(\alpha)) - \gamma^2} \quad . \tag{11.41}$$

Our goal is to map the flow in figure 11.6 to flow around an airfoil with several free parameters that can be used to choose the shape of the airfoil. First we will itemize the steps in the mapping then demonstrate the process with an example.

Step 1 – Translate the circle in the $z_1$ plane to a circle in the $z_2$ plane with its origin shifted from $(0,0)$. The mapping is

$$z_2 = z_1 + z_{2c} \quad \text{where} \quad z_{2c} = x_{2c} + iy_{2c} \quad . \tag{11.42}$$

Specify the coordinates of the center of the circle in the $z_2$ plane $(x_{2c}, y_{2c})$. In addition specify the coordinates of the trailing edge $(x_{2t}, y_{2t})$ in the $z_2$ plane. Now the radius of the circle in the $z_1$ plane is determined.

$$R = \left( \left( x_{2c} - x_{2t} \right)^2 + \left( y_{2c} - y_{2t} \right)^2 \right)^{1/2} \tag{11.43}$$

Step 2 – Map the circle in $z_2$ to an oval in the $z_3$ plane passing through the point $\left( x_3, y_3 \right) = \left( 1,0 \right)$. The mapping is

$$z_3 = z_2 - \frac{\varepsilon}{z_2 - \Delta} \tag{11.44}$$

where $\varepsilon$ is complex and $\Delta$ is real. We require that the coordinates of the trailing edge $z_{2t} = x_{2t} + iy_{2t}$ satisfy

$$z_{3t} = z_{2t} - \frac{\varepsilon_r + i\varepsilon_i}{z_{2t} - \Delta} = 1 + i(0) \quad . \tag{11.45}$$

Solve (11.45) for $\left( \varepsilon_r, \varepsilon_i \right)$ using $x_{3t} = 1$ and $y_{3t} = 0$.

$$\begin{aligned} \varepsilon_r &= \left( x_{2_{TE}} - 1 \right)\left( x_{2_{TE}} - \Delta \right) - y_{2_{TE}}^2 \\ \varepsilon_i &= y_{2_{TE}} \left( 2x_{2_{TE}} - 1 - \Delta \right) \end{aligned} \tag{11.46}$$

Step 3 – Map the oval to an airfoil using the Joukowsky transformation.

$$z = z_3 + \frac{1}{z_3} \tag{11.47}$$

The overall transformation from the circle to airfoil coordinates is

$$z = \left( \left( z_1 + z_{2c} \right) - \frac{\varepsilon_r + i\varepsilon_i}{\left( z_1 + z_{2c} \right) - \Delta} \right) + \frac{1}{\left( \left( z_1 + z_{2c} \right) - \dfrac{\varepsilon_r + i\varepsilon_i}{z_1 + z_{2c} - \Delta} \right)} \quad . \tag{11.48}$$

***Example – Joukowsky airfoil.***

Choose the following values.

$$\alpha = \pi / 9 \quad x_{2c} = -0.07 \quad y_{2c} = 0.02 \quad x_{2_{TE}} = 1.03 \quad y_{2_{TE}} = -0.02 \quad \Delta = 0.2 \tag{11.49}$$

The coordinates of the center of the circle $\left( x_{2c}, y_{2c} \right)$ and the trailing edge $\left( x_{2t}, y_{2t} \right)$ in the $z_2$ plane determine the radius of the circle in the $z_1$ plane.

$$R = \left( \left( x_{2c} - x_{2t} \right)^2 + \left( y_{2c} - y_{2t} \right)^2 \right)^{1/2} = \left( \left( 0.07 - 1.03 \right)^2 + \left( 0.02 + 0.02 \right)^2 \right)^{1/2} = 1.10073 \tag{11.50}$$

The location of the trailing edge in the $z_1$ plane is determined using (11.42).

$$
\begin{aligned}
z_{1_{TE}} &= z_{2_{TE}} - z_{2_c} = 1.10 - 0.04i \\
\theta_{10_{TE}} &= ArcTan\left(\frac{-0.04}{1.10}\right) = -0.03635
\end{aligned}
\quad . \tag{11.51}
$$

Now apply the Kutta condition. The dimensionless circulation (11.35) is chosen to insure that the rear stagnation point on the circle in the $z_1$ plane depicted in Figure 11.6 is located at the point (11.51) designated as the trailing edge. In other words, when we choose the parameters (11.49) the radius of the circle in the $z_1$ plane, the angle of the trailing edge and the value of $\gamma$ in (11.41) that gives the angle calculated in (11.51) as the root are all determined. The circulation is determined to be

$$
\gamma = Sin(\alpha)Cos\left(\theta_{10_{TE}}\right) - Cos(\alpha)Sin\left(\theta_{10_{TE}}\right) - Sin\left(\theta_{10_{TE}}\right) = 0.412282 \ . \tag{11.52}
$$

The angle of the forward stagnation point in the $z_1$ plane is determined from (11.41). The roots are

$$
\theta_{10} = ArcSin\left(-\frac{\gamma}{2} \pm \frac{Sin(\alpha)}{2\left(1+Cos(\alpha)\right)}\sqrt{2\left(1+Cos(\alpha)\right)-\gamma^2}\right) = \left(-2.7562, -0.03635\right) \tag{11.53}
$$

The position of the nose stagnation point in the $z_1$ plane is

$$
z_{1nose} = -1.01998 - 0.41381i \tag{11.54}
$$

and the position in the $z_2$ plane is

$$
z_{2nose} = -1.08998 - 0.39381i \ . \tag{11.55}
$$

Using the translation (11.42) map the circle in the $z_1$ to a shifted circle of the same radius in the $z_2$ plane.

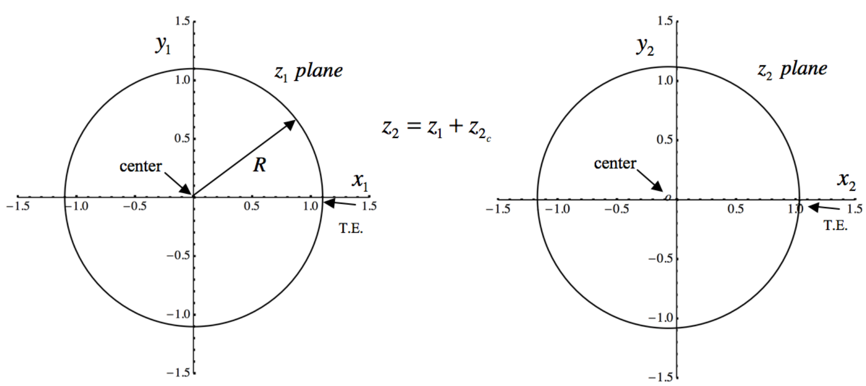



*Figure 11.7 Mapping from $z_1$ to $z_2$ .*

Now map the circle in the $z_2$ plane to a slightly vertically elongated oval in the $z_3$ plane.

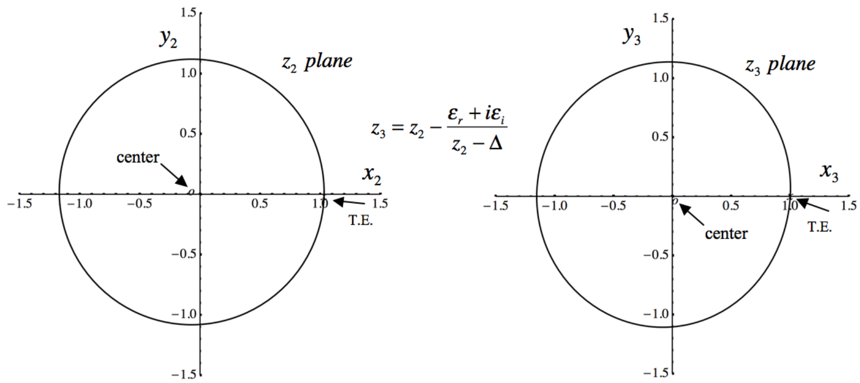



*Figure 11.8 Mapping from $z_2$ to $z_3$.*

The parameter delta is specified independently and the real and imaginary parts of $\varepsilon$ are determined by the condition that $\left(x_{2_{TE}}, y_{2_{TE}}\right)$ maps to $(1,0)$ in the $z_3$ plane. Referring to (11.46)

$$
\begin{aligned}
\varepsilon_r &= \left(x_{2_{TE}} - 1\right)\left(x_{2_{TE}} - \Delta\right) - y_{2_{TE}}^2 = (1.03-1)(1.03-0.2) - (0.02)^2 = 0.0245 \\
\varepsilon_i &= y_{2_{TE}}\left(2x_{2_{TE}} - 1 - \Delta\right) = -0.02\left(2(1.03) - 1 - 0.2 - 0.02\right) = -0.0172
\end{aligned} \quad . \tag{11.56}
$$

Now map the oval in the $z_3$ plane to an airfoil in the physical plane $z$ using (11.48).

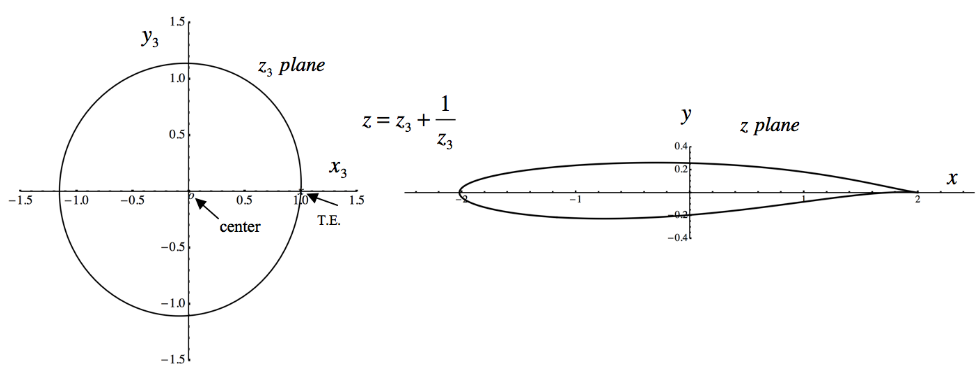



*Figure 11.9 Mapping from $z_3$ to $z$.*

The mapping used to generate the airfoil in Figure 11.9 is

$$x+iy=\left(RCos(\theta)+iRSin(\theta)+x_{2c}+iy_{2c}-\frac{\varepsilon_r+i\varepsilon_i}{RCos(\theta)+iRSin(\theta)+x_{2c}+iy_{2c}-\Delta}\right)+$$
$$\frac{1}{\left(RCos(\theta)+iRSin(\theta)+x_{2c}+iy_{2c}-\frac{\varepsilon_r+i\varepsilon_i}{RCos(\theta)+iRSin(\theta)+x_{2c}+iy_{2c}-\Delta}\right)} \quad . \tag{11.57}$$

where $(x,y)$ are the coordinates of the airfoil surface. All the constants in (11.57) are known from the values specified in (11.49) and calculated in (11.50) and (11.56). The angle $\theta$ in (11.57) runs from $0$ to $2\pi$ in the $z_1$ plane.

Ultimately we want the pressure distribution on the airfoil and for this we need to determine the mapping of the complex velocity (11.37) from the $z_1$ plane to the $z$ plane. Use the chain rule

$$\frac{dW}{dz}=\frac{dW}{dz_1}\frac{dz_1}{dz_2}\frac{dz_2}{dz_3}\frac{dz_3}{dz} \quad . \tag{11.58}$$

The required derivatives in (11.58) are

$$\begin{aligned}
&\frac{dz_1}{dz_2}=1\\
&\frac{dz_2}{dz_3}=\frac{1}{\left(\frac{dz_3}{dz_2}\right)}=\frac{1}{\left(1+\frac{\varepsilon}{(z_2-\Delta)^2}\right)}=\frac{(z_2-\Delta)^2}{(z_2-\Delta)^2+\varepsilon} \quad .\\
&\frac{dz_3}{dz}=\frac{1}{\left(\frac{dz}{dz_3}\right)}=\frac{z_3^2}{z_3^2-1}
\end{aligned} \tag{11.59}$$

Combine (11.58) and (11.59).

$$U_z=U_{z_1}\left(\frac{(z_2-\Delta)^2}{(z_2-\Delta)^2+\varepsilon}\right)\frac{z_3^2}{z_3^2-1} \tag{11.60}$$

The velocity field in the complex $z_1$ plane (11.37), repeated here for convenience, is

$$\frac{dw(z_1)}{dz_1} = \frac{e^{-i\alpha}}{R} - \left(\frac{R}{z_1^2}\right) + i\frac{\gamma}{z_1} \ . \tag{11.61}$$

Zero velocities occur at:

i) The stagnation points in $z_1$

ii) $z_2 = \Delta$, $z_3 = \infty$, $z = \infty$

iii) $z_3 = 0$, $z = \infty$.

Infinite velocities occur at $z_1 = 0$ and:

i) $z_3 = 1$, $z = z_3 + 1/z_3 = 2$. The infinity multiplies the zero at the stagnation point to produce a finite velocity at the trailing edge.

ii) $z_3 = 1$, $z = -2$. This point lies inside the airfoil.

iii) The point

$$\begin{aligned} &(z_2 - \Delta)^2 + \varepsilon = 0 \ \Rightarrow \ \mathrm{z}_2 = \Delta \pm (-\varepsilon)^{1/2} \\ &z_3 = \Delta \pm 2(-\varepsilon)^{1/2} \end{aligned} \ . \tag{11.62}$$

The singularity (11.62) transforms to

$$z_{Singularity} = \Delta + 2(-\varepsilon)^{1/2} + \frac{1}{\Delta + 2(-\varepsilon)^{1/2}} \ , \ \Delta - 2(-\varepsilon)^{1/2} + \frac{1}{\Delta - 2(-\varepsilon)^{1/2}} \ . \tag{11.63}$$

We need to be concerned about the position of the singularity (11.63) since it may lie outside of and close to the airfoil where it can affect the pressure distribution. For the example above

$$z_{Singularity} = 1.81465 - 1.30801i \ , \ 0.906875 + 2.46541i \tag{11.64}$$

These locations are relatively far from the airfoil and so we should expect a reasonably smooth pressure distribution. The pressure coefficient is

$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} = \left(1 - |U_z|^2\right) = 1 - \left(U_x^2 + U_y^2\right) \tag{11.65}$$

plotted below.

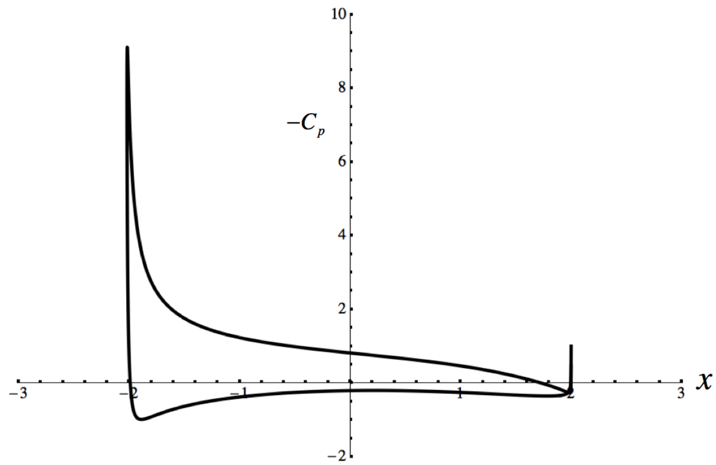



*Figure 11.10 pressure distributions on the example Joukowsky airfoil at* $\alpha = \pi / 9$.

## 11.3 THIN AIRFOIL THEORY

The figure below depicts a generic two-dimensional airfoil. The chord line of the airfoil is aligned with the $x$-axis. The upper surface is defined by $y = f(x)$ and the lower surface by $y = g(x)$. The flow is steady, inviscid and incompressible.

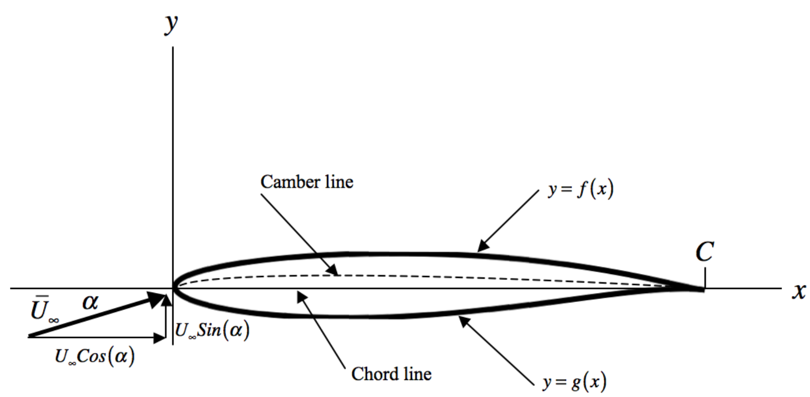



*Figure 11.11 Two-dimensional airfoil at angle of attack* $\alpha$.

The real flow about such an airfoil is viscous and there is no possibility of flow around the sharp trailing edge. The no slip condition and viscous dissipation prevents this from happening just as is it does in the wedge flow shown in Figure 11.1. In potential flow, the Kutta condition is used to mimic this behavior by *requiring* that the flow leave the trailing edge smoothly with no singularity. As we learned in the discussion of Joukowsky airfoils this requirement sets the lift on the airfoil.

The stagnation pressure is constant throughout a steady potential flow.

$$\frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^2 = \frac{P_{TE}^+}{\rho} + \frac{1}{2}\left(U_{TE}^+\right)^2 = \frac{P_{TE}^-}{\rho} + \frac{1}{2}\left(U_{TE}^-\right)^2 \qquad (11.66)$$

where the superscript "+" refers to the flow leaving the upper surface of the airfoil and the superscript "−" refers to the flow leaving the lower surface of the airfoil. Since the flow is incompressible, the Kutta condition boils down to

$$\frac{P_{TE}^{+}}{\rho} = \frac{P_{TE}^{-}}{\rho} \quad . \tag{11.67}$$

The static pressure is the same above and below the airfoil at the trailing edge. The immediate implication of (11.67) and (11.66) is that

$$U_{TE}^{+} = U_{TE}^{-} \quad . \tag{11.68}$$

The flow velocity must also be the same above and below the trailing edge.

### 11.3.1 THE THIN AIRFOIL APPROXIMATION

The Joukowsky transformation is a very useful way to generate interesting airfoil shapes. However the range of shapes that can be generated is limited by range available for the parameters that define the transformation. In practice the thickness of most airfoils is small compared to the chord. This enables a simplification of the surface boundary condition for Laplace's equation that allows very general, airfoil shapes to be produced relatively easily. Decompose the velocity potential into the free stream and perturbation potential.

$$\Phi = U_{\infty} x Cos(\alpha) + U_{\infty} y Sin(\alpha) + \phi(x,y) \tag{11.69}$$

The velocities are

$$\begin{aligned} U &= U_{\infty} Cos(\alpha) + u(x,y) \\ V &= U_{\infty} Sin(\alpha) + v(x,y) \end{aligned} \tag{11.70}$$

where

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \quad . \tag{11.71}$$

The disturbance potential satisfies Laplace's equation.

$$\nabla^2 \phi = 0 \tag{11.72}$$

with far field conditions

$$\frac{\partial \phi}{\partial x} \to 0 \qquad \frac{\partial \phi}{\partial y} \to 0 \quad \text{at} \quad \infty \tag{11.73}$$

and surface conditions

$$v\left(x,f(x)\right)=\left.\frac{\partial\phi}{\partial y}\right|_{y=f(x)}=U_\infty\left(\frac{df}{dx}-Tan(\alpha)\right)\quad\text{on}\quad 0\le x\le C \tag{11.74}$$

$$v\left(x,g(x)\right)=\left.\frac{\partial\phi}{\partial y}\right|_{y=g(x)}=-U_\infty\left(\frac{dg}{dx}-Tan(\alpha)\right)\quad\text{on}\quad 0\le x\le C\quad . \tag{11.75}$$

Within the thin airfoil approximation the surface conditions can be evaluated on $y=0$ instead of on the actual airfoil surface. The linearized surface conditions are

$$v\left(x,0^+\right)=\left.\frac{\partial\phi}{\partial y}\right|_{y=0^+}=U_\infty\left(\frac{df}{dx}-\alpha\right)\quad\text{on}\quad 0\le x\le C \tag{11.76}$$

$$v\left(x,0^-\right)=\left.\frac{\partial\phi}{\partial y}\right|_{y=0^-}=-U_\infty\left(\frac{dg}{dx}-\alpha\right)\quad\text{on}\quad 0\le x\le C\quad . \tag{11.77}$$

Define the thickness function of the airfoil.

$$\eta_{Thickness}(x)=\frac{1}{2}\left(f(x)-g(x)\right) \tag{11.78}$$

and the camber function

$$\eta_{Camber}(x)=\frac{1}{2}\left(f(x)+g(x)\right). \tag{11.79}$$

The linearized problem is now defined.

$$\begin{aligned}
&\nabla^2\phi=0\\
&\lim_{r\to\infty}\frac{\partial\phi}{\partial x}=\lim_{r\to\infty}\frac{\partial\phi}{\partial y}=0\\
&\left.\frac{\partial\phi}{\partial y}\right|_{x,y=0^+}=U_\infty\frac{d\eta_{Thickness}}{dx}+U_\infty\frac{d\eta_{Camber}}{dx}-U_\infty\alpha\\
&\left.\frac{\partial\phi}{\partial y}\right|_{x,y=0^-}=-U_\infty\frac{d\eta_{Thickness}}{dx}+U_\infty\frac{d\eta_{Camber}}{dx}-U_\infty\alpha
\end{aligned} \tag{11.80}$$

The surface condition breaks into three terms for the effect of thickness, camber and the angle of attack. Treat the problem as the superposition of three independent potentials.

$$\phi=\phi_{Thickness}+\phi_{Camber}+\phi_\alpha \tag{11.81}$$

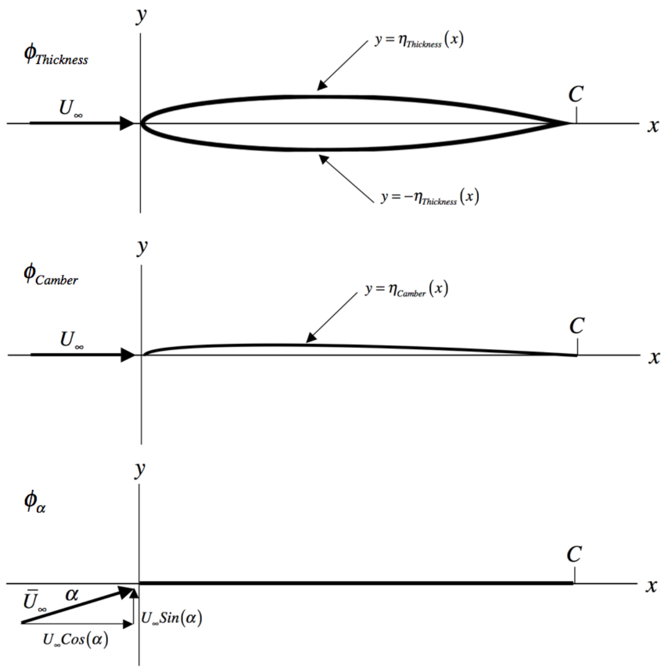



*Figure 11.12 The three basic problems of thin airfoil theory.*

***The thickness problem***

Let the thickness disturbance field be represented by a distribution of mass sources along the $x$-axis in the range $0 \le x \le C$.

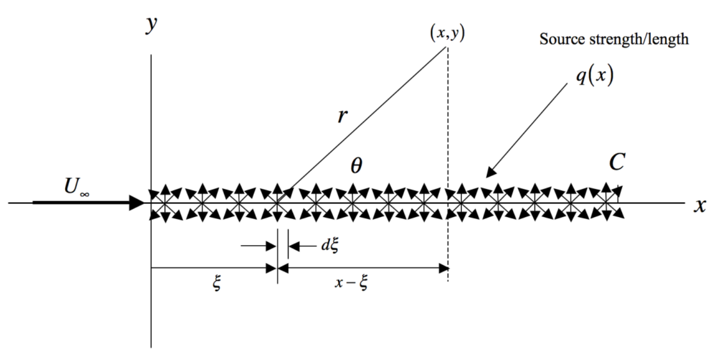



*Figure 11.13 Distribution of sources generating thickness.*

The velocity field due to such a distribution satisfies Laplace's equation and $\nabla\phi = 0$ at infinity. For a source of strength $q(\xi)d\xi$ (units $L^2/T$) at $(x,y) = (\xi,0)$ the complex potential is

$$dW_{Thickness} = \frac{q(\xi)d\xi}{2\pi} Ln\left(re^{i\theta}\right) \tag{11.82}$$

where

$$r^2 = (x-\xi)^2 + y^2 \quad . \tag{11.83}$$

The differential velocity potential is

$$d\phi_{Thickness} = \frac{q(\xi)}{2\pi} Ln\left((x-\xi)^2 + y^2\right)^{1/2} d\xi \quad . \tag{11.84}$$

The potential at $(x,y)$ is given by the sum of the potentials due to all the source increments.

$$\phi_{Thickness} = \frac{1}{2\pi}\int_0^C q(\xi) Ln\left((x-\xi)^2 + y^2\right)^{1/2} d\xi \tag{11.85}$$

with velocity components

$$\begin{aligned} u_{Thickness}(x,y) &= \frac{\partial \phi_{Thickness}}{\partial x} = \frac{1}{2\pi}\int_0^C q(\xi)\frac{x-\xi}{(x-\xi)^2 + y^2} d\xi \\ v_{Thickness}(x,y) &= \frac{\partial \phi_{Thickness}}{\partial y} = \frac{1}{2\pi}\int_0^C q(\xi)\frac{y}{(x-\xi)^2 + y^2} d\xi \end{aligned} \quad . \tag{11.86}$$

We wish to use the surface condition (11.80) on the thickness

$$\begin{aligned} v(x,0^+) &= U_\infty \frac{d\eta_{Thickness}}{dx} \\ v(x,0^-) &= -U_\infty \frac{d\eta_{Thickness}}{dx} \end{aligned} \quad . \tag{11.87}$$

to determine the source distribution $q(x)$. Let $y$ in (11.86) be a very small but fixed number $y = \varepsilon$. The vertical velocity component in (11.86) becomes

$$v_{Thickness}(x,\varepsilon) = \frac{1}{2\pi}\int_0^C q(\xi)\frac{\varepsilon}{(x-\xi)^2 + \varepsilon^2} d\xi \quad . \tag{11.88}$$

For very small values of $\varepsilon$ the only contribution to the integral (11.88) comes from the range of $\xi$ very close to $x$ where the function $\varepsilon / \left((x-\xi)^2 + \varepsilon^2\right) \to 1/\varepsilon$. This allows $q(x)$ to be pulled out of the integral (11.88). Now

$$v_{Thickness}(x,\varepsilon) = \frac{q(x)}{2\pi}\int_0^C \frac{\varepsilon}{(x-\xi)^2 + \varepsilon^2} d\xi \quad . \tag{11.89}$$

Now evaluate the remaining integral in (11.89).

$$\int_0^C \frac{\varepsilon}{(x-\xi)^2+\varepsilon^2}d\xi = \int_0^{C/\varepsilon} \frac{1}{\left(\frac{x-\xi}{\varepsilon}\right)^2+1} d\left(\frac{\xi}{\varepsilon}\right) = \int_{\frac{x-C}{\varepsilon}}^{x/\varepsilon} \left(\frac{1}{\lambda^2+1}\right) d\lambda =$$

$$ArcTan\left(\frac{x}{\varepsilon}\right) - ArcTan\left(\frac{x-C}{\varepsilon}\right) \quad . \qquad (11.90)$$

$$\lim_{\varepsilon \to 0}\left( ArcTan\left(\frac{x}{\varepsilon}\right) - ArcTan\left(\frac{x-C}{\varepsilon}\right)\right) = \pi$$

Now

$$v_{Thickness}\left(x, 0^{\pm}\right) = \pm\frac{q(x)}{2} = \pm U_{\infty}\frac{d\eta_{Thickness}}{dx} \quad . \qquad (11.91)$$

Finally the thickness problem reduces to choosing the distribution of source strength to be

$$q(x) = U_{\infty}\frac{d\eta_{Thickness}}{dx} \quad . \qquad (11.92)$$

The thickness potential is

$$\phi_{Thickness} = \frac{U_{\infty}}{2\pi}\int_0^C \frac{d\eta_{Thickness}(\xi)}{d\xi} Ln\left(\left(x-\xi\right)^2+y^2\right)^{1/2} d\xi \quad . \qquad (11.93)$$

Note that the total source strength due to thickness is zero.

$$\int_0^C q(x)dx = \int_0^C U_{\infty}\frac{d\eta_{Thickness}}{dx}dx = U_{\infty}\left(\eta_{Thickness}(C) - \eta_{Thickness}(0)\right) = 0 \qquad (11.94)$$

***The camber problem***

Let the camber disturbance field be represented by a distribution of vortices along the $x$-axis in the range $0 \le x \le C$.

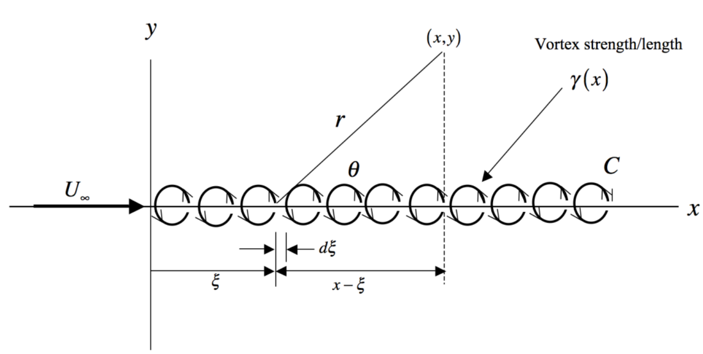



*Figure 11.14 Distribution of vortices generating camber.*

Again, the velocity field due to such a distribution satisfies Laplace's equation and $\nabla\phi = 0$ at infinity. The source function $\gamma(x)$ is the circulation per unit length. Physically it is the velocity difference between the upper and lower surface at the position $x$.

For a source of strength $\gamma(\xi)d\xi$ (units $L/T$) at $(x,y)=(\xi,0)$ the complex potential is

$$dW_{Camber} = \frac{\gamma(\xi)d\xi}{2\pi} iLn\left(re^{i\theta}\right) \tag{11.95}$$

where

$$r^2 = (x-\xi)^2 + y^2 \quad . \tag{11.96}$$

Take the real part of (11.95). The differential velocity potential is

$$d\phi_{Camber} = -\frac{\gamma(\xi)}{2\pi} ArcTan\left(\frac{y}{x-\xi}\right)d\xi \ . \tag{11.97}$$

The potential at $(x,y)$ is given by the sum of the potentials due to all the vortex increments.

$$\phi_{Camber} = -\frac{1}{2\pi}\int_0^C \gamma(\xi) ArcTan\left(\frac{y}{x-\xi}\right)d\xi \tag{11.98}$$

with velocity components

$$\begin{aligned} u_{Camber}(x,y) &= \frac{\partial\phi_{Camber}}{\partial x} = \frac{1}{2\pi}\int_0^C \gamma(\xi)\frac{y}{(x-\xi)^2+y^2}d\xi \\ v_{Camber}(x,y) &= \frac{\partial\phi_{Camber}}{\partial y} = -\frac{1}{2\pi}\int_0^C \gamma(\xi)\frac{x-\xi}{(x-\xi)^2+y^2}d\xi \end{aligned} \quad . \tag{11.99}$$

Check $\lim_{y\to 0^{\pm}} u_{Camber}(x,y)$.

$$u_{Camber}\left(x,0^{\pm}\right) = \lim_{y\to 0^{\pm}} \frac{1}{2\pi}\int_0^C \gamma(\xi)\frac{y}{(x-\xi)^2+y^2}d\xi = \pm\frac{\gamma(x)}{2} . \tag{11.100}$$

From (11.100).

$$\gamma(x) = u_{Camber}\left(x,0^{+}\right) - u_{Camber}\left(x,0^{-}\right) \ . \tag{11.101}$$

The vortex strength source function $\gamma(x)$ is precisely the velocity difference between the upper and lower surface of the camber line (and the airfoil) and goes to zero at the trailing edge as required by the Kutta condition.

The surface condition for camber is

$$v_{Camber}\left(x,0^{\pm}\right)=\lim_{y\to 0^{\pm}}\left(-\frac{1}{2\pi}\int_0^C \gamma(\xi)\frac{x-\xi}{(x-\xi)^2+y^2}d\xi\right)=U_\infty\frac{d\eta_{Camber}}{dx}\ . \qquad (11.102)$$

Taking the limit in (11.102)

$$v_{Camber}\left(x,0^{\pm}\right)=-\frac{1}{2\pi}\int_0^C \gamma(\xi)\frac{1}{(x-\xi)}d\xi=U_\infty\frac{d\eta_{Camber}}{dx}. \qquad (11.103)$$

The pressure on the camber line is

$$\begin{aligned}&P_\infty+\frac{1}{2}\rho U_\infty^2=P_{Camber}\left(x,0^{\pm}\right)+\frac{1}{2}\rho\left(\left(U_\infty+u_{Camber}\right)^2+\left(v_{Camber}\right)^2\right)\cong\\&P_{Camber}\left(x,0^{\pm}\right)+\frac{1}{2}\rho U_\infty+\rho U_\infty u_{Camber}\left(x,0^{\pm}\right)\end{aligned}\ . \qquad (11.104)$$

The lift on the airfoil due to camber is

$$L=\int_0^C\left(P_{Camber}\left(x,0^-\right)-P_{Camber}\left(x,0^+\right)\right)dx\ . \qquad (11.105)$$

Use (11.100) and (11.104) in (11.105). The lift on the airfoil is

$$\begin{aligned}&L=\int_0^C\left(\rho U_\infty\frac{\gamma(x)}{2}+\rho U_\infty\frac{\gamma(x)}{2}\right)dx=\rho U_\infty\int_0^C\gamma(x)dx\\&L=\rho U_\infty\Gamma\end{aligned}\ . \qquad (11.106)$$

The moment acting on the camber line about the leading edge is

$$M=\int_0^C\left(P_{Camber}\left(x,0^-\right)-P_{Camber}\left(x,0^+\right)\right)x\,dx \qquad (11.107)$$

which becomes

$$M=\rho U_\infty\int_0^C\gamma(x)\,x\,dx\ . \qquad (11.108)$$

We still need to determine the source distribution $\gamma(x)$. That is we need to solve

$$\int_0^C \frac{\gamma(\xi)}{x-\xi} d\xi = -2\pi U_\infty \frac{d\eta_{Camber}}{dx} \qquad 0 \le x \le C \qquad \gamma(C) = 0 \ . \tag{11.109}$$

1) Introduce

$$\begin{aligned} x &= \frac{C}{2}\left(1 + Cos(\theta)\right) \\ \xi &= \frac{C}{2}\left(1 + Cos(\varsigma)\right) \end{aligned} \ . \tag{11.110}$$

This change of variables is illustrated below.

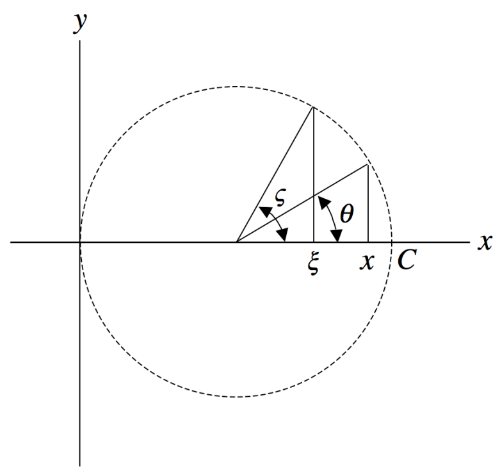



*Figure 11.15 The change of variables (11.110).*

2) Express $\eta_{Camber}(x)$ as a Fourier series in $\theta$.

$$\frac{d\eta_{Camber}}{dx} = \sum_{n=0}^{\infty} B_n Cos(n\theta) \tag{11.111}$$

The coefficients are determined from the orthogonality of the expansion functions.

$$B_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{d\eta_{Camber}}{dx}\right) Cos(n\theta)\, d\theta \ . \tag{11.112}$$

3) Solve the integral (11.109) in the form

$$\frac{1}{2\pi U_\infty}\int_{\pi}^{0} \frac{\gamma(\varsigma)}{Cos(\varsigma) - Cos(\theta)} d\varsigma = \sum_{n=0}^{\infty} B_n Cos(n\theta) \ . \tag{11.113}$$

The result is

$$\gamma(\theta) = -2U_\infty \sum_{n=1}^{\infty} B_n Sin(n\theta) + 2\frac{U_\infty B_0}{Sin(\theta)}\left(\frac{K}{2U_\infty B_0} + Cos(\theta)\right) . \qquad (11.114)$$

The constant $K$ is determined by the Kutta condition $\gamma(\theta = 0) = 0$. The result is $K = -2U_\infty B_0$ which cancels the singularity at $\theta = 0$ in (11.114). Thus

$$\gamma(\theta) = -2U_\infty \left( B_0 \left( \frac{1 - Cos(\theta)}{Sin(\theta)} \right) + \sum_{n=1}^{\infty} B_n Sin(n\theta) \right) . \qquad (11.115)$$

Note that in general there is a singularity in the circulation at the leading edge where $\theta = \pi$.

Now all the zero angle of attack aerodynamic properties of the cambered airfoil are known; namely the lift coefficient

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 C} = \frac{1}{U_\infty}\int_0^\pi \gamma(\theta) Sin(\theta) d\theta = -\pi(2B_0 + B_1) \qquad (11.116)$$

and the moment coefficient.

$$C_M = \frac{M}{\frac{1}{2}\rho U_\infty^2 C^2} = \frac{1}{2U_\infty}\int_0^\pi \gamma(\theta)(1 + Cos(\theta)) Sin(\theta) d\theta = -\frac{\pi}{4}(2B_0 + B_1) - \frac{\pi}{4}(B_1 + B_2) . \qquad (11.117)$$

The moment coefficient can be expressed as

$$C_M = \frac{C_L}{4} - \frac{\pi}{4}(B_1 + B_2) . \qquad (11.118)$$

The force system on the airfoil is equivalent to a pure moment about the leading edge tending to pitch the nose up plus the moment that would be generated by the lift force acting at the one-quarter chord point tending to pitch the nose down.

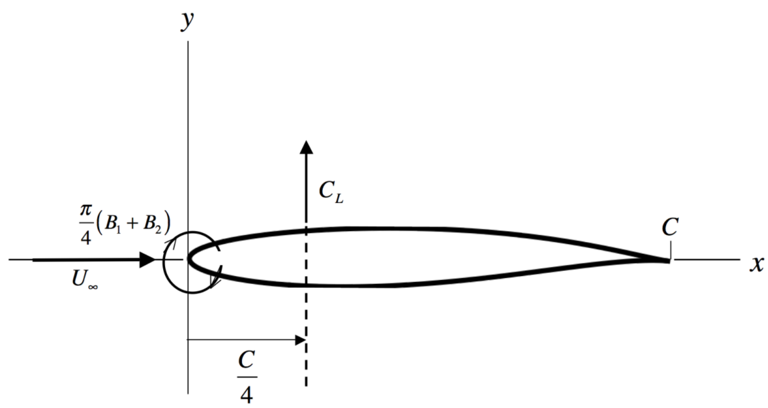



*Figure 11.16 Forces and moments on a thin cambered airfoil at zero angle of attack.*

***The angle-of-attack problem***

Finally we look at the incompressible potential flow past a flat plate at a small angle of attack illustrated below. The source is modeled as a distribution of vortices as in the camber problem.

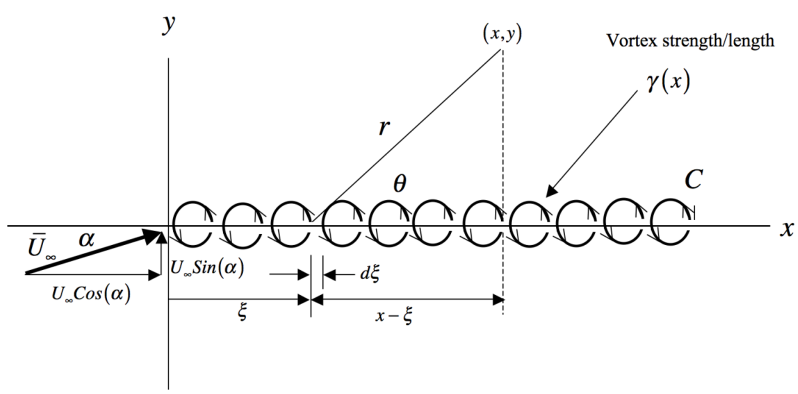



*Figure 11.17 Distribution of vortices generating lift on a flat plate at angle-of-attack* $\alpha$ *.*

In this case the surface condition is especially simple since $d\eta_{Camber} / dx = -\alpha$ .

$$v_{Camber}\left(x,0^{\pm}\right) = -U_{\infty}\alpha \tag{11.119}$$

Equation (11.109) becomes

$$\int_0^C \frac{\gamma(\xi)}{x-\xi}d\xi = 2\pi U_{\infty}\alpha \qquad 0 \le x \le C \qquad \gamma(C) = 0\,. \tag{11.120}$$

In (11.111) all coefficients higher than $n = 0$ are zero so the series truncates at the first term.

$$B_0 = -\frac{2}{\pi}\int_0^{\pi} \alpha \, d\theta = -2\alpha \,, \quad B_1, B_2, B_3, \ldots, B_n = 0 \tag{11.121}$$

The circulation distribution (11.115) on the flat plate is

$$\gamma(\theta) = 2U_{\infty}\alpha\left(\frac{1 - Cos(\theta)}{Sin(\theta)}\right). \tag{11.122}$$

From (11.116) and (11.117) the lift and moment coefficients for the flat plate at angle of attack are

$$\begin{aligned} C_L &= 2\pi\alpha \\ C_M &= \frac{\pi}{2}\alpha \end{aligned} \tag{11.123}$$