

2.0 BOUNDARY LAYER THEORY

2.1 Introduction

Introduced by Prandtl in 1904. In 1908, **H. Blasius** (1883–1970), one of Prandtl's students, was able to solve these simplified equations for the boundary layer flow past a flat plate parallel to the flow.

The boundary layer is needed to allow for the no-slip boundary condition that requires the fluid to cling to any solid surface that it flows past. Outside of the boundary layer the velocity gradients normal to the flow are relatively small, and the fluid acts as if it were inviscid, even though the viscosity is not zero. A necessary condition for this structure of the flow is that the Reynolds number be large. Large Reynolds number flow fields may be divided into viscous and inviscid regions.

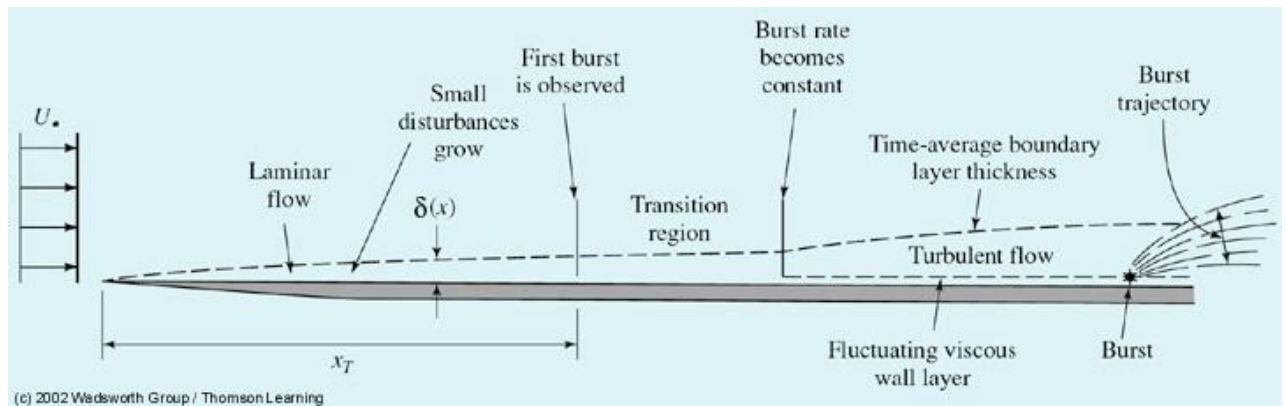


Fig. 2.1 Boundary layer with transition.

In this section we consider the simplest situation, one in which the boundary layer is formed on an infinitely long flat plate along which flows a viscous, incompressible fluid as is shown in Fig. 2.2. If the surface were curved (i.e., a circular cylinder or an airfoil), the boundary layer structure would be more complex. If the Reynolds number is sufficiently large, only the fluid in a relatively thin boundary layer on the plate will feel the effect of the plate. That is, except in the region next to the plate the flow velocity will

be essentially $V = U\hat{i}$ the upstream velocity. For the infinitely long flat plate extending from $x = 0$ to $x = \infty$, it is not obvious how to define the Reynolds number because there is no characteristic length. The plate has no thickness and is not of finite length!

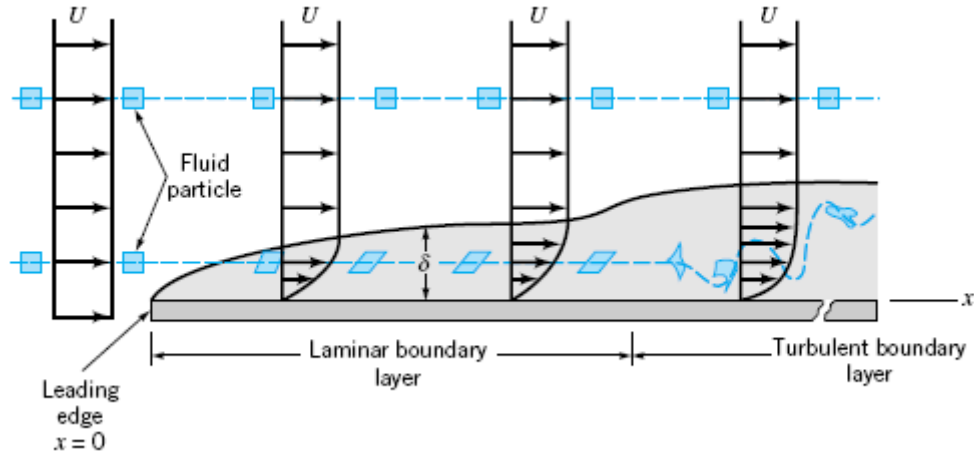


Fig. 2.2 Distortion of a fluid particle as it flows within the boundary layer.

2.2 Thickness of Boundary Layer.

The purpose of the boundary layer on the plate is to allow the fluid to change its velocity from the upstream value of U to zero on the plate. Thus, $V = 0$ at $y = 0$ and $V \approx U\hat{i}$ at $y = \delta$ with the velocity profile, $u = u(x, y)$ bridging the boundary layer thickness. In actuality (both mathematically and physically), there is no sharp “edge” to the boundary layer. That is, $u \rightarrow U$ as we get farther from the plate; it is not precisely $u = U$ at $y = \delta$. We define the *boundary layer thickness*, δ as that distance from the plate at which the fluid velocity is within some arbitrary value of the upstream velocity. Typically, as indicated in Fig. 2.3,

$$\delta = y \text{ where } u = 0.99 U$$

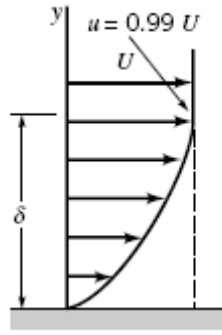


Fig. 2.3 Boundary layer thickness

2.3 Boundary Layer Displacement Thickness, δ^*

The displacement thickness represents the amount that the thickness of the body must be increased so that the fictitious uniform inviscid flow has the same mass flow rate properties as the actual viscous flow. It represents the outward displacement of the streamlines caused by the viscous effects on the plate. This idea allows us to simulate the presence that the boundary layer has on the flow outside of the boundary layer by adding the displacement thickness to the actual wall and treating the flow over the thickened body as an inviscid flow. The displacement thickness concept is illustrated in Fig. 2.4

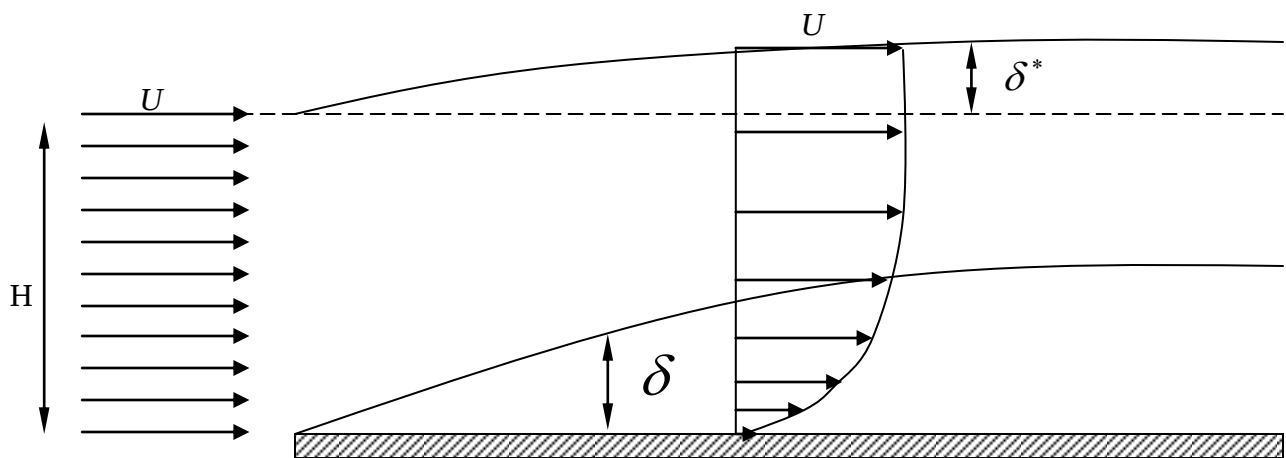


Fig. 2.4 Boundary layer displacement thickness

$$U\delta^* = \int_0^{\infty} (U - u) dy$$

$$\delta^* = \int_0^{\infty} \left[\frac{U-u}{U} \right] dy$$

$$\delta^* = \int_0^{\infty} \left(1 - \frac{u}{U} \right) dy$$

2.4 Boundary Layer Momentum Thickness, θ

Boundary Layer Momentum Thickness, θ is often used when determining the drag on an object. The velocity deficit, $U - u$, in the boundary layer, the momentum flux across section $b-b$ in Fig. 2.5 is less than that across section $a-a$. This deficit in momentum flux for the actual boundary layer flow is given by

$$\int \rho u(U-u)dA = \rho b \int u(U-u)dy$$

which by definition is the momentum flux in a layer of uniform speed U and thickness θ
That is,

$$\rho b U^2 \theta = \rho b \int_0^{\infty} u(U-u)dy$$

or

$$\theta = \int_0^{\infty} \frac{u}{U} \left(1 - \frac{u}{U} \right) dy$$

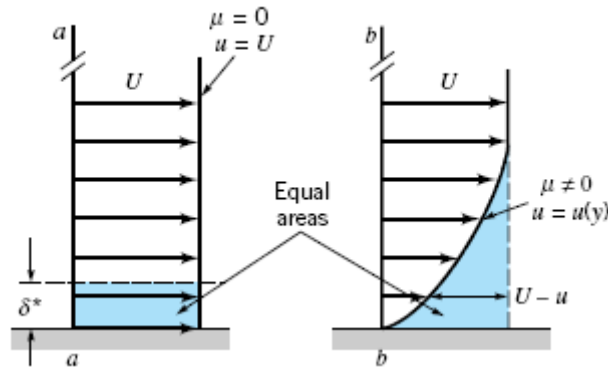


Fig. 2.5 Boundary layer displacement thickness

Example 1

If the velocity of a flow profile is

$$\frac{u}{U} = \left(\frac{y}{\delta} \right)^{1/2}$$

find

$$\frac{\delta^*}{\delta} \text{ and } \frac{\theta}{\delta}$$

Solution

$$\delta^* = \int_0^{\delta} \left(1 - \frac{u}{U} \right) dy$$

$$\begin{aligned} \delta^* &= \int_0^{\delta} \left(1 - \left(\frac{y}{\delta} \right)^{1/2} \right) dy = \left[y - \frac{2}{3} \frac{y^{3/2}}{\delta^{1/2}} \right]_0^{\delta} = \delta - \frac{2}{3} \delta^{(3/2-1/2)} = \delta - \frac{2}{3} \delta = \delta \left(\frac{1}{3} \right) \\ \frac{\delta^*}{\delta} &= \frac{1}{3} \end{aligned}$$

$$\theta = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U} \right) dy$$

$$\begin{aligned} \theta &= \int_0^{\delta} \frac{y^{1/2}}{\delta^{1/2}} \left(1 - \frac{y^{1/2}}{\delta^{1/2}} \right) dy = \int_0^{\delta} \left(\frac{y^{1/2}}{\delta^{1/2}} - \frac{y^{(1/2+1/2)}}{\delta^{(1/2+1/2)}} \right) dy = \int_0^{\delta} \left(\frac{y^{1/2}}{\delta^{1/2}} - \frac{y}{\delta} \right) dy = \left[\frac{2}{3} \frac{y^{3/2}}{\delta^{1/2}} - \frac{1}{2} \frac{y^2}{\delta} \right]_0^{\delta} \\ \theta &= \frac{2}{3} \delta^{(3/2-1/2)} - \frac{1}{2} \delta^{(2-1)} \\ \theta &= \frac{\delta}{6} \\ \frac{\theta}{\delta} &= 1/6 \end{aligned}$$

2.5 Integral Momentum Equation

One of the important aspects of boundary layer theory is the determination of the drag caused by shear forces on a body.

It was introduced by **T. von Karman** (1881–1963), a Hungarian/German aerodynamicist.

Note that this equation is valid for laminar or turbulent flows.

We consider the uniform flow past a flat plate and the fixed control volume as shown in Fig. 2.6. One element of flow volume with length dx , width in one unit and high $\delta(x)$.

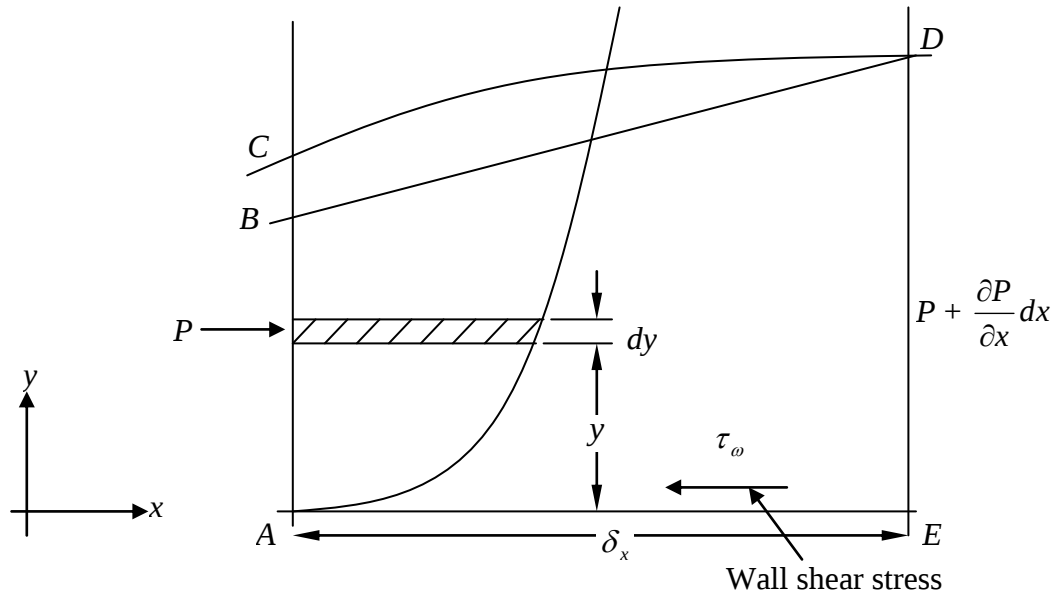


Fig. 2.6 Uniform flow past a flat plate and the fixed control volume

Newton's Second Law:

time rate of change of the linear momentum of the system = sum of external forces acting on the system

If we apply the x component of the momentum to the steady flow of fluid within this control volume we obtain

$$P(AC \times 1) + \left[P + \frac{\partial P}{\partial x} dx \right] [ED - AC] - \left[P + \frac{\partial P}{\partial x} \right] ED \times 1 - \tau_w \delta_x \times 1 = K \frac{d}{dt} (\text{momentum})$$

$$P(AC) - P(ED) - \frac{\partial P}{\partial x} \delta x (ED) - \tau_w \delta x + P(ED) - P(AC) + \frac{1}{2} \frac{\partial P}{\partial x} \delta x ED - \frac{1}{2} \frac{\partial P}{\partial x} \delta x (AC) = \frac{d}{dt} (\text{momentum})$$

$$\therefore -\frac{1}{2} \frac{\partial P}{\partial x} \delta x (ED + AC) - \tau_w \delta x = \frac{d}{dt} (\text{momentum})$$

when $\delta x \rightarrow 0$

$$AC \rightarrow ED$$

$$-\frac{\partial P}{\partial x} \delta x (ED) - \tau_w \delta x = \frac{d}{dt} (\text{momentum})$$

we know that

$$\frac{\delta}{\delta t} (\text{momentum}) = [\rho \times 1 \times u \delta y] u = \rho u^2 \delta y$$

$$\text{inlet momentum} = \int_0^{\delta} \rho u^2 dy + \rho U^2 (BC)$$

$$\text{outlet momentum} = \int_0^{\delta} \rho u^2 dy + \left[\int_0^{\delta} \rho u^2 dy \right] \delta x$$

rate of change of the momentum = outlet momentum – inlet momentum

$$-\frac{\partial P}{\partial x} \delta x (ED) - \tau_w \delta x = \frac{\partial}{\partial x} \left[\int_0^{\delta} \rho u^2 dy \right] \delta x - \rho U^2 (BC) \quad (1)$$

By conservation of mass

mass flow rate BC = mass flow rate ED = mass flow rate AB

$$\begin{aligned} \rho U BC &= \int_0^{\delta} \rho u dy + \frac{\partial}{\partial x} \left[\int_0^{\delta} \rho u dy \right] \delta x - \int_0^{\delta} \rho u dy \cdot 1 \\ BC &= \frac{1}{\rho U} \left[\frac{\partial}{\partial x} \left\{ \int_0^{\delta} \rho u dy \right\} \delta x \right] \end{aligned} \quad (2)$$

Equation (1) and (2) give

$$\begin{aligned} -\frac{\partial P}{\partial x} \delta x (ED) - \tau_w \delta x &= \frac{\partial}{\partial x} \left[\int_0^{\delta} \rho u^2 dy \right] \delta x - \rho U^2 \cdot \frac{1}{\rho U} \left[\frac{\partial}{\partial x} \left\{ \int_0^{\delta} \rho u dy \right\} \delta x \right] \\ -\frac{\partial P}{\partial x} \delta x (ED) - \tau_w \delta x &= U \frac{\partial}{\partial x} \left[\int_0^{\delta} \rho u dy \right] \delta x - \frac{\partial}{\partial x} \left[\int_0^{\delta} \rho u^2 dy \right] \delta x \end{aligned} \quad (i)$$

we know that

$$P + \frac{1}{2} \rho U^2 = \text{constant}$$

so

$$\frac{dP}{dx} + \rho U \frac{dU}{dx} = 0$$

$$\frac{dP}{dx} = -\rho U \frac{dU}{dx} \quad (ii)$$

and

$$ED = \int_0^{\delta} dy = \delta \quad (iii)$$

Take equation (ii) and (iii) into equation (i)

$$\begin{aligned} \tau_w - \rho U \frac{dU}{dx} \int_0^{\delta} dy &= \rho U \frac{\partial}{\partial x} \int_0^{\delta} u dy - \rho \frac{\partial}{\partial x} \int_0^{\delta} u^2 dy \\ U \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} (U \cdot u) - u \frac{\partial U}{\partial x} \end{aligned}$$

$$\begin{aligned}
\therefore \tau_w &= \rho \frac{\partial}{\partial x} \int_0^\delta U \cdot u dy - \rho \frac{\partial U}{\partial x} \int_0^\delta u dy - \rho \frac{\partial}{\partial x} \int_0^\delta U \cdot u dy + \rho U \frac{\partial U}{\partial x} \int_0^\delta dy \\
\therefore \tau_w &= \rho \frac{\partial}{\partial x} \int_0^\delta (U \cdot u) dy + \rho \frac{\partial U}{\partial x} \int_0^\delta (U \cdot u) dy \\
\tau_w &= \rho \frac{\partial}{\partial x} \int_0^\delta (U^2 \theta) dy + \rho \frac{\partial U}{\partial x} U \delta^*
\end{aligned}$$

because

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy \quad \text{and also} \quad \theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

finally

$$\frac{\tau_w}{\rho U^2} = \frac{d\theta}{dx} + \frac{1}{U} (\delta^* 2\theta) \frac{dU}{dx} \quad (3)$$

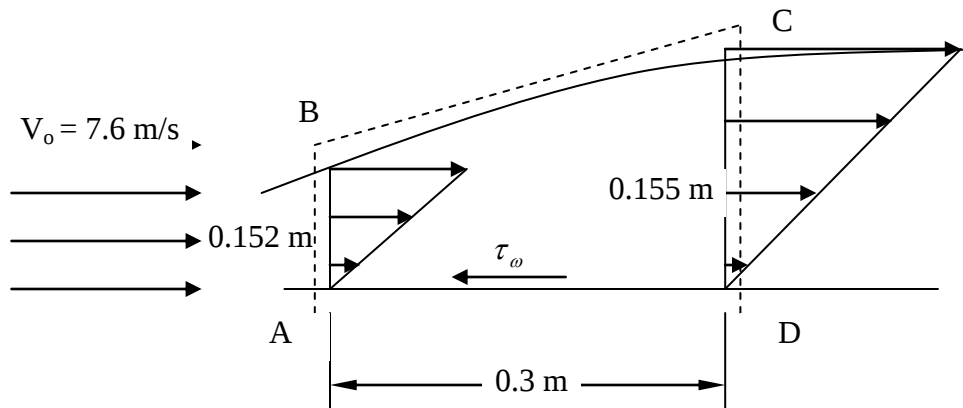
equation (3) is **Integral Momentum Equation (I.M.E) Von Karman**

for uniform flow, $\frac{dU}{dx} = 0$

and for that IME Von Karman can be simplify

$$\boxed{\frac{\tau_w}{\rho U^2} = \frac{d\theta}{dx}}$$

Example 2



Find the value of the wall shear stress, τ_o on the flat plate. Given, $\rho = 999.1 \text{ kg/m}^3$ at 15°C .

Solution

$$\rho = 999.1 \text{ kg/m}^3$$

$$V_o = 7.6 \text{ m/s}$$

$$dx = 0.3 \text{ m}$$

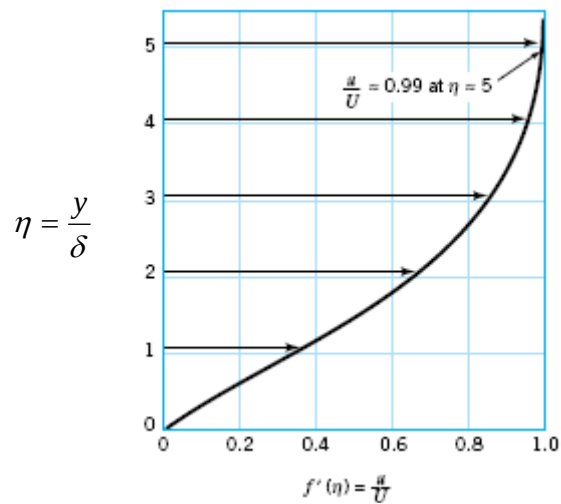
$$d\delta = 0.155 - 0.152 = 0.003 \text{ m}$$

$$\tau_w = ??$$

By using Von Karman Equation

$$\tau_w = \rho U^2 \frac{d\theta}{dx}$$

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d}{dx} \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$



Blasius boundary layer profile boundary layer profile in dimensionless form using the similarity variable, η

$$\begin{aligned} u &= 0 & y &= 0 \\ u &= U & y &\geq \delta \end{aligned}$$

$$\frac{u}{U} = \frac{y}{\delta} = \eta \quad \Rightarrow \quad \eta = \frac{y}{\delta}$$

$$d\eta = \frac{dy}{\delta} \Rightarrow dy = \delta d\eta$$

$$\tau_o = \rho U \frac{d\delta}{dx} \int_0^1 \eta(1-\eta) d\eta$$

$$\tau_o = \rho U^2 \frac{d\delta}{dx} \int_0^1 (\eta - \eta^2) d\eta$$

$$\tau_o = \rho U^2 \frac{d\delta}{dx} \left[\frac{\eta^3}{2} - \frac{\eta^3}{3} \right]_0^1$$

$$\tau_o = \rho U^2 \frac{d\delta}{dx} \left[\frac{1}{2} - \frac{1}{3} \right]$$

$$\tau_o = (999.1)(7.6)^2 \frac{0.003}{0.3} \left[\frac{1}{2} - \frac{1}{3} \right]$$

$$\tau_o = 96.18 \text{ N/m}^2$$

2.6 Analysis of Lamina Boundary Layer flows along a flat plate with zero pressure gradient.

Consider lamina boundary layer flows along a flat plate with uniform flow is zero and pressure gradient is also zero.

$$\frac{dU}{dx} = 0 \text{ (uniform flow)} \quad \text{and} \quad \frac{dp}{dx} = 0 \text{ (pressure gradient)}$$

Prandtl assume that velocity profile is

$$\frac{u}{U} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad \text{for } 0 \leq y \leq \delta$$

$$\frac{u}{U} = 1 \quad \text{for } y \geq \delta$$

with

$$\eta = \frac{y}{\delta}$$

using I.M.E Von Karman Equations

$$\frac{\tau_o}{\rho U^2} = \frac{d\theta}{dx}$$

or

$$\tau_{\omega} = \rho U^2 \frac{d}{dx} \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

$$\tau_{\omega} = \rho U^2 \frac{d\delta}{dx} \left[\int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) d\eta \right]$$

$$\tau_{\omega} = \rho U^2 \frac{d\delta}{dx} \left[\int_0^{\delta} \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3 \right) \left(1 - \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3 \right) \right) d\eta \right]$$

$$\tau_{\omega} = \rho U^2 \frac{d\delta}{dx} \left[\int_0^{\delta} \left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 + \frac{3}{4}\eta^4 - \frac{1}{2}\eta^3 + \frac{3}{4}\eta^4 - \frac{1}{6}\eta^6 \right) d\eta \right]$$

$$\tau_{\omega} = \rho U^2 \frac{d\delta}{dx} \left[\int_0^{\delta} \left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 + \frac{6}{4}\eta^4 - \frac{1}{2}\eta^3 - \frac{1}{4}\eta^6 \right) d\eta \right]$$

$$\tau_{\omega} = \rho U^2 \frac{d\delta}{dx} \left[\int_0^{\delta} \left(-\frac{3}{2}\eta + \frac{6}{4}\eta^4 - \frac{1}{2}\eta^3 - \frac{1}{4}\eta^6 \right) d\eta \right]$$

$$\tau_{\omega} = 0.139 \rho U^2 \frac{d\delta}{dx} \rightarrow (1)$$

2.7 Analysis of Turbulent Boundary Layer flows along a flat.

Consider turbulent boundary layer flows along a flat plate with pressure gradient is zero.

$$\frac{\partial p}{\partial x} = 0$$

For turbulent flow, Prandtl law 1/7 is used.

Prandtl assume that velocity profile is

$$\frac{u}{U} = \left(\frac{y}{\delta} \right)^{1/7} = \eta^{1/7} \quad \text{with} \quad \frac{y}{\delta} = \eta$$

using I.M.E Von Karman Equations

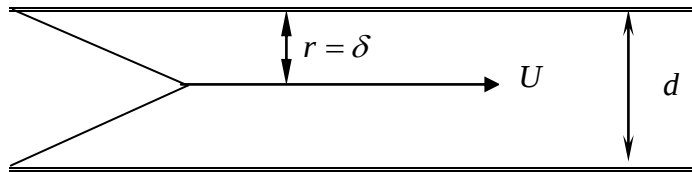
$$\tau_w = \rho U^2 \frac{\partial \theta}{\partial x}$$

$$\tau_w = \rho U^2 \frac{\partial}{\partial x} \left[\int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U} \right) dy \right]$$

know that

$$dy = \delta d\eta$$

Velocity profile is,



Value for experimental in turbulence flow, $\bar{u} = 0.8U$ (3)

Substitute equation (3) into equation (2)

$$\therefore \tau_w = \frac{0.079}{\left(\frac{\rho \cdot 0.8U \cdot d}{\mu} \right)^{1/4}} \cdot \frac{1}{2} \rho (0.8U)^2$$

we know that $r = \delta$, so diameter of pipe is $d = 2\delta$

$$\therefore \tau_w = 0.0225 \rho U^2 \left(\frac{\mu}{\rho U \delta} \right)^{1/4} \dots\dots\dots(4)$$

equation (4) = equation (1)

$$\therefore \tau_w = 0.0225 \rho U^2 \left(\frac{\mu}{\rho U \delta} \right)^{1/4} = \frac{7}{72} \rho U^2 \frac{d\delta}{dx} \text{ ,}$$

$$\delta^{1/4} d\delta = 0.234 \left(\frac{\mu}{\rho U} \right)^{1/4}$$

$$\frac{4}{5} \delta^{5/4} = 0.234 \left(\frac{\mu}{\rho U} \right)^{1/4} x + A$$

boundary condition; $\delta = 0$ at $x = 0$. So $A = 0$.

$$\therefore \delta^{5/4} = \frac{5}{4} (0.234) \left(\frac{\mu}{\rho U} \right)^{1/4} .x$$

$$\delta = 0.378 \left(\frac{\mu}{\rho U} \right)^{1/5} .x^{4/5}$$

$$\delta = 0.378 \left(\frac{\mu}{\rho U} \right)^{1/5} .x$$

$$\delta = \frac{0.378.x}{\text{Re}_x^{1/5}} \quad \text{with} \quad \text{Re}_x = \frac{\rho U .x}{\mu} \quad \dots\dots\dots(5)$$

Drag force can be determined by

$$F_D = \int_0^l \tau_w \delta x \text{.1} \quad \dots\dots\dots(6)$$

for one unit width of element.

From equation (5),

$$\delta = \frac{0.378.x}{\text{Re}_x^{1/5}} = 0.378 x^{4/5} \left(\frac{\mu}{\rho U} \right)^{1/5}$$

differentiate equation (5),

$$\therefore \frac{d\delta}{dx} = 0.378 \times \frac{4}{5} x^{-1/5} \left(\frac{\mu}{\rho U} \right)^{1/5}$$

$$\frac{d\delta}{dx} = \frac{0.3024}{\text{Re}_x^{1/5}} \quad \dots\dots\dots(7)$$

substitute equation (7) into equation (1)

$$\begin{aligned}\tau_w &= \frac{7}{72} \rho U^2 \frac{d\delta}{dx} \\ \tau_w &= \frac{7}{72} \rho U^2 \frac{0.3024}{\text{Re}_x^{1/5}} \\ \tau_w &= 0.0294 \rho U^2 \left(\frac{1}{\text{Re}_x} \right)^{1/5} \dots\dots\dots(8)\end{aligned}$$

From equation (6), drag force,

$$\begin{aligned}F_D &= \int_0^l \tau_w \delta x \cdot 1 \\ F_D &= \int_0^l 0.0294 \rho U^2 \left(\frac{\mu}{\rho U x} \right)^{1/5} dx \\ F_D &= \left[0.0294 \rho U^2 \left(\frac{\mu}{\rho U} \right)^{1/5} \cdot \left(\frac{5}{4} x^{4/5} \right) \right]_0^l \\ F_D &= 0.037 \rho U^2 \left(\frac{\mu}{\rho U} \right)^{1/5} \cdot \left(l \cdot \frac{1}{l^{1/5}} \right) \circ \circ \circ \left(x \cdot \frac{1}{x^{1/5}} \right) \\ F_D &= 0.037 \rho U^2 \left(\frac{\mu}{\rho U \cdot l} \right)^{1/5} \cdot l \\ F_D &= 0.037 \rho U^2 l \cdot \left(\frac{1}{\text{Re}_l} \right)^{1/5}\end{aligned}$$

so Drag force,

$$F_D = 0.037 \rho U^2 \cdot l \cdot \text{Re}_l^{-1/5} \dots\dots\dots(9)$$

with $\text{Re}_l = \frac{\rho U \cdot l}{\mu}$

Drag coefficient, $C_D = \frac{F_D/A}{\frac{1}{2} \rho U^2}$

So

$$C_D = \frac{0.037 \rho U^2 \cdot l \cdot (\text{Re}_l)^{-1/5} / l \times 1}{\frac{1}{2} \rho U^2}$$

$$C_D = 0.074 \operatorname{Re}_l^{-1/5} \dots\dots\dots(10)$$

Equation (10) can only be used for $\operatorname{Re} < 10^7$.

If Re between $10^7 < \operatorname{Re} < 10^9$, the suitable drag coefficient can be used is

$$C_D = 0.455(\log \operatorname{Re}_l)^{-2.58}$$