SOLVING ONE-DIMENSIONAL DAMPED WAVE EQUATION USING FINITE DIFFERENCE METHOD

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ABSTRACT: This study is aimed to solve one-dimensional damped wave equation using explicit Finite Difference Method (FDM). In this work, the damped vibration of a string with fixed ends is considered. The FDM proceeds by replacing the derivatives in the damped wave equations by finite difference approximations. This gives a large algebraic system of equations to be solved, which easily can be solved on a computer. The computational experiments are performed using MATLAB Distributed Computing R2009a software. In the experiments, the effect of the non-negative damping coefficient, time step and the step size are investigated. The computational experiments show compromising results, in such that the comparison between FDM and analytical solutions gives small tolerable errors.

Introduction

Partial differential equations (PDE) are those which contain one or more partial derivatives, usually with respect to two or more independent variables. Moreover, partial differential equation is a many-faceted subject that was created to describe the mechanical behavior of objects such as vibrating strings and blowing winds. In mathematics, partial differential equation has developed into a body of material that interacts with many branches such as differential geometry, complex analysis and harmonic analysis as well as severe factor in the description and explanation of problems in mathematical physics.

In partial differential equation, there exist three types of equation namely elliptic, parabolic and hyperbolic equation. Hyperbolic equations put less stringent constraints on explicit method. Hyperbolic equations arise in wave mechanics, gas dynamics, vibrations and other areas.

The most frequent hyperbolic equation that always been discussed is wave equation. However, in this research, we are interested to study the damped wave equation.

Methodology

3.1 Introduction

In real world, most of the problems in science and engineering are complicated enough that they can only be solved numerically. Real mathematics problems do not have analytical solutions. However, they do have the real answers for each problem. In case of analytical

solution of partial differential equation, numerical values at different intervals of variables are required.

Solving partial differential equation is one of the main concerns of scientists and engineers, so it is important to understand at least the main principles of the approximate solution of partial differential equations. In this study, we are used finite difference method in solving hyperbolic partial differential equations for damped wave equation.

3.2 Damped Wave Equation

Consider a piece of thin flexible string of length L of negligible weight. Suppose the two ends of the string are firmly secured (clamped) at some supports so they will not move. Assume the set-up has damping. Then, the vertical displacement of the string, $0 < x < L$ and at any time $t > 0$, is given by the displacement function $u(x,t)$. It satisfies the one dimensional damped wave equation:

$$u_{tt} = c^2 u_{xx} - a u_t$$

Consider the damped wave equation is given by,

$$u_{tt} = c^2 u_{xx} - a u_t , \qquad 0 < x < L, \qquad t > 0,$$

with boundary conditions: $u(0,t) = 0,$ $u(L,t) = 0,$ $t > 0,$

and initial conditions: $u(x,0) = f(x),$ $u_t(x,0) = g(x),$ $0 < x < L$.

3.3 Finite Difference Method

To solve differential equations numerically we can replace the derivatives in the equation with finite difference approximations on a discretized domain. In this study, we focused on solving damped wave equation by using explicit methods which is can be solved directly from unknown value of previous time step and all future time terms appear on one side of an equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - a \frac{\partial u}{\partial t} \qquad\qquad (3.0)$$

At any interior mesh point (x_j, t_i), the damped wave equation of equation (3.0) can be simplified and written as

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = c^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} - a \left(\frac{\partial u}{\partial t}\right)_{i,j} \tag{3.1}$$
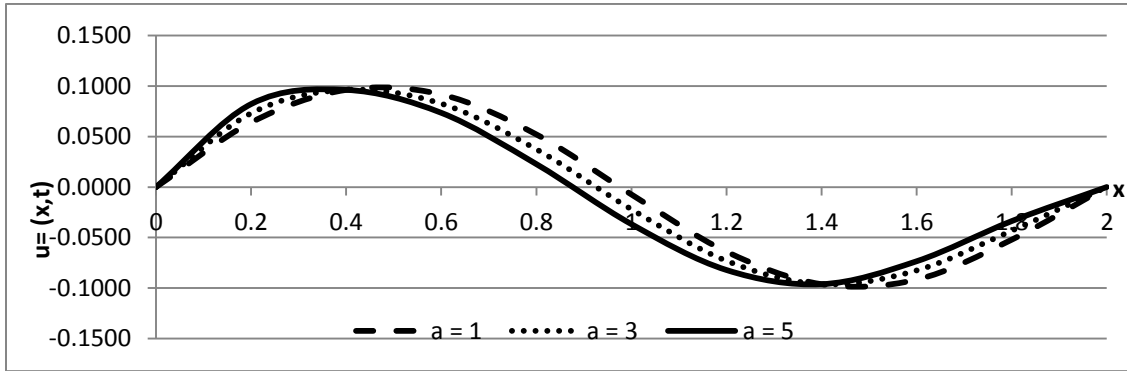
To solve equation (3.1), we will use finite difference approximations which are forward and central formulae. Then, the equation becomes,

$$u_{i,j+1} = \left(\frac{\lambda^2}{1+ak}\right) u_{i-1,j} + \left(\frac{2 - 2\lambda^2 + \alpha k}{1+ak}\right) u_{i,j} + \left(\frac{\lambda^2}{1+ak}\right) u_{i+1,j} \tag{3.3}$$

$$- \left(\frac{1}{1+ak}\right) u_{i,j-1} \quad .$$

Results and Discussion

4.1 The Effect of Different Values of Non-negative Damping Coefficient, a.

The experiment is done for different values of a when $a = 1$, $a = 3$, and $a = 5$.

Figure 4.1: Graph of Different Values of Non-negative Damping Coefficient, a.

**4.2 The Effect of Time Step, k and Step Size, h to the Solution of Finite Difference Method**

In this section, we solved the damped wave equation using finite difference method at time step $T = 0.2$, where the value of k is $k = 0.05$, while at step size $h = 0.2$, $h = 0.1$ and $h = 0.05$. The accuracy will be obtained by comparing the value of finite difference method with the analytical solution.
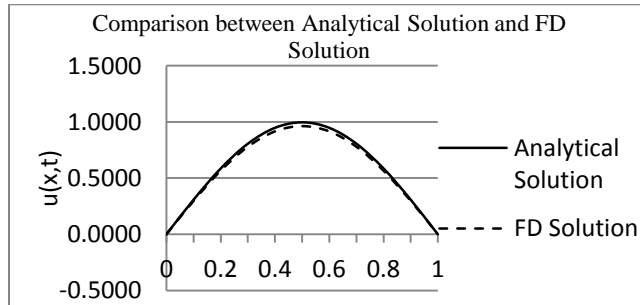
4.3 Effect of Time Step, k.

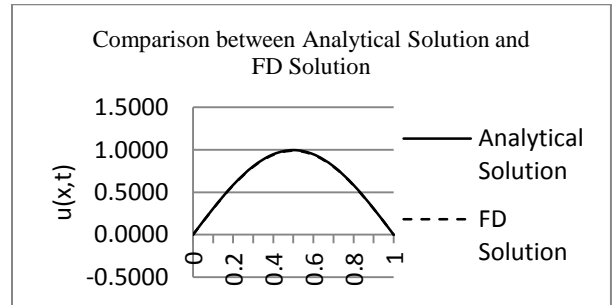We set the maximum time step is $T = 0.2$ where $t = 0(0.05)0.2$ and $x = 0(0.1)1$.
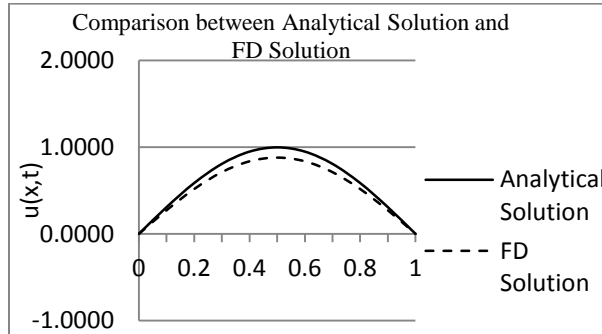
t=0.2
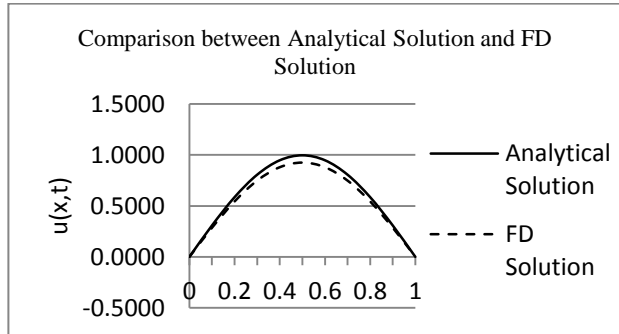
t=0.15

t=0.1

t=0.05

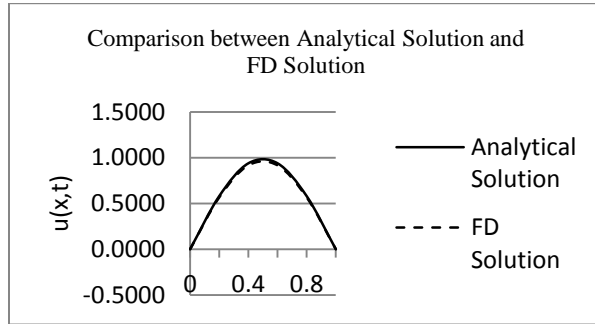### Table 4.1: Mean Square Error (MSE)

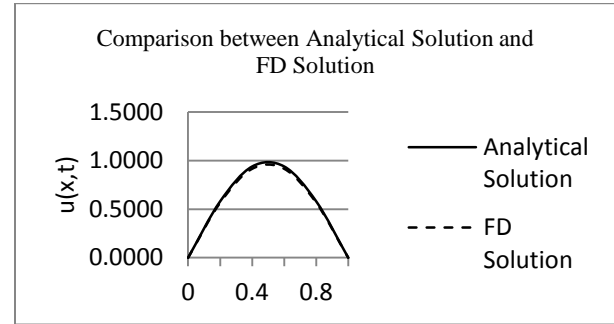| t	Mean Square Error: $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(exact - approximate)^2$
0.20	6.31×10^{-3}
0.10	2.33×10^{-3}
0.15	0.51×10^{-3}
0.05	0.02×10^{-3}

4.4 Effect of Step Size, h.

In this experiment, we have tested three different value of step size, h when $h = 0.2$, $h = 0.1$ and $h = 0.05$ with the fixed value of time step, t. However, in this problem we only focused the point x at x-axis at $0, 0.2, 0.4, 0.6, 0.8$ and 1.0 to prevent from biased results of absolute error,
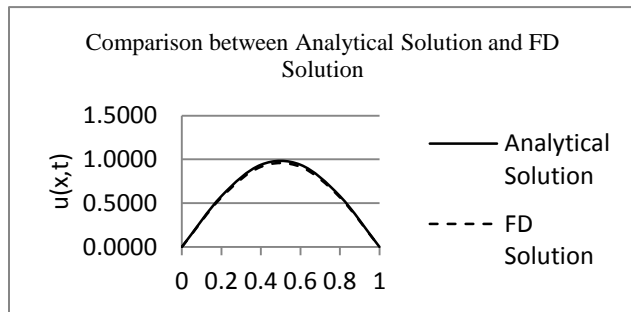
relative error and mean square error. The following setting are considered: $T = k = 0.1$, $c = 1$ and $a = 2\pi$.

$$h = 0.2$$

$$h = 0.1$$

$$h = 0.05$$

Table 4.2: Mean Square Error (MSE)

h	Mean Square Error: $\frac{1}{n}\sum_{i=1}^{n}(exact - approximate)^2$
0.2	0.227×10^{-3}
0.1	0.248×10^{-3}
0.05	0.254×10^{-3}

4.5 Summary

According to the experiment, it can be summarized that the increased value of non-negative damping coefficient, the earlier time of damping took to oscillate. The different values of time step, k and step size, h gives different values of absolute error, relative error and mean square error. When the value of k is decreased, the mean square error also decreased. It concludes that the smaller value of k gives a better approximation to the analytical values. However, the

decreasing value of step size, h does not affect the accuracy of the finite difference method. As a summary, it can be concluded that finite difference method is a reliable method which is capable to solve the one-dimensional damped wave equation. Based on the findings, it can be clearly seen that finite difference method is well approximated and gives better results with smaller time step, k.

Conclusion and Further Works

As the results from this study, the damping took the earlier time to oscillate as the number of the non-negative damping coefficient is increased. Besides that, it can be identified that different values of time step, k and different values of step size, h gives different results of absolute error, relative error and mean square error. As the value of k decreased, the value of absolute error, relative error and mean square error are also decreased. Conversely, as the value of h decreased, the value of absolute error, relative error and mean square error are increased. Regarding to the findings, it can be clearly seen that the finite difference method is well approximated and gives better results with smaller of time step, k.

It is suggested that the Finite Element Method and Finite Volume Method should be applied to solve damped wave equation. It also suggested that the method should be expanded to higher dimensional of the damped wave equation. The researcher also can use other programming such as *C programming, C++ programming* and others.

References

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