Dimensional Analysis

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- Dimensional homogeneity
- Dimensional analysis techniques
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  - **Buckingham π theorem**
    - Manipulation of the groups
    - Common groups
    - Similarity
    - Model

Introduction

- Case: Engineer designing a pipeline system
- Interest point: Pressure drop (Δp)
- Planning of an experiment to study the problem:

  Parameters influencing Δp:

  - Pipe diameter (d)
  - Fluid density (ρ)
  - Fluid viscosity (μ)
  - Mean velocity (v)

  $$\Delta p = \mathrm{f}(\mathrm{d}, \rho, \mu, v)$$

  Condition:

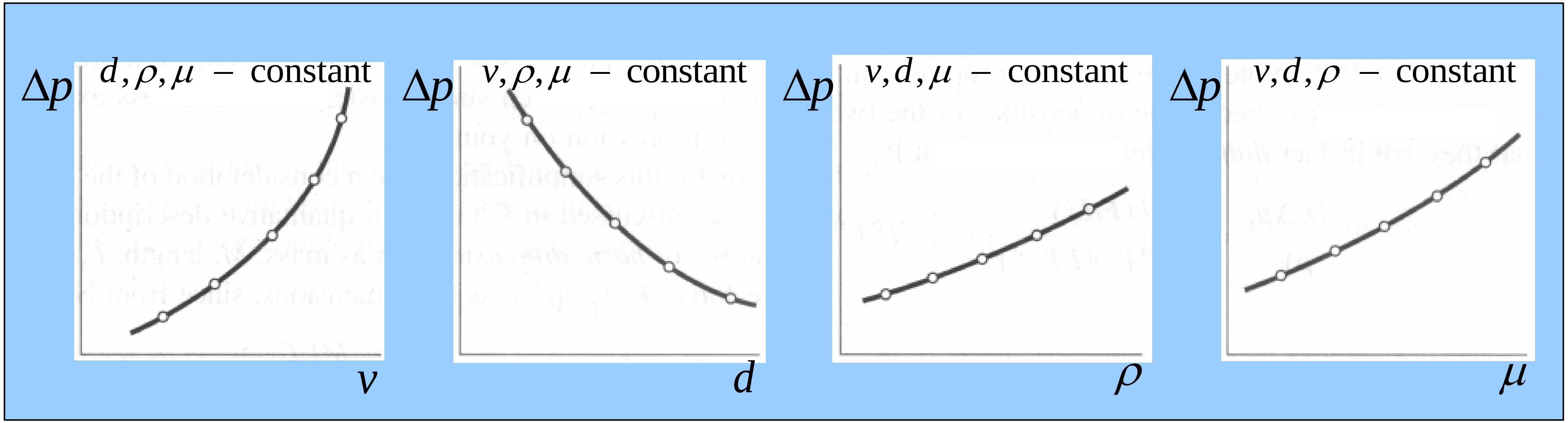
  Smooth-welled pipe and **incompressible Newtonian** fluid

Introduction

- Experimentally:

  Relationship between Δp and (d, ρ, μ, v)

  determined by **changing one** of the variables, while **holding all others constant**



Introduction

- **Limitations:**
  - Some of the experiments would be **hard to carry out**:

    Example: it would be necessary to vary ρ while holding μ constant. How would we do this ?

  - How could we combine these data to obtain the desired **general function relationship** between variables?

  - Would the derived relationship be **valid** for any similar piping system?

Introduction

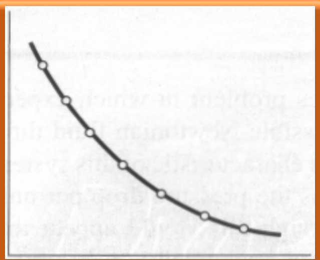
Solution

- Dimensional Analysis Method

Dimensionless Group

$$\frac{d\Delta p}{\rho v^2} = \phi\left(\frac{\rho v d}{\mu}\right)$$

$\frac{d\Delta p}{\rho v^2}$

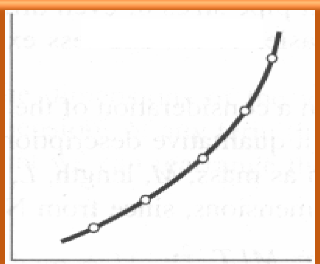


$\frac{\rho v d}{\mu}$

Experiment

- Simplified
- Single-run

Δp



d, ρ, μ – constant

V

Introduction

- Advantages:
  - **Valid** curve for any similar piping system & incompressible Newtonian fluid
  - **Simple** experiment & **less expensive**
- Simplification basis:

  Consideration: **dimensions** of the variables involved
- Dimensional analysis:

  Laboratory model ≈ actual system (field application)

  This concept is called **similarity**

  (will be discussed in the last section of this chapter)

Dimensions and Units (Definition)

Dimension

- A measure of physical quantity (quality)

Units

- Standard elements used to quantify the dimension (quantity)

Dimensions and Units (Dimensions System)

Dimensional Analysis	• Concerned with the nature of the dimension (**quality**)
Mass-Length-Time (MLT) System	• **Mass (M)** • **Length (L)** • **Time (T)**
Force-Length-Time (FLT) System	• Force (MLT^{-2}) • Length (L) • Time (T)

Dimensions and Units (Fundamental & Derived Dimensions / Units)

FUNDAMENTAL

- Mass (**M**), Length (**L**), Time (**T**), Force (F/MLT^{-2}), Temperature (**θ**)

DERIVED

- Other quantities derived from fundamental. Example: Volume (L^3), Density (ML^{-3}), Velocity (LT^{-1})

Dimensions and Units
(Fundamental & Derived Dimensions / Units)

Fundamental Dimension	Symbol	SI Unit	Imperial Unit	Notes
Mass	M	kg	slug	All other dimensions/units can be derived from these fundamental units **Example**: Density (ML^{-3}) Volume (L^3) Velocity (LT^{-1})
Length	L	m	ft	
Time	T	s	s	
Force	F	N	lb	
Temperature	θ	°C / K	°F / R	

Table 7-1 Dimensions of quantities in mechanics (based on Newton 2nd law).

Quantity	Defining Equation	Dimensions, MLT System
Geometrical		
Angle, θ	Arc/radius (a ratio)	[$M^0 L^0 T^0$]
Length, L	(including all linear measurement)	[L]
Area, A	Length x Length	[L^2]
Volume, V	Area x Length	[L^3]
First moment of area,	Area x Length	[L^3]
Permeability, k	Area	[L^2]
Second moment of area, I	Area x $Length^2$	[L^4]
Strain, ε	Extension/Length	[L^0]
Kinematic		
Time, t	-	[T]
Velocity linear, v	Distance/Time	[LT^{-1}]
Acceleration linear, a or g	Linear velocity/Time	[LT^{-2}]
Velocity angular, ω	Angle/Time : radian/sec	[T^{-1}]
Acceleration angular,	Angular velocity/Time	[T^{-2}]
Volume rate of discharge, Q	Volume/Time	[$L^3 T^{-1}$]
Reynolds Number, ***Re***	A ratio of inertia and viscous forces	[$M^0 L^0 T^0$]
Mach Number, ***Ma***	A ratio of fluid velocity to sonic velocity	[$L^0 T^0$]

Quantity	Defining Equation	Dimensions, MLT System
Dynamic		
Mass, ***m***	Force/Acceleration	[M]
Force, ***F***	Mass x acceleration	[MLT^{-2}]
Weight, ***w***	Force	[MLT^{-2}]
Mass density, ρ	Mass/Volume	[ML^{-3}]
Specific weight, γ	Weight/Volume	[$ML^{-2}\,T^{-2}$]
Specific gravity, ***S.G.***	Density/Density of water	[$M^0\,L^0\,T^0$]
Pressure intensity, ***p***	Force/Area	[$ML^{-1}\,T^{-2}$]
Sheering Stress, τ	Force/Area	[$ML^{-1}\,T^{-2}$]
Elastic modulus, ***E***	Stress/Strain	[$ML^{-1}\,T^{-2}$]
Impulse	Force x Time	[MLT^{-1}]
Torque, ***T***	Length x Force	[$ML^{-2}\,T^{-2}$]
Mass momentum of Area	Mass x Length^2	[ML^2]
Momentum linear,	Mass x Linear velocity	[MLT^{-1}]
Momentum angular	Moment of area x Angular velocity	[$ML^2\,T^{-1}$]
Work, energy	Force x Distance	[$ML^2\,T^{-2}$]
Power, ***P***	Work/Time	[$ML^2\,T^{-3}$]
Moment of a force	Force x distance	[$ML^2\,T^{-2}$]
Viscosity, dynamic, μ	Shear stress/Velocity gradient	[$ML^{-1}\,T^{-1}$]
Viscosity, kinematic, ν	Dynamic viscosity/Mass density	[$L^2\,T^{-1}$]
Surface tension, σ	Energy/Area	[MT^{-2}]

Dimensional Homogeneity & Analysis Method (Homogeneity)

- Dimensional homogeneity:

  Left side = right side equation (**same dimensions**)

  All additive **separate term** possess **same dimensions**

- Property of dimensional homogeneity can be use for:

  Checking units of equations

  Converting between two sets of units

  Defining dimensionless relationship

- Result of performing dimensional analysis: **single equation**
- Knowledge (dimensionless groups):

  often helps in deciding what **experimental measurements** should be taken

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical)

- Performed by:

  Identifying the relationships and the influences of **each variables individually**

- The equation is solved by:

  Using a **numerical constant *K***

- The method is rather **lengthy**, especially if there are a **large number of variables** involved

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical: Problem 7.1)

The thrust (F) of a screw propeller is known to depend upon the diameter (d), speed of advance (v), fluid density (ρ), revolutions per second (N), and the coefficient of viscosity (μ) of the fluid. Using indicial method, determine an expression for F

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical: Problem 7.1 solution)

- General relationship: $F = \phi\,(d, v, \rho, N, \mu)$

  Expanded as the sum of an infinite series of terms:

$$F = A\left(d^{m} v^{p} \rho^{q} N^{r} \mu^{s}\right) + B\left(d^{m'} v^{p'} \rho^{q'} N^{r'} \mu^{s'}\right) + C\left(d^{m''} v^{p''} \rho^{q''} N^{r''} \mu^{s''}\right).......,$$

  A, *B*, *C* etc.,: numerical constants

  m, *p*, *q*, *r*, s: unknown powers

- Dimensional homogeneity, all terms must be dimensionally the same, this can be reduced to:

$$F = K\, d^{m}\; v^{p}\; \rho^{q}\; N^{r}\; \mu^{s} \qquad ----(1)$$

  K: numerical constants

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical: Problem 7.1 solution)

- Dimensions system (MLT):

  Dependent variable: *F*

  Independent variables: *d*, *v*, *ρ*, *N*, *μ*

F = Force = $[MLT^{-2}]$,

d = Diameter = $[L]$,

v = Velocity = $[LT^{-1}]$,

ρ = Mass density = $[ML^{-3}]$,

N = Rotational speed = $[T^{-1}]$,

μ = Dynamic viscosity = $[ML^{-1}T^{-1}]$

	F	d	v	ρ	N	μ
M	1	0	0	1	0	1
L	1	1	1	-3	0	-1
T	-2	0	-1	0	-1	-1

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical: Problem 7.1 solution)

- Substituting the dimensions for the variables in Eq. (1):

$$F = K\ d^m\ v^p\ \rho^q\ N^r\ \mu^s \quad ----(1)$$

$$[\mathrm{MLT}^{-2}] = [\mathrm{L}]^m [\mathrm{LT}^{-1}]^p [\mathrm{ML}^{-3}]^q [\mathrm{T}^{-1}]^r [\mathrm{ML}^{-1}\mathrm{T}^{-1}]^s$$

- Equating powers of [M], [L] and [T]:

$$[\mathrm{M}], \quad 1 = q + s \quad ----(2) \qquad q = 1 - s$$

$$[\mathrm{L}], \quad 1 = m + p - 3q - s \quad ----(3) \qquad p = 2 - r - s$$

$$[\mathrm{T}], \quad -2 = -p - r - s \quad ----(4) \qquad m = 1 - p + 3q + s = 2 + r - s$$

- Substituting these value in Eq. (1):

$$F = K\ d^{2+r-s}\ v^{2-r-s}\ \rho^{1-s}\ N^r\ \mu^s$$

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical: Problem 7.1 solution)

- Regrouping the powers:

$$F = K\rho v^2 d^2 \left(\frac{\rho v d}{\mu}\right)^{-s} \left(\frac{dN}{v}\right)^{r}$$

- *s* and *r* are unknown, this can be written as:

$$F = \rho v^2 d^2 \phi\left(\frac{\rho v d}{\mu}, \frac{dN}{v}\right) \quad ----(5)$$

ϕ:'a function of'

- Eq. (5) indicates that:

$$F = C\rho v^2 d^2 \quad ----(6)$$

C: constant (determined experimentally)

Dimensional Homogeneity & Analysis Method (Analysis Method-Indical: Problem 7.1 solution)

- Eq. (5) could have been written:

$$\frac{F}{\rho v^2 d^2} = \phi\left(\frac{\rho v d}{\mu}, \frac{dN}{v}\right) \quad \text{or} \quad \phi\left(\frac{F}{\rho v^2 d^2}, \frac{\rho v d}{\mu}, \frac{dN}{v}\right) = 0$$

- Each of the terms in the bracket forms a dimensionless group:

$$\frac{F}{\rho v^2 d^2} = \frac{[\mathrm{MLT^{-2}}]}{[\mathrm{ML^{-3}}][\mathrm{L^2T^{-2}}][\mathrm{L^2}]} = [\mathrm{M^0L^0T^0}] = [1]$$

$$\frac{\rho v d}{\mu} = \frac{[\mathrm{ML^{-3}}][\mathrm{LT^{-1}}][\mathrm{L}]}{[\mathrm{ML^{-1}T^{-1}}]} = [\mathrm{M^0L^0T^0}] = [1]$$

$$\frac{dN}{v} = \frac{[\mathrm{L}][\mathrm{T^{-1}}]}{[\mathrm{LT^{-1}}]} = [\mathrm{L^0T^0}] = [1]$$

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem)

- Also known as **group method**
  - If an equation involving (***m***) variables is dimensionally homogeneous: Relationship among (***m-n***) = Independent dimensionless products (***π***)

n:Number of fundamental dimensions (describe the variables)

1
- List down: All variables (***m***) involved in the problem
- Express them: Fundamental dimensions (***n*=3**; MLT system)
- Determine: Required number of ***π*** terms (***m-n***)
- Select: A number of repeating variables = MLT system (3)

2
- Form a ***π* term**: One of the nonrepeating variables x product of the repeating variables (combination: dimensionless)

3
- Repeat step 2: Remaining nonrepeating variables
- Check: All ***π*** terms (dimensionless)
- Express final form: relationship among ***π*** terms

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem)

- Choice of repeating variables:
  - Appear in most of the π terms (not exactly all)
  - Combination: Variables must contain all of dimensions (M, L, T*).* That is not to say that each must contain M, L and T. Not form a dimensionless group
  - Measurable in an experimental investigation (major interest)

    Example: Pipe diameter (dimension L) is more useful and measurable than roughness height (dimension L)
  - In fluids (normally): ***ρ***, ***v*** and ***d***
  - Freedom of choice: Many possible different π terms can be formed. All are valid (not really a wrong choice)

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem: Problem 7.2)

The thrust (***F***) of a screw propeller is known to depend upon the diameter (***d***), speed of advance (***v***), fluid density (***r***), revolutions per second (***N***), and the coefficient of viscosity (***m***) of the fluid. Using Buckingham π Theorem method, determine an expression for ***F.***

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem: Problem 7.2 solution)

- General relationship: $F = \phi(d, v, \rho, N, \mu)$ or $0 = \phi(F, d, v, \rho, N, \mu)$
- MLT system (n = 3), variables (m) = 6, π terms = 6 - 3 = 3

$$\phi(\pi_1, \pi_2, \pi_3) = 0$$

- Repeating variables according to MLT system = 3

  ρ, ***v*** and ***d*** (measurable/major interest)

  (combination: contain all M, L, T)

- Form 3 π terms (dimensionless- $M^0L^0T^0$):

$$\pi_1 = \rho^{a_1} v^{b_1} d^{c_1} F \qquad \pi_2 = \rho^{a_2} v^{b_2} d^{c_2} N \qquad \pi_3 = \rho^{a_3} v^{b_3} d^{c_3} \mu$$

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem: Problem 7.2 solution)

- Apply dimensional homogeneity concept (equating dimensions)

$$\pi_1 = \rho^{a_1} v^{b_1} d^{c_1} F$$

$$[M^0L^0T^0] = [ML^{-3}]^{a_1} [LT^{-1}]^{b_1} [L]^{c_1} [MLT^{-2}]$$

M

$$0 = a_1 + 1$$

$$a_1 = -1$$

L

$$0 = -3a_1 + b_1 + c_1 + 1$$

$$0 = 4 + b_1 + c_1$$

$$c_1 = -4 - b_1 = -2$$

T

$$0 = -b_1 - 2$$

$$b_1 = -2$$

$$\pi_1 = \rho^{-1} v^{-2} d^{-2} F$$

$$\therefore \pi_1 = \frac{F}{\rho v^2 d^2}$$

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem: Problem 7.2 solution)

$$\pi_2 = \rho^{a_2} v^{b_2} d^{c_2} N$$

$$[M^0L^0T^0] = [ML^{-3}]^{a_2} [LT^{-1}]^{b_2} [L]^{c_2} [T^{-1}]$$

M	L	T
$0 = a_2$	$0 = -3a_2 + b_2 + c_2$	$0 = -b_2 - 1$
	$0 = b_2 + c_2$ $\quad c_2 = 1$	$b_2 = -1$

$$\pi_2 = \rho^0 v^{-1} d^1 N$$

$$\therefore \pi_2 = \frac{Nd}{v}$$

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem: Problem 7.2 solution)

$$\pi_3 = \rho^{a_3} v^{b_3} d^{c_3} \mu \qquad [M^0L^0T^0] = [ML^{-3}]^{a_3}[LT^{-1}]^{b_3}[L]^{c_3}[ML^{-1}T^{-1}]$$

M	L		T
$0 = a_3 + 1$	$0 = -3a_3 + b_3 + c_3 - 1$		$0 = -b_3 - 1$
$a_3 = -1$	$b_3 + c_3 = -2$	$c_3 = -1$	$b_3 = -1$

$$\pi_3 = \rho^{-1} v^{-1} d^{-1} \mu \qquad \therefore \pi_3 = \frac{\mu}{\rho v d}$$

Dimensional Homogeneity & Analysis Method (Analysis Method-Buckingham π Theorem: Problem 7.2 solution)

- Thus, the problem may be described by the following function of the 3 π terms:

$$\phi(\pi_1, \pi_2, \pi_3) = 0$$

$$\phi\left(\frac{F}{\rho v^2 d^2}, \frac{Nd}{v}, \frac{\mu}{\rho v d}\right) = 0 \quad \text{or} \quad \frac{F}{\rho v^2 d^2} = \phi\left(\frac{Nd}{v}, \frac{\mu}{\rho v d}\right)$$

Manipulation of the Groups (π Term)

- Once identified: π groups manipulation is permitted
- Manipulations: same number of groups involved, change their appearance drastically
- Defining equation as:

$$\phi(\pi_1, \pi_2, \pi_3, ..., \pi_{m-n}) = 0$$

- Following manipulations are permitted:

1) Any number of groups can be combined by multiplication or division to form a new group which replaces one of the existing. E.g. π_1 and π_2 may be combined to form

   $\pi_{1a} = \pi_1 / \pi_2$ so the defining equation becomes:

$$\phi(\pi_{1a}, \pi_2, \pi_3, ..., \pi_{m-n}) = 0$$

Manipulation of the Groups (π Term)

- Following manipulations are permitted:
  2) The **reciprocal** of any dimensionless group is valid

     So $\phi(\pi_1, 1/\pi_2, \pi_3, ..., 1/\pi_{m-n}) = 0$ is valid
  3) Any dimensionless group may be raised to any **power**

     So $\phi((\pi_1)^2, (\pi_2)^{1/2}, \pi_3, ..., \pi_{m-n}) = 0$ is valid
  4) Any dimensionless group may be multiplied by a constant
  5) Any group may be expressed as a **function** of the other groups. Example: $\pi_2 = \phi(\pi_1, \pi_3, ..., \pi_{m-n})$

- In general the defining equation could look like

$$\phi(\pi_{1a}, 1/\pi_2, (\pi_3)^i, ..., 0.5\pi_{m-n}) = 0$$

Common Groups (π Term)

- During dimensional analysis several **groups will appear again and again** for different problems (often have names)
- You will recognize the Reynolds number
- Some common non-dimensional numbers (groups) are listed below:

Reynolds number	$\mathrm{Re} = \frac{\rho v d}{\mu}$	inertial, viscous force ratio
Euler number	$\mathrm{En} = \frac{p}{\rho v^2}$	pressure, inertial force ratio
Froude number	$\mathrm{Fn} = \frac{v^2}{gd}$	inertial, gravitational force ratio
Weber number	$\mathrm{We} = \frac{\rho v d}{\sigma}$	inertial, surface tension force ratio
Mach number	$\mathrm{Mn} = \frac{v}{c}$	Local velocity, local velocity of sound ratio

Common Groups (π Term): Problem 7.3

The discharge *Q* through an orifice is a function of the diameter *d*, the pressure difference *p*, the density *ρ*, and the viscosity *μ*, show that

$$Q = \frac{d^2 p^{1/2}}{\rho^{1/2}} \phi \left(\frac{d \rho^{1/2} p^{1/2}}{\mu} \right)$$

where ϕ is some unknown function

Common Groups (π Term): Problem 7.3 solution

- Dimensions of the variables:

$$\begin{array}{llll} \rho: & [ML^{-3}] & Q: & [L^3T^{-1}] \\ d: & [L] & \mu: & [ML^{-1}T^{-1}] \\ p: & [ML^{-1}T^{-2}] & & \end{array}$$

- 5 variables involved in the problem: *d, p, r, μ* and *Q*
- Choose the three recurring (governing) variables: *Q, d, ρ*
- Buckingham's π theorem: $m - n = 5 - 3 = 2$ non-dimensional groups

where $$\phi(\pi_1, \pi_2) = 0$$

$$\pi_1 = Q^{a_1} d^{b_1} \rho^{c_1} \mu \qquad \pi_2 = Q^{a_2} d^{b_2} \rho^{c_2} p$$

- For the first group, π_1:

$$[M^0L^0T^0] = [L^3T^{-1}]^{a_1}[L]^{b_1}[ML^{-3}]^{c_1}[ML^{-1}T^{-1}]$$

$$[M] \qquad 0 = c_1 + 1, \quad c_1 = -1$$

$$[L] \qquad 0 = 3a_1 + b_1 - 3c_1 - 1, \quad -2 = 3a_1 + b_1$$

$$[T] \qquad 0 = -a_1 - 1, \quad a_1 = -1, \quad b_1 = 1$$

$$\therefore \pi_1 = Q^{-1} d^1 \rho^{-1} \mu = \frac{d\mu}{\rho Q}$$

Common Groups (π Term): Problem 7.3 solution

- For the second group, π_2:

$$[M^0L^0T^0] = [L^3T^{-1}]^{a_2}[L]^{b_2}[ML^{-3}]^{c_2}[ML^{-1}T^{-2}]$$

[M] $$0 = c_2 + 1 \qquad c_2 = -1$$

[L] $$0 = 3a_2 + b_2 - 3c_2 - 1 \qquad -2 = 3a_2 + b_2$$

[T] $$0 = -a_2 - 2 \qquad a_2 = -2 \qquad b_2 = 4$$

$$\therefore \pi_2 = Q^{-2}d^4\rho^{-1}p = \frac{d^4 p}{\rho Q^2}$$

- So the physical situation is described by this function of non-dimensional numbers:

$$\phi(\pi_1, \pi_2) = \phi\left(\frac{d\mu}{Qp}, \frac{d^4 p}{\rho Q^2}\right) = 0$$

or

$$\frac{d\mu}{Qp} = \phi\left(\frac{d^4 p}{\rho Q^2}\right)$$

Common Groups (π Term): Problem 7.3 solution

- The question wants us to show:

$$Q = \frac{d^2 p^{1/2}}{\rho^{1/2}} \phi\left(\frac{d\rho^{1/2} p^{1/2}}{\mu}\right)$$

- Take the reciprocal of square root of π_2:

$$\frac{1}{\sqrt{\pi_2}} = \frac{p^{1/2} Q}{d^2 p^{1/2}} = \pi_{2a}$$

- Convert π_1 by multiplying by this new group, π_{2a}:

$$\pi_{1a} = \pi_1 \pi_{2a} = \frac{d\mu}{Q\rho} \frac{\rho^{1/2} Q}{d^2 p^{1/2}} = \frac{\mu}{d\rho^{1/2} p^{1/2}}$$

- Then, we can say

$$\phi(1/\pi_{1a}, \pi_{2a}) = \phi\left(\frac{d\rho^{1/2} p^{1/2}}{\mu}, \frac{d^2 p^{1/2}}{Q\rho^{1/2}}\right) = 0$$

or

$$Q = \frac{d^2 p^{1/2}}{\rho^{1/2}} \phi\left(\frac{d\rho^{1/2} p^{1/2}}{\mu}\right)$$

Similarity

- Hydraulic models may be either true or distorted models

  True models reproduce features of the prototype but at a scale - that is they are ***geometrically*** similar

- There are three types of similarity:

1) Geometric
2) Kinematic
3) Dynamic

Similarity (Geometric)

- Geometric similarity exists between model and prototype if the ratio of all corresponding dimensions in the model and prototype are equal

$$\frac{L_{\text{model}}}{L_{\text{prototype}}} = \frac{L_m}{L_p} = \lambda_L$$

where λ_L is the scale factor for length

- For area:

$$\frac{A_{\text{model}}}{A_{\text{prototype}}} = \frac{L_m^2}{L_p^2} = \lambda_L^2$$

- All corresponding angles are the same

Similarity (Kinematic)

- Kinematic similarity is the similarity of time as well as geometry. It exists between model and prototype
  1) If the paths of moving particles are geometrically similar
  2) If the rations of the velocities of particles are similar
- Some useful ratios are:

1) Velocity

$$\frac{v_m}{v_p} = \frac{L_m/T_m}{L_p/T_p} = \frac{\lambda_L}{\lambda_T} = \lambda_v$$

2) Acceleration

$$\frac{a_m}{a_p} = \frac{L_m/T_m^2}{L_p/T_p^2} = \frac{\lambda_L}{\lambda_T^2} = \lambda_a$$

3) Discharge

$$\frac{Q_m}{Q_p} = \frac{L_m^3/T_m}{L_p^3/T_p} = \frac{\lambda_L^3}{\lambda_T} = \lambda_Q$$

- This has the consequence that streamline patterns are the same

Similarity (Dynamic)

- Dynamic similarity exists between geometrically and kinematically similar systems if the ratios of all forces in the model and prototype are the same

Force Ratio

$$\frac{F_m}{F_p} = \frac{M_m a_m}{M_p a_p} = \frac{\rho_m L_m^3}{\rho_p L_p^3} \times \frac{\lambda_L}{\lambda_T^2} = \lambda_\rho \lambda_L^2 \left(\frac{\lambda_L}{\lambda_T}\right)^2 = \lambda_\rho \lambda_L^2 \lambda_v^2$$

- This occurs when the controlling dimensionless group on the right-hand side of the defining equation is the same for model and prototype

Model

- Hydraulic structure is build: Analysis in the design stage
- Complex structures: For simple mathematical analysis
- A hydraulic model is build: Help engineer (analyse the problem)
- Usually: Model < full size (but it may be greater)
- The real structure (prototype)
- The model is usually built to an exact geometric scale of the prototype but in some cases - notably river model - this is not possible
- Measurements can be taken from the model and a suitable scaling law applied to predict the values in the prototype
- **To illustrate how these scaling laws can be obtained we will use the relationship for resistance of a body moving through a fluid**

Model: Problem 7.4

- The resistance, *R,* is dependent on the following physical properties:

ρ (density): ML^{-3} v (velocity): LT^{-1} l (length): L μ (dynamic viscosity): $ML^{-1}T^{-1}$

Hint

Equation: $\phi(R, \rho, v, l, \mu) = 0$

m = 5, n = 3, so there are 5-3 = 2 π groups (Buckingham π theorem)

$$\pi_1 = \rho^{a_1} v^{b_1} l^{c_1} R \qquad \pi_2 = \rho^{a_2} v^{b_2} l^{c_2} \mu$$

Model: Problem 7.4 solution

- For the π_1 group, $\pi_1 = \rho^{a_1} v^{b_1} l^{c_1} R$

  $[M^0L^0T^0] = [ML^{-3}]^{a_1} [LT^{-1}]^{b_1} [L]^{c_1} [MLT^{-2}]$

  Leading to π_1 as, $\pi_1 = \dfrac{R}{\rho v^2 l^2}$

- For the π_2 group, $\pi_2 = \rho^{a_2} v^{b_2} l^{c_2} \mu$

  $[M^0L^0T^0] = [ML^{-3}]^{a_2} [LT^{-1}]^{b_2} [L]^{c_2} [ML^{-1}T^{-1}]$

  Leading to π_1 as, $\pi_2 = \dfrac{\mu}{\rho v l}$

- Notice how $1/\pi_2$ is the Reynolds number. We can call this π_{2a}.
- So the defining equation for resistance to motion is

$$\phi(\pi_1, \pi_{2a}) = 0$$

- We can write

$$\frac{R}{\rho v^2 l^2} = \phi\left(\frac{\mu}{\rho v l}\right)$$

or,

$$R = \rho v^2 l^2 \phi\left(\frac{\rho v l}{\mu}\right)$$

- This equation applies whatever the size of the body i.e. it is applicable to a to the prototype and a geometrically similar model. Thus for the model

$$\frac{R_m}{\rho_m v_m^2 l_m^2} = \phi\left(\frac{\rho_m v_m l_m}{\mu_m}\right)$$

  and for the prototype

$$\frac{R_p}{\rho_p v_p^2 l_p^2} = \phi\left(\frac{\rho_p v_p l_p}{\mu_p}\right)$$

- Dividing these two equations gives

$$\frac{R_m / \rho_m v_m^2 l_m^2}{R_p / \rho_p v_p^2 l_p^2} = \frac{\phi(\rho_m v_m l_m / \mu_m)}{\phi(\rho_p v_p l_p / \mu_p)}$$

- At this point we can go no further unless we make some assumptions. One common assumption is to assume that the Reynolds number is the same for both the model and prototype i.e.

$$\rho_m v_m l_m / \mu_m = \rho_p v_p l_p / \mu_p$$

- Dividing these two equations gives

$$\frac{R_m}{R_p} = \frac{\rho_m v_m^2 l_m^2}{\rho_p v_p^2 l_p^2}$$

Model: Problem 7.4 solution

- Which gives this scaling law for resistance force:

$$\lambda_R = \lambda_\rho \lambda_v^2 \lambda_L^2$$

- That the Reynolds numbers were the same, was an essential assumption for this analysis. The consequence of this should be explained

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{\rho_m v_m l_m}{\mu_m} = \frac{\rho_p v_p l_p}{\mu_p}$$

$$\frac{v_m}{v_p} = \frac{\rho_p}{\rho_m} \frac{\mu_m}{\mu_p} \frac{l_p}{l_m}$$

$$\lambda_v = \frac{\lambda_\mu}{\lambda_\rho \lambda_L}$$

- So the force on the prototype can be predicted from measurement of the force on the model
- But only if the fluid in the model is moving with same Reynolds number as it would in the prototype. That is to say the *Rp* can be predicted by

$$R_p = \frac{\rho_p v_p^2 l_p^2}{\rho_m v_m^2 l_m^2} R_m$$

provided that $v_p = \frac{\rho_m}{\rho_p} \frac{\mu_p}{\mu_m} \frac{l_m}{l_p} v_m$

- In this case the model and prototype are **dynamically similar**
- Formally this occurs when the controlling dimensionless group on the right hand side of the defining equation is the same for model and prototype
- In this case the controlling dimensionless group is the Reynolds number

Model: Problem 7.5

An underwater missile, diameter 2 m and length 10 m is tested in a water tunnel to determine the forces acting on the real prototype. A 1/20th scale model is to be used. If the maximum allowable speed of the prototype missile is 10 m/s, what should be the speed of the water in the tunnel to achieve dynamic similarity?

Model: Problem 7.5 solution

- For dynamic similarity the Reynolds number of the model and prototype must be equal:

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\left(\frac{\rho v d}{\mu}\right)_m = \left(\frac{\rho v d}{\mu}\right)_p$$

- So the model velocity should be

$$v_m = v_p \frac{\rho_p}{\rho_m} \frac{d_p}{d_m} \frac{\mu_m}{\mu_p}$$

- As both the model and prototype are in water then, $\mu_m = \mu_p$ and $\rho_m = \rho_p$

$$v_m = v_p \frac{d_p}{d_m} = 10 \frac{1}{1/20} = 200\,\mathrm{m/s}$$

- Note that this is a **very** high velocity
- This is one reason why model tests are not always done at exactly equal Reynolds numbers
- Some relaxation of the equivalence requirement is often acceptable when the Reynolds number is high
- Using a wind tunnel may have been possible in this example
- If this were the case then the appropriate values of the and ratios need to be used in the above equation

Model: Problem 7.6

A model aeroplane is built at 1/10 scale and is to be tested in a wind tunnel operating at a pressure of 20 times atmospheric. The aeroplane will fly at 500 km/h. At what speed should the wind tunnel operate to give dynamic similarity between the model and prototype? If the drag measure on the model is 337.5 N, what will be the drag on the plane?

Model: Problem 7.6 solution

- From earlier we derived the equation for resistance on a body moving through air:

$$R = \rho v^2 l^2 \phi\left(\frac{\rho v l}{\mu}\right) = \rho v^2 l^2 \phi(\mathrm{Re})$$

- For dynamic similarity $\mathrm{Re}_m = \mathrm{Re}_p$, so

$$v_m = v_p \frac{\rho_p}{\rho_m}\frac{d_p}{d_m}\frac{\mu_m}{\mu_p}$$

- The value of μ does not change much with pressure so, $\mu_m = \mu_p$
- The equation of state for an ideal gas is $p = \rho RT$
- As temperature is the same then the density of the air in the model can be obtained from

$$\frac{p_m}{p_p} = \frac{\rho_m RT}{\rho_p RT} = \frac{\rho_m}{\rho_p}$$

$$\frac{20 p_p}{p_p} = \frac{\rho_m}{\rho_p}$$

$$\rho_m = 20\rho_p$$

- So the model velocity is found to be

$$v_m = v_p \frac{1}{20}\frac{1}{1/10} = 0.5 v_p$$

$$v_m = 250 \text{ km/h}$$

- The ratio of forces is found from

$$\frac{R_m}{R_p} = \frac{(\rho v^2 l^2)_m}{(\rho v^2 l^2)_p}$$

$$\frac{R_m}{R_p} = \frac{20}{1}\frac{(0.5)^2}{1}\frac{(0.1)^2}{1} = 0.05$$

- So the drag force on the prototype will be

$$R_p = \frac{1}{0.05} R_m = 20 \text{ X } 337.5 = 6750 \text{ N}$$

Model: Problem 7.7

1) The loss of pressure, *p*, when a fluid flows through geometrically similar pipes is a function of the pipe diameter, *d*, the length of the pipe, *l*, the mean velocity of flow through the pipe, *v*, the mass density of the fluid, ρ, and the dynamic viscosity of the fluid, μ. Use *d*, *v*, and ρ as repeating variables. By applying Buckingham pi (π) theorem and *MLT* system, show that:

$$p = \frac{\rho v^2 l}{d}\phi\left(\frac{\rho v d}{\mu}\right)$$

2) The measured loss of head in a 50 mm diameter pipe conveying water at 0.6 m/s is 800mm of water per 100 m length. Calculate the loss of head in millimeters of water per 400 m length when air flows through a 200 mm diameter pipe at the "corresponding speed". Assume that the pipes have geometrically similar roughness and take the densities of air and water as 1.23 and 1000 kg/m^3 and the absolute viscosities as 1.8 x 10^{-4} and 1.2 x 10^{-2} poise respectively

Model: Problem 7.7 solution

1) The physical problem can be expressed as,

$$0 = \phi(p, d, l, v, \rho, \mu)$$

- The dimensions of the variables

p:	$[ML^{-1}T^{-2}]$	v:	$[LT^{-1}]$
d:	$[L]$	ρ:	$[ML^{-3}]$
l:	$[L]$	μ:	$[ML^{-1}T^{-1}]$

- There are 6 variables, so $m = 6$, and we are using MLT system, so $n = 3$
- We are told by the question to use d, v, ρ as repeating variables:
- From Buckingham's π theorem, we have $m - n = 6 - 3 = 3$ non-dimensional groups.

$$\phi(\pi_1, \pi_2, \pi_3) = 0$$

where

$$\pi_1 = d^{a_1} v^{b_1} \rho^{c_1} p$$

$$\pi_2 = d^{a_2} v^{b_2} \rho^{c_2} l$$

$$\pi_3 = d^{a_3} v^{b_3} \rho^{c_3} \mu$$

- For the π_1 group, $\pi_1 = d^{a_1} v^{b_1} \rho^{c_1} p$

$$[M^0L^0T^0] = [L]^{a_1}[LT^{-1}]^{b_1}[ML^{-3}]^{c_1}[ML^{-1}T^{-2}]$$

[M]
$$0 = c_1 + 1$$
$$c_1 = -1$$

[L]
$$0 = a_1 + b_1 - 3c_1 - 1$$
$$-2 = a_1 + b_1$$

[T]
$$0 = -b_1 - 2$$
$$b_1 = -2$$
$$a_1 = 0$$

$$\therefore \pi_1 = d^{-0} v^{-2} \rho^{-1} p = \frac{p}{\rho v^2}$$

- For the π_2 group, $\pi_2 = d^{a_2} v^{b_2} \rho^{c_2} l$

$$[M^0L^0T^0] = [L]^{a_2}[LT^{-1}]^{b_2}[ML^{-3}]^{c_2}[L]$$

[M]
$$0 = c_2$$
$$c_2 = 0$$

[L]
$$0 = a_2 + b_2 - 3c_2 + 1$$
$$-1 = a_2 + b_2$$

[T]
$$0 = -b_2$$
$$b_2 = 0$$
$$a_2 = -1$$

$$\therefore \pi_2 = d^{-1} v^0 \rho^0 l = \frac{l}{d}$$

Model: Problem 7.7 solution

- For the π_3 group, $\pi_3 = d^{a_3} v^{b_3} \rho^{c_3} \mu$

$$[M^0L^0T^0] = [L]^{a_3}[LT^{-1}]^{b_3}[ML^{-3}]^{c_3}[ML^{-1}T^{-1}]$$

$$\begin{aligned}
[M] \quad & 0 = c_3 + 1 \\
& c_3 = -1 \\
[L] \quad & 0 = a_3 + b_3 - 3c_3 - 1 \\
& -2 = a_3 + b_3 \\
[T] \quad & 0 = -b_3 - 1 \\
& b_3 = -1 \\
& a_3 = -1
\end{aligned}$$

$$\therefore \pi_3 = d^{-1} v^{-1} \rho^{-1} \mu = \frac{\mu}{\rho v d}$$

- So the physical situation is described by this function of non-dimensional numbers,

$$\phi(\pi_1, \pi_2, \pi_3) = \phi\left(\frac{p}{\rho v^2}, \frac{l}{d}, \frac{\mu}{\rho v d}\right) = 0$$

or

$$p = \rho v^2 \phi\left(\frac{l}{d}, \frac{\mu}{\rho v d}\right)$$

- The question wants us to show:

$$p = \frac{\rho v^2 l}{d} \phi\left(\frac{\rho v d}{\mu}\right)$$

- Take reciprocal of π_3: $\frac{1}{\pi_3} = \frac{\mu}{\rho v d} = \pi_{3a}$

then we can say

$$\phi(\pi_1, \pi_2, \pi_{3a}) = \phi\left(\frac{p}{\rho v^2}, \frac{l}{d}, \frac{\rho v d}{\mu}\right) = 0$$

- The above equation can written as

$$\frac{p}{\rho v^2} = \phi\left(\frac{l}{d}, \frac{\rho v d}{\mu}\right)$$

$$p = K \rho v^2 \left(\frac{l}{d}\right)^x \left(\frac{\rho v d}{\mu}\right)^y$$

$$p = K \rho v^2 \left(\frac{l}{d}\right)\left(\frac{l}{d}\right)^{x-1} \left(\frac{\rho v d}{\mu}\right)^y \quad ----(1)$$

- For geometrically similar pipes (l/d) is constant and $(l/d)^{x-1}$ can be combined with K to give another constant C, where

$$C = K\left(\frac{l}{d}\right)^{x-1}$$

- Therefore, Equation (1) becomes

$$p = C\frac{\rho v^2 l}{d}\left(\frac{\rho v d}{\mu}\right)^y$$

- Since neither K nor y are known, this can simply written as

$$p = \frac{\rho v^2 l}{d} \phi\left(\frac{\rho v d}{\mu}\right)$$

Model: Problem 7.7 solution

b. For dynamic similarity the Reynolds number of the model and prototype must be equal:

$$\left(\frac{\rho v d}{\mu}\right)_m = \left(\frac{\rho v d}{\mu}\right)_p$$

$$v_p = \frac{\rho_m}{\rho_p}\frac{\mu_p}{\mu_m}\frac{d_m}{d_p}v_m$$

and the ratio of pressure drops

$$\left(\frac{pd}{\rho v^2 l}\right)_m = \left(\frac{pd}{\rho v^2 l}\right)_p$$

$$\frac{p_p}{p_m} = \frac{\rho_p}{\rho_m}\frac{v_p^2}{v_m^2}\frac{l_p}{l_m}\frac{d_m}{d_p}$$

- The velocity of flow in the prototype pipe required to make the Re number the same in both is known as the *corresponding speed*.
- Therefore, corresponding speed for air, v_p

$$v_p = \frac{1000}{1.23} X \frac{1.8 X 10^{-4}}{1.2 X 10^{-2}} X \frac{50}{200} X 0.6 = 1.83 \text{ m/s}$$

- Ratio of pressure drops

$$\frac{p_p}{p_m} = \frac{1.23}{1000} X \frac{1.83^2}{0.6^2} X \frac{400}{100} X \frac{50}{200} = 0.01144$$

- If loss p_f head per 100 m in 50 mm pipe is 800 mm water, loss of head per 400 m in 200 mm pipe is

$$= 0.01144 X 800 = 9.15 \text{ mm of water}_{\#}$$

THANK YOU

Stay safe!