

1.0 POTENTIAL FLOW

One of the most important applications of potential flow theory is to aerodynamics and marine hydrodynamics.

Key assumption.

1. Incompressibility – The density and specific weight are to be taken as constant.
2. Irrotationality – This implies a nonviscous fluid where particles are initially moving without rotation.
3. Steady flow – All properties and flow parameters are independent of time.

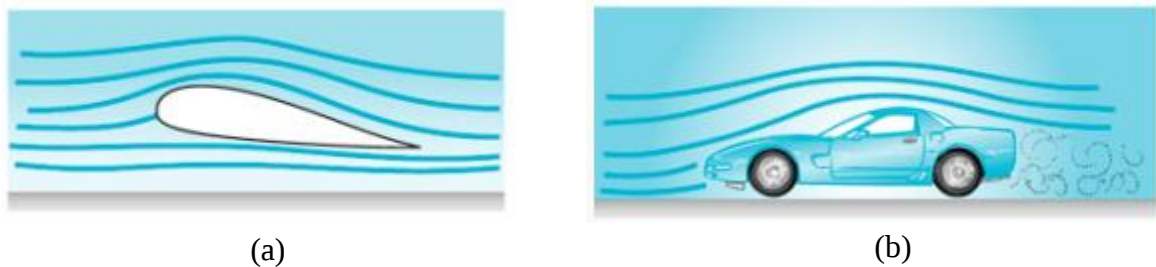


Fig. 1.1 Examples of complicated immersed flows: (a) flow near a solid boundary; (b) flow around an automobile.

In this section we will be concerned with the mathematical description of the motion of fluid elements moving in a flow field. A small fluid element in the shape of a cube which is initially in one position will move to another position during a short time interval as illustrated in Fig.1.1.

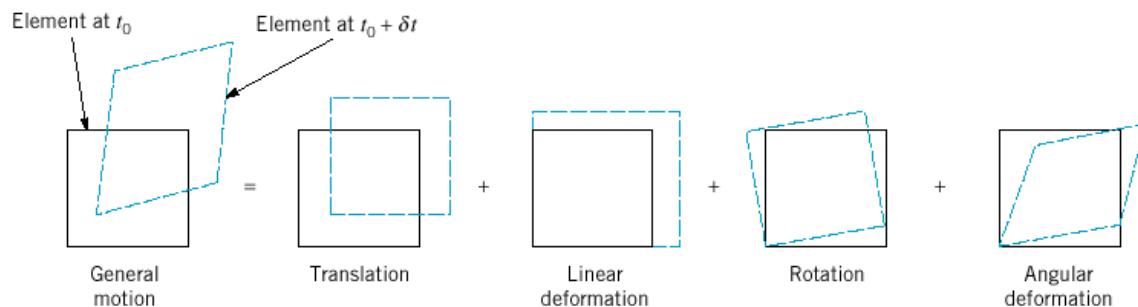
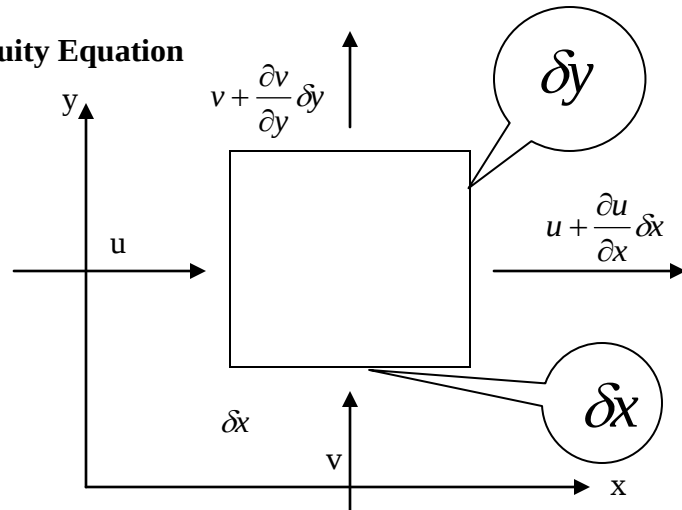


Fig. 1.2

1.1 Continuity Equation



u = velocity component x direction
 v = velocity component y direction

Continuity Equation

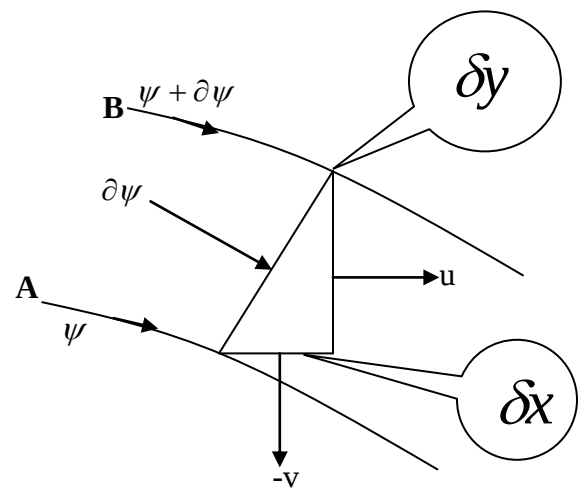
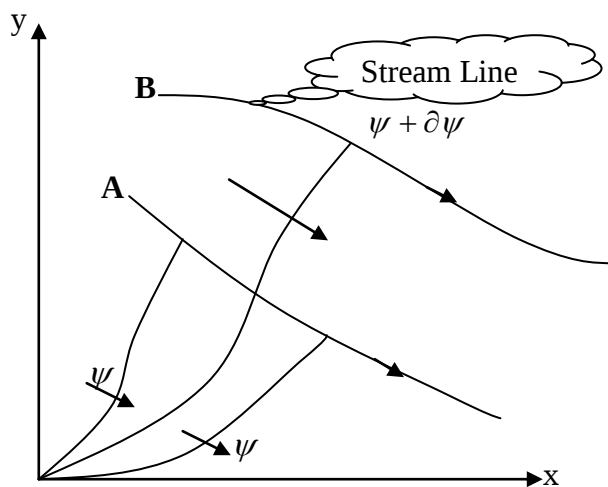
Flow inwards = Flow outwards

$$u \delta y + v \delta x = \left(u + \frac{\partial u}{\partial x} \delta x \right) \delta y + \left(v + \frac{\partial v}{\partial y} \delta y \right) \delta x$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad - 2D$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad - 3D$$

1.2 Stream Function, ψ (psi)



The stream is continuity

$$d\psi = -vdx + udy$$

if

$$\psi = \psi(x, y)$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$\therefore$

$$u = \frac{\partial \psi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial y}$$

Integrated the equations

$$\psi = \int \left(\frac{\partial \psi}{\partial x} \right) dx + \int \left(\frac{\partial \psi}{\partial y} \right) dy + C$$

$$\psi = \int -v dx + \int u dy + C$$

$\therefore$ Continuity equation in ψ

$$\frac{\partial \left(\frac{\partial \psi}{\partial y} \right)}{\partial x} + \frac{\partial \left(-\frac{\partial \psi}{\partial x} \right)}{\partial y} = 0$$

or

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

if $\neq 0$ not continuity

Vorticity equation, ξ (rotational flow)

$$\xi = 2\omega$$

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right); \text{ angular velocity (rad/s)}$$

$$\therefore \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

or substitute with

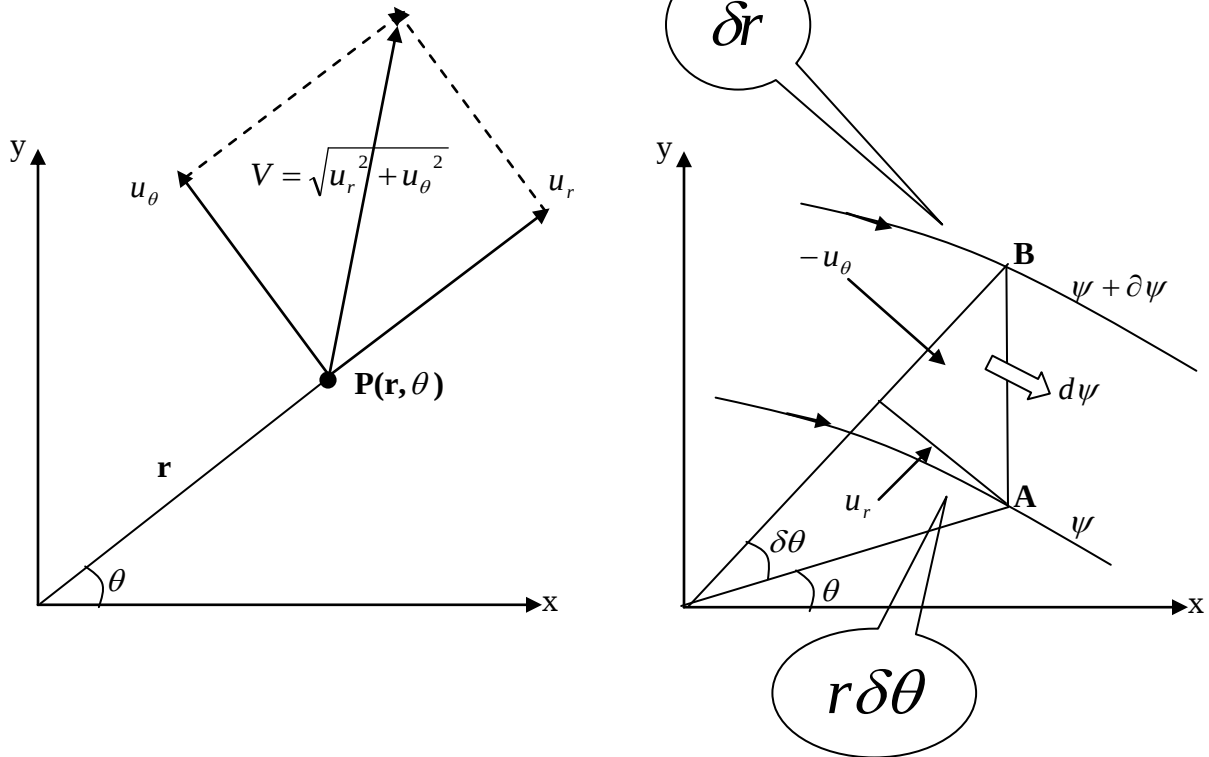
$$\xi = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

Irrotational flow, $\xi = 0$

Rotational flow. $\xi \neq 0$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

1.3 Cylindrical Polar Coordinate



The stream is continuity

$$d\psi = u_r (r\delta\theta) - u_\theta (\delta r) \text{ -----(1) 2-D equation}$$

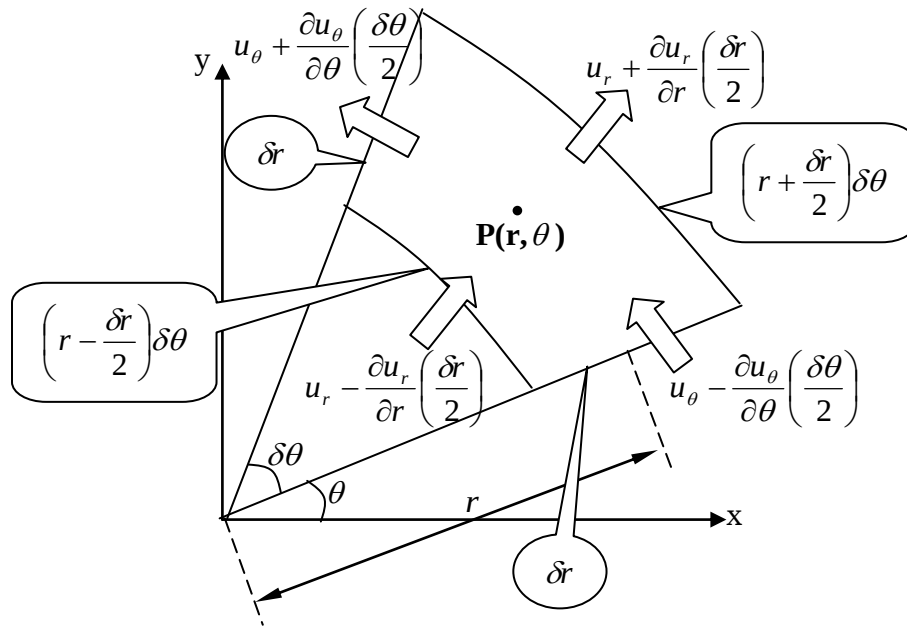
if $\psi = \psi(r, \theta)$

$$d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial r} dr \text{ -----(2)}$$

compare equation (1) and (2)

$$\therefore \quad u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

1.3.1 Continuity Equation



Flow inwards = Flow outwards

$$\left[\left(u_r - \frac{\partial u_r}{\partial r} \cdot \frac{\delta r}{2} \right) \left(r - \frac{\delta r}{2} \right) \delta \theta \right] + \left[\left(u_\theta - \frac{\partial u_\theta}{\partial \theta} \left(\frac{\delta \theta}{2} \right) \right) (\delta r) \right] = \left[\left(u_r + \frac{\partial u_r}{\partial r} \cdot \frac{\delta r}{2} \right) \left(r + \frac{\delta r}{2} \right) \delta \theta \right] + \left[\left(u_\theta + \frac{\partial u_\theta}{\partial \theta} \cdot \frac{\delta \theta}{2} \right) \delta r \right]$$

$$\therefore \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_\theta}{r \partial \theta} = 0$$

1.4 Velocity Potential, ϕ (phi)

Occur when some vorticity flow are irrotational, $\xi = 0$

Velocity component;

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad \text{and} \quad w = \frac{\partial \phi}{\partial z}$$

1.4.1 Potential Flow

The flow that consist potential flow

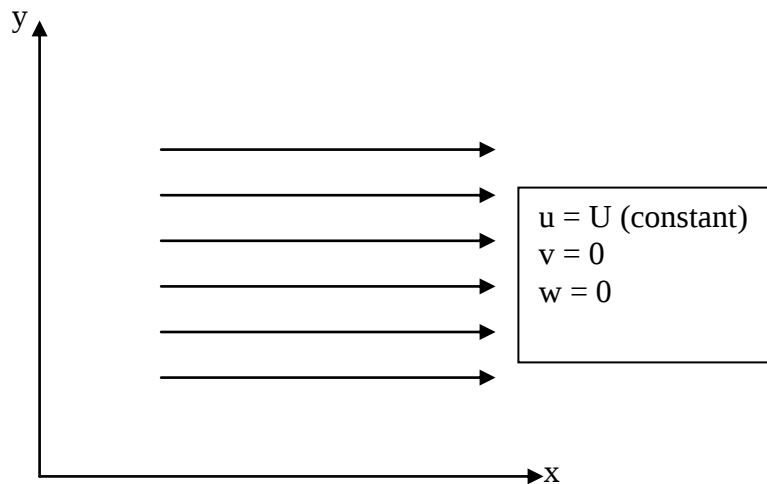
Occur when some flow are irrotational and consist potential flow.

The relationship between velocity potential and stream function are known as **Cauchy-Rieman Equation**.

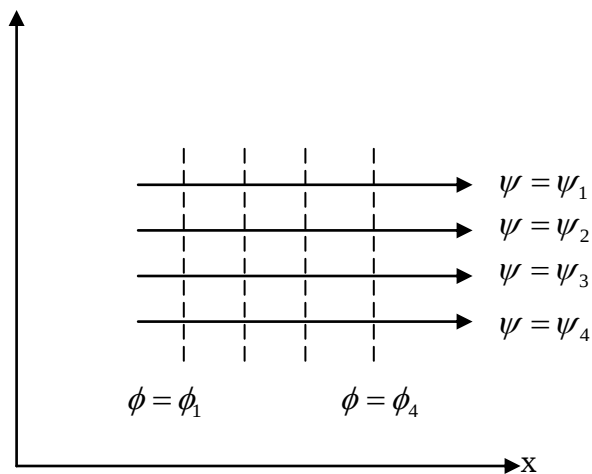
$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

1.4 Summary

1.4.1 Uniform Flow in 3-D



1.4.2 The relationship between potential flow and stream function

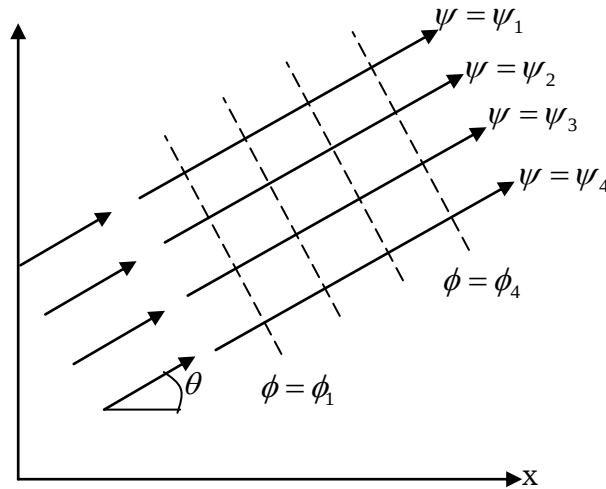


$$\frac{\partial \phi}{\partial x} = U, \frac{\partial \phi}{\partial y} = 0$$

$$\therefore \phi = U_x + C$$

$$\frac{\partial \psi}{\partial y} = U, \frac{\partial \psi}{\partial x} = 0$$

$$\therefore \psi = U_y + C$$



$$\phi = U(x \cos \theta + y \sin \theta)$$

$$\psi = U(y \cos \theta - x \sin \theta)$$

1.4.3 Continuity Equation

1.4.3.1 Cartesian Coordinate.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

1.4.3.2 Polar Coordinate

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0$$

1.4.4 Stream Function

1.4.4.1 Cartesian Coordinate

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\text{Resultant velocity, } q = \sqrt{u^2 + v^2}$$

If we know velocity component

$$\text{So} \quad \psi = \int -v \delta x + \int u \delta y + C$$

1.4.4.2 Polar Coordinate

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

$$\text{Resultant velocity, } q = \sqrt{u_r^2 + u_\theta^2}$$

If we know velocity component

$$\psi = \int -u_\theta \delta r + \int r u_r \delta \theta$$

1.4.5 Circulation, Γ

$$\Gamma = 2\pi \omega r^2$$

1.4.6 Vorticity, ξ

$$\xi = \frac{\partial \Gamma}{\partial A} = \frac{2\pi \omega r^2}{\pi r^2} = 2\omega$$

1.4.7 The relationship between vorticity with velocity component and stream function.

1.4.7.1 Cartesian Coordinate.

$$\xi = \frac{\Gamma_{ABCD}}{\partial x \partial y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

we know that,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\therefore \quad \xi = \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

1.4.7.2 Polar Coordinate

$$\xi = \frac{\partial \Gamma}{\partial A} = \frac{\partial \Gamma}{r \delta \theta \delta r}$$

$$\xi = \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta}$$

if we know that,

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

$$\therefore \quad \xi = -\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] = -\nabla^2 \psi$$

1.4.8 Potential Flow, ϕ

1.4.8.1 Cartesian Coordinate

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}$$

1.4.8.2 Polar Coordinate

$$u_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

1.4.9 The relationship between Stream Function and Potential Flow

1.4.9.1 Cartesian Coordinate

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

1.4.9.2 Polar Coordinate

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

Example 1

Given a flow field $\psi = k(x^2 - y^2)$. Shows that a flow is irrotational.

Solution

Irrotational is $\xi = 0$

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \text{----- irrotational}$$

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\psi = k(x^2 - y^2) = kx^2 - ky^2$$

$$u = \frac{\partial \psi}{\partial y} = -2ky \quad \text{so} \quad \frac{\partial u}{\partial y} = -2k$$

$$v = -\frac{\partial \psi}{\partial x} = -2kx \quad \text{so} \quad \frac{\partial v}{\partial x} = -2k$$

$$\therefore \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -2k - (-2k) = 0 = \text{irrotational}$$

Examples 2

For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = 4xy\hat{i} + 2(x^2 - y^2)\hat{j}$$

is this flow irrotational?

Solution

$$u = 4xy \quad \text{and} \quad v = 2(x^2 - y^2) = 2x^2 - 2y^2$$

$$\frac{\partial v}{\partial y} = 4x \quad \frac{\partial v}{\partial x} = 4x$$

$$\xi = 0 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 4x - 4x = 0 = \text{irrotational}$$

1.1 Source and Sink

Consider a fluid flowing radially outward and inward from a line through the origin perpendicular to the x - y plane as is shown in Fig. 1.3.

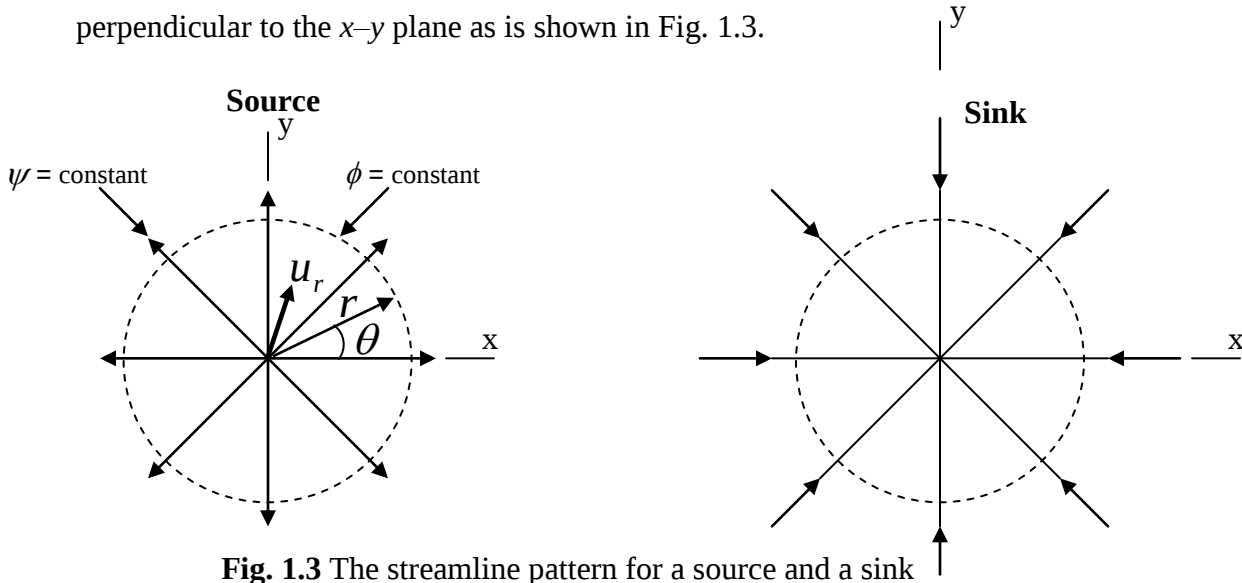


Fig. 1.3 The streamline pattern for a source and a sink

1.1.1 Stream Function, ψ

Let Q be the volume rate of flow emanating from the line (per unit length), and therefore to satisfy conservation of mass

$$Q = (2\pi r)u_r$$

or

$$u_r = \frac{Q}{2\pi r}$$

we know that

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r} = 0$$

$$\psi = \psi(r, \theta)$$

$$\partial \psi = \frac{\partial \psi}{\partial r} \delta r + \frac{\partial \psi}{\partial \theta} \delta \theta = (0 \times \delta r) + (r \times \frac{Q}{2\pi r} \delta \theta)$$

$$\psi = \int r \left(\frac{Q}{2\pi r} \right) \partial \theta + C$$

$$\psi = \frac{Q\theta}{2\pi} + C$$

If $\psi = 0$ at $\theta = 0$ $\therefore C = 0$

$$\psi = \frac{Q\theta}{2\pi} \quad \text{FOR SOURCE}$$

$$\psi = -\frac{Q\theta}{2\pi} \quad \text{FOR SINK}$$

1.1.2 Velocity potential, ϕ

$$u_r = \frac{\partial \phi}{\partial r} = \frac{Q}{2\pi r} \quad , \quad U_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$$

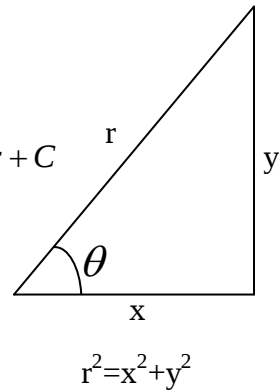
$$\partial \phi = \frac{\partial \phi}{\partial r} \delta r + \frac{\partial \phi}{\partial \theta} \delta \theta$$

$$\partial \phi = \frac{Q}{2\pi r} \partial r + (0) \partial \theta$$

$$\therefore \phi = \int \left(\frac{Q}{2\pi r} \cdot \frac{1}{r} \right) dr + \int (0) d\theta + C$$

$$= \frac{Q}{2\pi} \log_e r + C \quad \text{atau} \quad \frac{Q}{2\pi} \ln r + C$$

$$= \frac{Q}{4\pi} \log_e (x^2 + y^2)$$



$$\phi_{\text{source}} = \frac{Q}{2\pi} \log_e r = \frac{Q}{4\pi} \log_e (x^2 + y^2)$$

$$\phi_{\text{sink}} = -\frac{Q}{2\pi} \log_e r = -\frac{Q}{4\pi} \log_e (x^2 + y^2)$$

E XAMPLE 1.1

A nonviscous, incompressible fluid flows between wedge-shaped walls into a small opening as shown in Fig. E1.1. The velocity potential (in ft^2/in), which approximately describes this flow is

$$\phi = -2 \ln r$$

Determine the volume rate of flow (per unit length) into the opening.

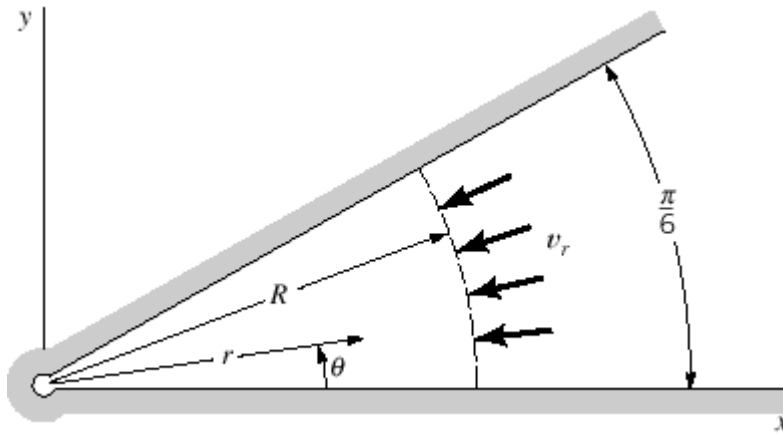


Fig.E1.1

Solution

The components of velocity are

$$u_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r} \quad , \quad U_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$$

which indicates we have a purely radial flow. The flow rate per unit width, q , crossing the arc of length $R\pi/6$ can thus be obtained by integrating the expression

$$Q = \int_0^{\pi/6} u_r R d\theta = - \int_0^{\pi/6} \frac{2}{R} R d\theta = -\frac{3}{\pi} = -1.05 \text{ ft}^2/\text{s}$$

Note that the radius R is arbitrary since the flow rate crossing any curve between the two walls must be the same. The negative sign indicates that the flow is toward the opening, that is, in the negative radial direction.

1.2 Free Vortex and Forced Vortex

The irrotational vortex is usually called a *free vortex*. The swirling motion of the water as it drains from a bathtub is similar to that of a free vortex, whereas the motion of a liquid contained in a tank that is rotated about its axis with angular velocity ω corresponds to a forced vortex. A *combined vortex* is one with a forced vortex as a central core and a velocity distribution corresponding to that of a free vortex outside the core.

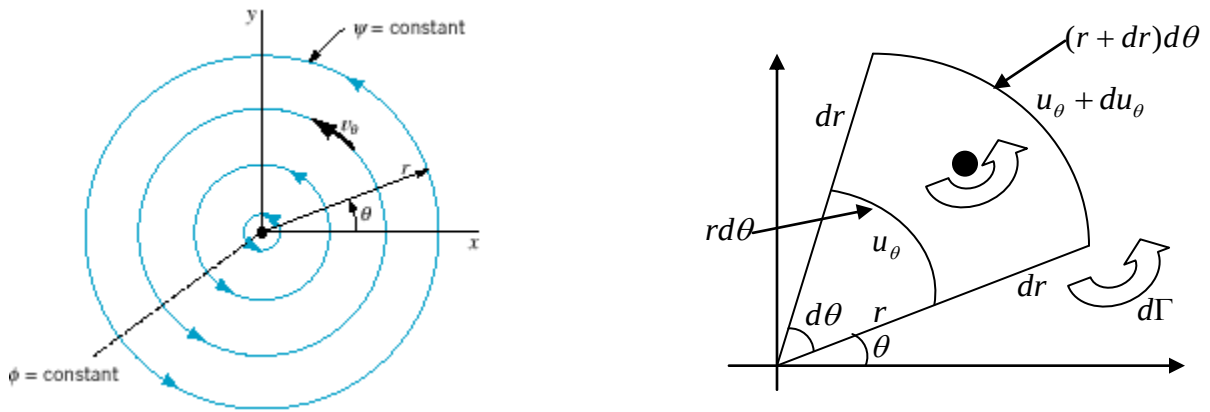


Fig. 1.4 A vortex represents a flow in which the streamlines are concentric circles.

$$d\Gamma = (u_\theta + du_\theta)(r + dr)d\theta - 0 - u_\theta(rd\theta) + 0$$

du_θ and $d\theta$ is a small components so it can be neglect.

A mathematical concept commonly associated with vortex motion is that of circulation. The circulation, Γ is defined as the line integral of the tangential component of the velocity taken around a closed curve in the flow field. In equation form, Γ can be expressed as

$$\begin{aligned} \Gamma &= \oint_C u_\theta \cdot dl \\ \therefore \Gamma &= 2\pi \cdot u_\theta \\ u_\theta &= -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi} \\ \psi &= \int \left(-\frac{\Gamma}{2\pi} \right) dr + C = -\frac{\Gamma}{2\pi} \ln r + C \end{aligned}$$

so ψ for free vortex is shows below

$$\therefore \quad \psi_{f.vortex} = -\frac{\Gamma}{2\pi} \ln r \quad (\text{ccw})$$

$$\psi_{f.vortex} = \frac{\Gamma}{2\pi} \ln r \quad (\text{cw})$$

$$\phi_{f.vortex} = \frac{\Gamma}{2\pi} \cdot \theta \quad (\text{ccw})$$

$$\phi_{f.vortex} = -\frac{\Gamma}{2\pi} \cdot \theta \quad (\text{cw})$$

and

ψ for force vortex is shows below

$$\psi_{\text{forced vortex}} = \frac{\omega r^2}{2}$$

E XAMPLE 1.2

A liquid drains from a large tank through a small opening as illustrated in Fig.E1.2. A vortex forms whose velocity distribution away from the tank opening can be approximated as that of a free vortex having a velocity potential

$$\phi = \frac{\Gamma}{2\pi} \cdot \theta$$

Determine an expression relating the surface shape to the strength of the vortex as specified by the circulation Γ .

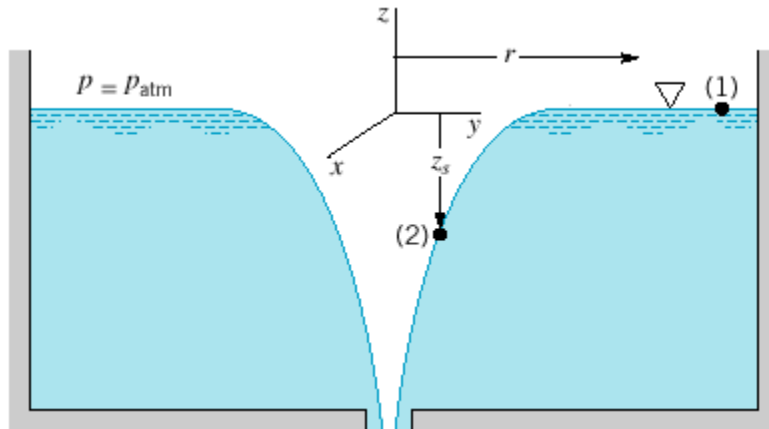


Fig E1.2

Solution

Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

can be written between any two points. If the points are selected at the free surface,

$p_1 + p_2 = 0$ so that

$$\frac{V_1^2}{2g} = Z_s + \frac{V_2^2}{2g} \quad (1)$$

The velocity is given by the equation

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

We note that far from the origin at point (1), $V_1 = u_\theta \approx 0$ so that Eq.1 becomes

$$Z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$

1.3 Doublet

The final, basic potential flow to be considered is one that is formed by combining a source and sink in a special way. Consider the equal strength, source-sink pair of Fig. 1.5.

The combined stream function for the pair is

$$\psi_{\text{Doublet}} = \psi_{\text{source}} + \psi_{\text{sink}}$$

$$\psi_{\text{Doublet}} = \frac{Q\theta_1}{2\pi} - \frac{Q\theta_2}{2\pi}$$

$$\psi_{\text{Doublet}} = \frac{Q}{2\pi} (\theta_1 - \theta_2)$$

$$\alpha = \theta_1 - \theta_2$$

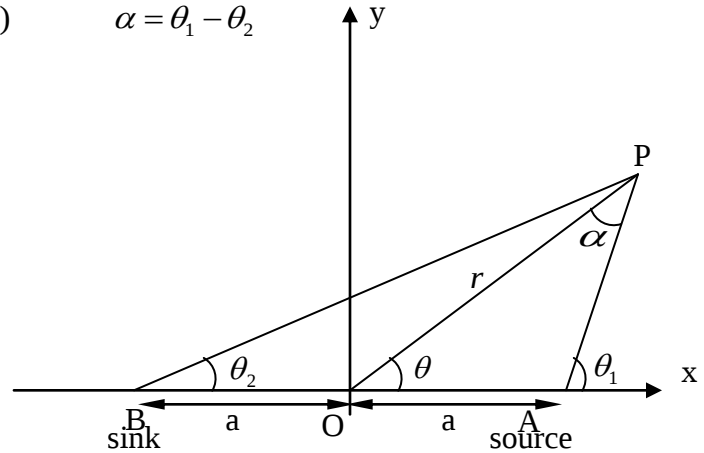
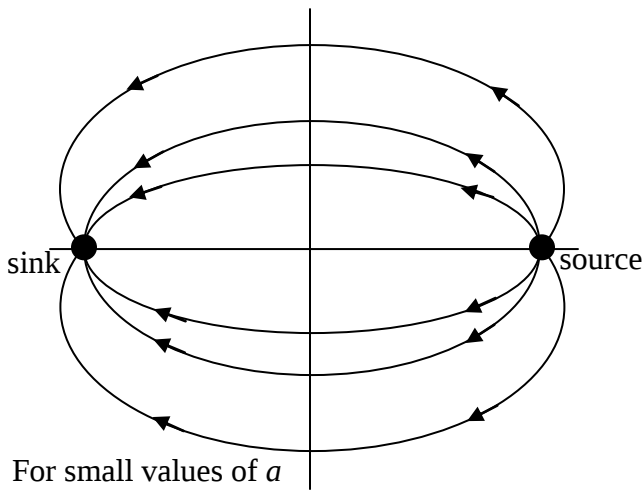


Fig 1.5 The combination of a source and sink of equal strength located along the x

$$r\alpha = OA \sin \theta_1 + OB \sin \theta_2$$

$$\therefore \alpha = \frac{\sin \theta(OA + OB)}{r}$$

$$\psi_{Doublet} = \frac{Q}{2\pi} \frac{AB \sin \theta}{r}$$

since the tangent of an angle approaches the value of the angle for small angles. The so-called *doublet* is formed by letting the source and sink approach one another $a \rightarrow 0$ while increasing the strength so that the product $AB \times Q$ remains constant. $AB \times Q$ is called the *strength* of the doublet, μ

$$\psi_{Doublet} = \frac{\mu}{2\pi r} \sin \theta$$

$$\phi_{Doublet} = -\frac{\mu}{2\pi r} \cos \theta$$

1.6 Source in a Uniform Stream —Half-Body

Consider the superposition of a source and a uniform flow as shown in Fig. 1.6. The resulting stream function is

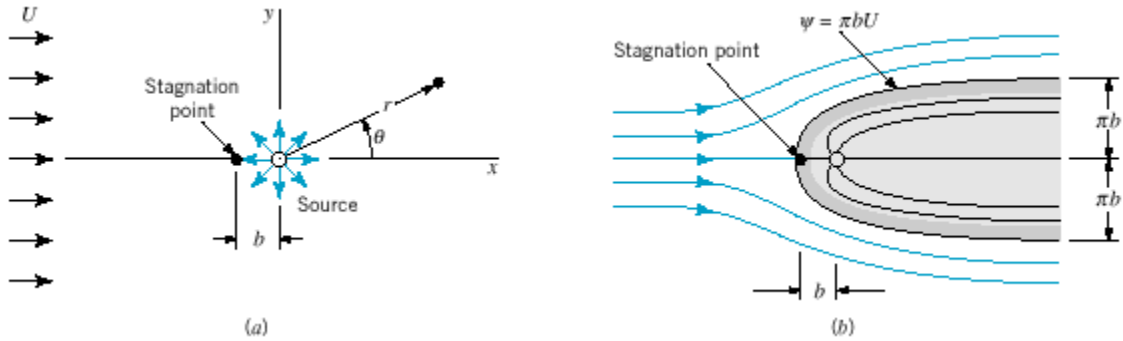


Fig. 1.6 The flow around a half-body: (a) superposition of a source and a uniform flow; (b) replacement of streamline $\psi = \pi b U$ with solid boundary to form half-body.

$$\psi = \psi_{u.f} + \psi_{source}$$

$$\psi = Ur \sin \theta + \frac{Q\theta}{2\pi}$$

and the corresponding velocity potential is

$$\phi = Ur \cos \theta + \frac{Q}{2\pi} \ln r$$

It is clear that at some point along the negative x axis the velocity due to the source will just cancel that due to the uniform flow and a stagnation point will be created. For the source alone

$$u_r = \frac{Q}{2\pi r}$$

so that the stagnation point will occur at $x = -b$ where

$$U = \frac{Q}{2\pi b}$$

or

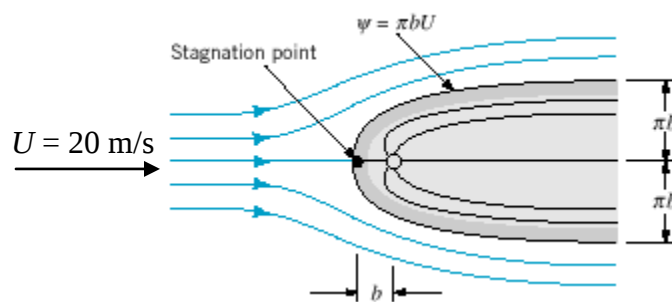
$$b = \frac{Q}{2\pi U}$$

The value of the stream function at the stagnation point can be obtained by valuating ψ at $r = b$ and $\theta = 0$, which yields from Eq.6.97

$$\psi_{\text{stagnation}} = \frac{Q}{2}$$

E XAMPLE 1.3

A certain body has the shape of a half-body with a thickness of 0.5 m. If this body is to be placed in an airstream moving at 20 m/s, what source strength is required to simulate flow around the body?



Solution

The width of half-body = $2\pi b$

So that

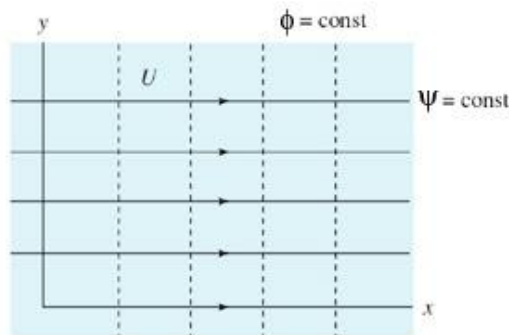
$$b = \frac{(0.5m)}{2\pi}$$

from Eq. 6.99, the distance between the source and the nose of the body is

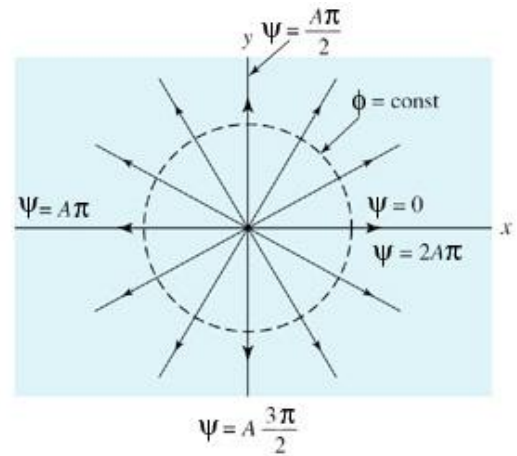
$$b = \frac{Q}{2\pi U}$$

where Q is the source strength, and therefore

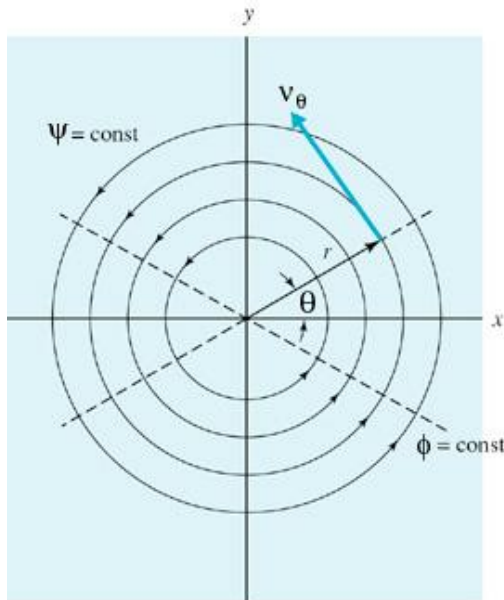
$$Q = 2\pi U b = 2\pi(20m/s)\left(\frac{0.5m}{2\pi}\right) = 10.0 \text{ m}^2/\text{s}.$$



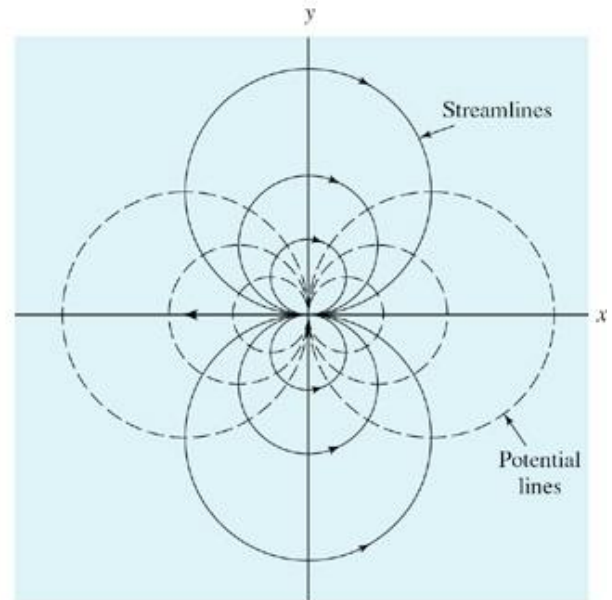
(a) Uniform flow in x -direction



(b) Line source



(c) Irrotational vortex



(d) Doublet

Fig. 1.6 Four simple potential flows.

1.7 Doublet in Uniform Flow (Flow Around a Circular Cylinder)

As was noted in the previous section, when the distance between the source-sink pair approaches zero, the shape of the Rankine oval becomes more blunt and in fact approaches a circular shape. Since the doublet described in Section 1.3 was developed by letting a source-sink pair approach one another, it might be

expected that a uniform flow in the positive x direction combined with a doublet could be used to represent flow around a circular cylinder. This combination gives for the stream function.

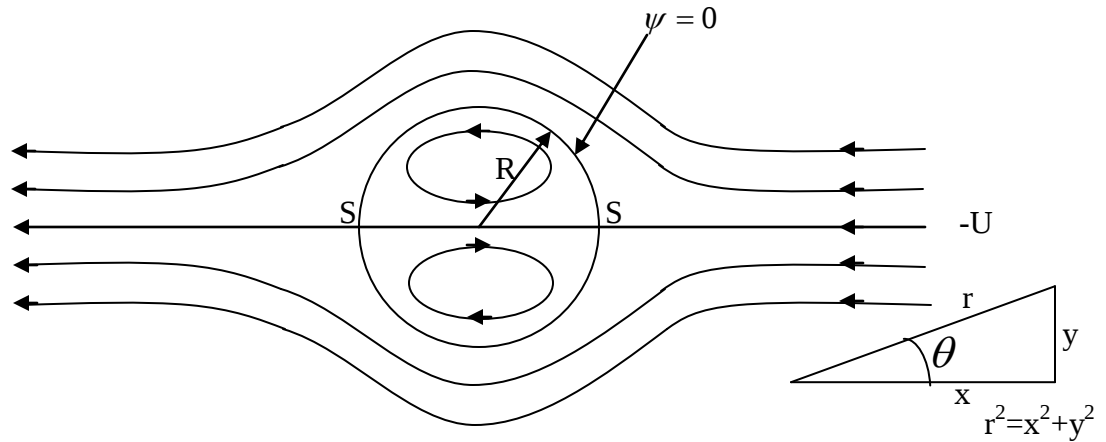


Fig. 1.7 A doublet combined with a uniform flow can be used to represent flow around a circular cylinder.

Stream Function

$$\psi_c = \psi_{u.f} + \psi_{doublet} = -Ur \sin \theta + \frac{\mu}{2\pi r} \sin \theta$$

it follows that $\psi = 0$

$$0 = -Ur \sin \theta + \frac{\mu}{2\pi r} \sin \theta$$

$$\sin \theta \left(\frac{\mu}{2\pi r} - Ur \right) = 0$$

$$\theta = 0^\circ \text{ or } \theta = 180^\circ \quad \text{so; } \sin \theta = 0$$

this is located on axis – x

$$\sin \theta \left(\frac{\mu}{2\pi r} - Ur \right) = 0$$

$$\therefore \quad r^2 = \frac{\mu}{2\pi U}$$

$$r = \sqrt{\frac{\mu}{2\pi U}} \quad \text{for all values of } \theta$$

if $\psi = 0$, $r = R$

$$\mu = 2\pi UR^2$$

$$\psi = -Ur \sin \theta + \frac{2\pi UR^2}{2\pi r} \sin \theta$$

$$\psi = -Ur \sin \theta + \frac{UR^2}{r} \sin \theta$$

$\therefore$

$$\psi = U \left(\frac{R^2}{r} - r \right) \sin \theta$$

Velocity components

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = U \left(\frac{R^2}{r^2} - 1 \right) \cos \theta$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -U \left(\frac{R^2}{r^2} + 1 \right) \sin \theta$$

$$\psi = 0; r = R; u_r = 0; u_\theta = 2U \sin \theta$$

$$\therefore \text{Resultant velocity } q = \sqrt{(u_r)^2 + (u_\theta)^2} = u_\theta = 2U \sin \theta$$

$$\theta = 0^\circ, 180^\circ$$

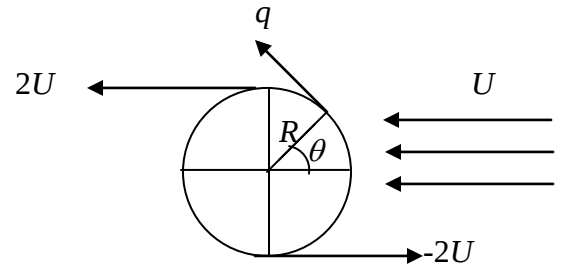
$$q = 0$$

$$\theta = 90^\circ$$

$$q = 2U$$

$$\theta = 270^\circ$$

$$q = -2U$$



We observe from this result that the maximum velocity occurs at the top and bottom of the Cylinder and has a magnitude of twice the upstream velocity, U to find pressure coefficient by using Bernoulli Equation.

$$p_1 + \frac{\rho U^2}{2} = p_2 + \frac{\rho q^2}{2}$$

$$p_1 + \frac{\rho U^2}{2} = p_2 + \frac{1}{2} \rho 4U^2 \sin^2 \theta$$

$$- \frac{\rho 4U^2 \sin^2 \theta}{2} + \frac{\rho U^2}{2} = p_2 - p_1$$

$$\frac{\rho U^2}{2} (1 - 4 \sin^2 \theta) = p_2 - p_1$$

$$(1 - 4 \sin^2 \theta) = \frac{p_2 - p_1}{\frac{1}{2} \rho U^2} = C_p$$

E XAMPLE 1.4

When a circular cylinder is placed in a uniform stream, a stagnation point is created on the cylinder as is shown in Fig. E1.4. If a small hole is located at this point, the stagnation pressure, p_{stag} , can be measured and used to determine the approach velocity, U . (a) Show how p_{stag} and U are related. (b) If the cylinder is misaligned by an angle α (Fig.E1.4a) but the measured pressure still interpreted as the stagnation pressure, determine an expression for the ratio of the true velocity, U , to the predicted velocity, U' . Plot this ratio as a function of α for the range $-20^\circ \leq \alpha \leq 20^\circ$.

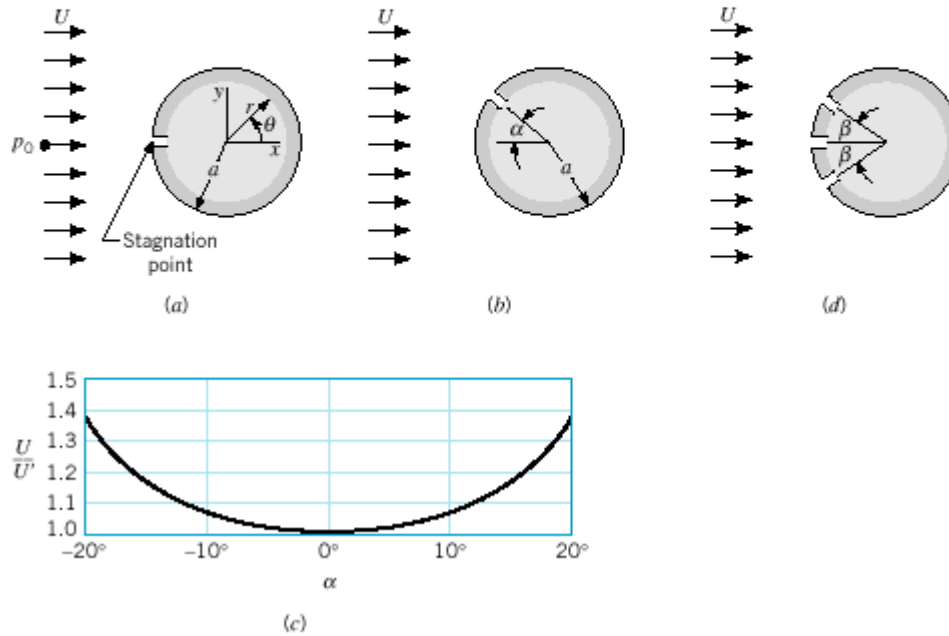


Fig. E1.4

Solution

- (a) The velocity at the stagnation point is zero so the Bernoulli equation written between a point on the stagnation streamline upstream from the cylinder and the stagnation point gives

$$\frac{p_0}{\gamma} + \frac{U^2}{2g} = \frac{p_{\text{stag}}}{\gamma}$$

Thus,

$$U = \left[\frac{2}{\rho} (p_{\text{stag}} - p_0) \right]^{1/2}$$

- (b) If the direction of the fluid approaching the cylinder is not known precisely, it is possible that the cylinder is misaligned by some angle, α . In this instance the pressure actually measured, p_α , will be different from the stagnation pressure, but if the misalignment is not recognized the predicted approach velocity, U' , would still be calculated as

$$U' = \left[\frac{2}{\rho} (p_\alpha - p_0) \right]^{1/2}$$

Thus,

$$\frac{U(\text{true})}{U'(\text{predicted})} = \left(\frac{p_{\text{stag}} - p_0}{p_\alpha - p_0} \right)^{1/2} \quad (1)$$

The velocity on the surface of the cylinder, v_θ , where $r = a$, is obtained from Eq. 6.115 as

$$v_\theta = -2U \sin \theta$$

If we now write the Bernoulli equation between a point upstream of the cylinder and the point on the cylinder where $r = a$, $\theta = \alpha$, it follows that

$$p_0 + \frac{1}{2} \rho U^2 = p_\alpha + \frac{1}{2} \rho (-2U \sin \alpha)^2$$

and, therefore,

$$p_\alpha - p_0 = \frac{1}{2} \rho U^2 (1 - 4 \sin^2 \alpha) \quad (2)$$

Since $p_{\text{stag}} - p_0 = \frac{1}{2} \rho U^2$ it follows from Eqs. 1 and 2 that

$$\frac{U(\text{true})}{U'(\text{predicted})} = (1 - 4 \sin^2 \alpha)^{-1/2}$$

1.8 Flow around a rotating cylinder

This type of flow field could be approximately created by placing a rotating cylinder in a uniform stream. Because of the presence of viscosity in any real fluid, the fluid in contact with the rotating cylinder would rotate with the same velocity as the cylinder, and the resulting flow field would resemble that developed by the combination of a uniform flow past a cylinder and a free vortex.

$$\psi_c = \psi_{\text{Doublet}} + \psi_{u.f} + \psi_{f.vortex}$$

$$\psi_c = -U \sin \theta \left(r - \frac{R^2}{r} \right) - \frac{\Gamma}{2\pi} \ln r$$

velocity components

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = -U \left(1 - \frac{R^2}{r^2} \right) \cos \theta$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -U \left(\frac{R^2}{r^2} + 1 \right) \sin \theta + \frac{\Gamma}{2\pi r}$$

on cylinder surface, $\psi = 0$, $r = R$;

$$u_r = 0; u_\theta = 2U \sin \theta + \frac{\Gamma}{2\pi R}$$

at Stagnation point, $u_r = 0$, $u_\theta = 0 \therefore q = 0$

$$\therefore \boxed{\sin \theta = -\frac{\Gamma}{4\pi RU}}$$

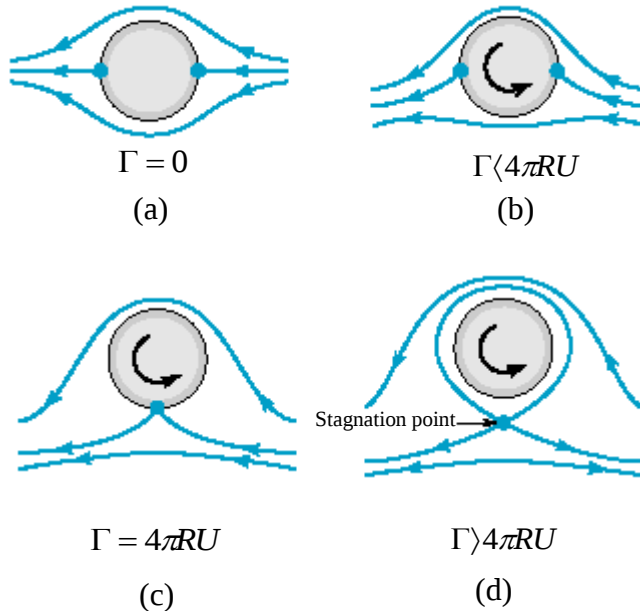


Fig. 1.8 The location of stagnation points on a circular cylinder: (a) without circulation; (b), (c) and (d) with circulation.

Thus, for the cylinder with circulation, lift is developed equal to the product of the fluid density, the upstream velocity, and the circulation. The negative sign means that if U is positive (in the positive x direction) and Γ is positive (a free vortex with counterclockwise rotation), the direction of the F_y is downward. Of course, if the

cylinder is rotated in the clockwise direction ($\Gamma < 0$) the direction of F_y would be upward. It is this force acting in a direction perpendicular to the direction of the approach velocity that causes baseballs and golf balls to curve when they spin as they are propelled through the air. The development of lift on rotating bodies is called the **Magnus effect**.

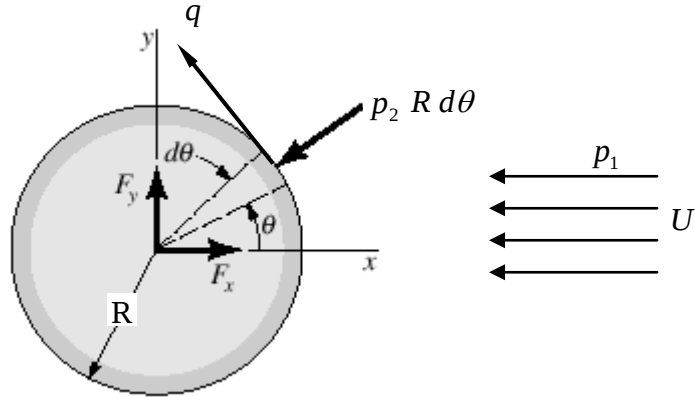


Fig. 1.9 The notation for determining lift and drag on a circular cylinder.

$$p_1 + \frac{\rho U^2}{2} = p_2 + \frac{\rho q^2}{2}$$

on cylinder surface

$$r = R, u_r = 0, u_\theta = 2U \sin \theta + \frac{\Gamma}{2\pi R}$$

$$\therefore q = u_\theta = 2U \sin \theta + \frac{\Gamma}{2\pi R}$$

$$p_2 - p_1 = \frac{1}{2} \rho U^2 - 2\rho U^2 \sin^2 \theta - \frac{\rho U \Gamma}{\pi R} \sin \theta - \frac{\rho \Gamma^2}{8\pi^2 R^2}$$

Drag Force, F_D

$$F_x = F_D = \int_0^{2\pi} p_2 R \cos \theta d\theta = [p_2 R \sin \theta]_0^{2\pi} = 0$$

Lift Force, F_L

$$F_y = F_L = - \int_0^{2\pi} p_2 R \sin \theta d\theta$$

$$F_y = F_L = -R \int_0^{2\pi} \left(p_1 \sin \theta + \frac{1}{2} \rho U^2 \sin \theta - 2 \rho U^2 \sin^2 \theta - \frac{\rho U \Gamma}{\pi R} \sin^2 \theta - \frac{\rho \Gamma^2 \sin \theta}{8 \pi^2 R^2} \right) d\theta$$

$$F_y = F_L = -R \left\{ 0 + 0 - 0 - \frac{\rho U \Gamma}{\pi R} \left[\frac{\theta}{2} - \frac{\sin^2 \theta}{4} \right]_0^{2\pi} - 0 \right\}$$

$F_L = \rho U \Gamma$	Nm
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The generalized equation relating lift to the fluid density, velocity, and circulation is called the *Kutta–Joukowski Theorem*.