Chapter 9: Complex Numbers

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9.1 Imaginary numbers

Consider:

\[ x^2 = -4 \]

This equation has no real solution. To solve the equation, we will introduce an imaginary number.

**Definition 9.1 (Imaginary Number)**

The imaginary number \( i \) is defined as:

\[ i^2 = -1 \]

Therefore, using the definition, we will get,

\[ x^2 = -4 \]
\[ x = \sqrt{-4} \]
\[ = \sqrt{4(-1)} \]
\[ = \sqrt{4}i^2 \]
\[ = \pm 2i \]

*Example:* Express the following as imaginary numbers

a) \( \sqrt{-25} \)  

b) \( \sqrt{-8} \)
9.2 Complex Numbers

**Definition 9.2 (Complex Numbers)**

If $z$ is a complex number, then it can be expressed in the form:

$$z = x + iy,$$

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

- $x$: real part
- $y$: imaginary part

Or frequently represented as:

$$Re(z) = x \text{ and } Im(z) = y$$

**Example:**

Find the real and imaginary parts of the following complex numbers

(a) $z_1 = 2 + 3i$

(b) $4i^2 + i - 2i^3$
9.2.1 Argand Diagram

We can graph complex numbers using an Argand Diagram.

\[ y = \Im (z) \]

\[ z = x + iy \]

Example:

Sketch the following complex numbers on the same axes.

(a) \( z_1 = 3 + 2i \) \hspace{1cm} (b) \( z_2 = 3 - 2i \)

(c) \( z_3 = -3 - 2i \) \hspace{1cm} (d) \( z_4 = -3 + 2i \)
9.2.2 Equality of Two Complex Numbers

Given that \( z_1 = a + bi \) and \( z_2 = c + di \)

where \( z_1, z_2 \in C \).

Two complex numbers are equal iff the real parts and the imaginary parts are respectively equal.

So, if \( z_1 = z_2 \), then \( a = c \) and \( b = d \).

Example 1:
Solve for \( x \) and \( y \) if given \( 3x + 4i = (2y + x) + xi \).

Example 2:
Solve \((3 + 4i)^2 - 2(x - iy) = x + iy\) for real numbers \( x \) and \( y \).

Example 3:
Solve the following equation for \( x \) and \( y \) where
\[
xy - 2i + x + 2xyi - 5 = \frac{3}{2} - 3i
\]
9.3 Algebraic Operations on Complex Numbers

9.3.1 Addition and subtraction

If \( z_1 = a + bi \) and \( z_2 = c + di \) are two complex numbers, then

\[
z_1 \pm z_2 = (a + c) \pm (b + d)i
\]

**Example:**

Given \( Z_1 = -2 + 2i \), \( Z_2 = 1 - \frac{\sqrt{3}}{2}i \) and \( Z_3 = 4 - 6i \). Find

a) \( Z_1 - Z_2 \) \hspace{0.5cm} b) \( Z_1 + Z_3 \)

9.3.2 Multiplication

If \( z_1 = a + bi \) and \( z_2 = c + di \) are two complex numbers, and \( k \) is a constant, then

(i) \( z_1 \cdot z_2 = (a + bi) \cdot (c + di) \)

\[
= (ac - bd) + (ad + bc)i
\]

(ii) \( kz_1 = ka + kbi \)

**Example:**

Given \( Z_1 = -2 + 2i \), and \( Z_2 = 4 - 6i \). Find \( Z_1Z_2 \).
9.3.3 **Complex Conjugate**

If $z = a + bi$ then the **conjugate** of $z$ is denoted as $\bar{z} = a - bi$.

Note that $z \cdot \bar{z} = a^2 + b^2$

9.3.4 **Division**

If we are dividing with a complex number, the denominator must be converted to a real number. In order to do that, multiply both the denominator and numerator by complex conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$
Example 1:

Given that \( z_1 = 1 - 2i, \ z_2 = -3 + 4i \). Find \( \frac{z_1}{z_2} \), and express it in \( a + bi \) form.

Example 2:

Given \( z_1 = 2 + i \) and \( z_2 = 3 - 4i \), find \( \frac{1}{z_1} + \frac{1}{z_2} \) in the form of \( a + ib \).

Example 3:

Given \( Z = \frac{-2+3i}{3-2} \). Find the complex conjugate, \( \bar{Z} \). Write your answer in \( a + ib \) form.

Example 4:

Given \( Z_1 = -2 + 2i \), and \( Z_2 = 4 - 6i \). Find \( \frac{2}{Z_1 + Z_2} \).
9.4 Polar Form of Complex Numbers

Modulus of $z$, 
\[ |z| = r = \sqrt{x^2 + y^2}. \]

Argument of $z$, 
\[ \text{arg} (z) = \theta \]

where 
\[ \tan \theta = \frac{y}{x}. \]

From the diagram above, we can see that
\[ x = r \cos \theta \quad y = r \sin \theta \]

Then, $z$ can be written as
\[ z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = rcis\theta \quad (z \text{ in polar form}) \]
Example 1:
Express $z = -2 - \sqrt{3} \, i$ in polar form.

Example 2:
Express $\frac{2+3i}{1-i}$ in polar form.

Example 3
Given that $z_1 = 2 + i$ and $z_2 = -2 + 4i$, find $z$ such that

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}.$$  

Give your answer in the form of $a + ib$. Hence, find the modulus and argument of $z$. 


9.5 De Moivre’s Theorem

9.5.1 The n-th Power Of A Complex Number

**Definition 9.5 (De Moivre’s Theorem)**

If \( z = r(\cos \theta + i \sin \theta) \) and \( n \in R \), then

\[
z^n = r^n(\cos n\theta + i \sin n\theta)
\]

**Example 1:**

a) Write \( z = 1 - i \) in the polar form.

Then, using De Moivre’s theorem, find \( z^4 \).

b) Use D’Moivre’s formula to write \((-1 - i)^{12}\) in the form of \( a + ib \).
9.5.2 The $n$-th Roots of a Complex Number

A complex number $w$ is a $n$-th root of the complex number $z$ if

$$w^n = z \text{ or } w = \frac{1}{z^{1/n}}.$$  
Hence

$$w = \frac{1}{z^{1/n}} = [r(\cos \theta + i \sin \theta)]^{1/n},$$

$$= r^{1/n} \left( \cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right) \text{ for } k = 0,1,2,\ldots,n-1$$

Substituting $k = 0,1,2,\ldots,n-1$ yields the $n$th roots of the given complex number.

**Example 1:**
Find all the roots for the following equations:

(a) $z^3 = 27$  \hspace{1cm} (b) $z^4 = (\sqrt{3} + i)$.

**Example 2:**
Solve $z^4 + (-1 + i) = 0$ and express them in $a + ib$ form.
Example 3:
Find all cube roots of \(-26 - 8i\).

Example 4:
Solve \(z^3 + 8 = 0\). Sketch the roots on the argand diagram.
9.5.3 De Moivre’s Theorem to Prove Trigonometric Identities

De-Moivre’s theorem can be used to prove some trigonometric identities. (with the help of Binomial theorem or Pascal triangle.)

Example:
Prove that
\[
\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta \quad \text{and} \quad 
\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta.
\]

Solution:
The idea is to write \((\cos \theta + i \sin \theta)^5\) in two different ways.
We use both the Pascal triangle and De Moivre’s theorem, and compare the results.
From Pascal triangle,
\[
(\cos \theta + i \sin \theta)^5 = \cos^5 \theta + 5 \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta \\
- i 10 \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta.
\]
\[
= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + \\
i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta).
\]
Also, by De Moivre’s Theorem, we have
\[(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta.\]
and so
\[\cos 5\theta + i \sin 5\theta = (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta).\]

Equating the real parts gives
\[\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta.\]
\[= \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2\]
\[= \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta - 10 \cos^3 \theta + 5 \cos^5 \theta.\]
\[= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta. \text{ (proved)}\]

Equating the imaginary parts gives
\[\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta\]
\[= 5(1 - \sin^2 \theta)^2 \sin \theta - 10(1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta\]
\[= 5(1 - 2 \sin^2 \theta + \sin^4 \theta) \sin \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta\]
\[= 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta\]
\[= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \text{ (proved)}.\]
9.6 Euler’s Formula

Definition 9.6

Euler’s formula states that

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

It follows that

\[ e^{i n \theta} = \cos n \theta + i \sin n \theta \]

From the definition, if \( z \) is a complex number with modulus \( r \) and Arg\( (z), \theta; \) then

\[ z = r(\cos \theta + i \sin \theta) \]

\[ = re^{i \theta} \quad \text{(z in euler form)} \]

Example:

Express the following complex numbers in the form of \( re^{i \theta} \)

(a) \( 3 + i \) \hspace{1cm} (b) \( 2 - 4i \)
9.6.1 The \( n \)-th Power Of A Complex Number

We know that a complex number can be express as \( z = re^{i\theta} \), then

\[
\begin{align*}
\theta^2 &= r^2 e^{i2\theta} \\
\theta^3 &= r^3 e^{i3\theta} \\
\theta^4 &= r^4 e^{i4\theta} \\
&\vdots \\
\theta^n &= r^n e^{in\theta}
\end{align*}
\]

**Example 1:**

Given \( z = 2 + 2\sqrt{3} i \). Find the modulus and argument of \( z^5 \).

**Example 2:**

Find \((\sqrt{3} - i)^{40}\) in the form of \( a + ib \).

**Example 3:**

Express the complex number \( z = -1 + \sqrt{3}i \) in the form of \( re^{i\theta} \).

Then find

(a) \( z^2 \) \hspace{1cm} (b) \( z^3 \) \hspace{1cm} (c) \( z^7 \)
9.6.2 The \( n \)-th Roots Of A Complex Number

The \( n \)-th roots of a complex number can be found using the Euler’s formula. Note that:

\[ z = re^{i(\theta + 2k\pi)} \]

Then,

\[ z^2 = r^2 e^{i\left(\frac{\theta + 2k\pi}{2}\right)}, \quad k = 0, 1 \]

\[ z^3 = r^3 e^{i\left(\frac{\theta + 2k\pi}{3}\right)}, \quad k = 0, 1, 2 \]

\[ \vdots \]

\[ z^n = r^n e^{i\left(\frac{\theta + 2k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots, n - 1 \]

**Example 1:**

Find the cube roots of \( z = 1 + i \).

**Example 2:**

Given \( z = -1 + i \). Find all roots of \( z^3 \) in Euler form.

**Example 3:**

Solve \( z^3 + 8i = 0 \) and sketch the roots on an Argand diagram.
### 9.6.3 Relationship between circular and hyperbolic functions.

Euler’s formula provides the theoretical link between circular and hyperbolic functions. Since

\[
e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta
\]

we deduce that

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\]  \hspace{1cm} (1)

In Chapter 8, we defined the hyperbolic function by

\[
\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}
\]  \hspace{1cm} (2)

Comparing (1) and (2), we have

\[
\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x
\]

\[
\sinh ix = \frac{e^{ix} - e^{-ix}}{2i} = i \sin x
\]

so that

\[
tanh ix = i \tan x.
\]
Also,

\[
\cos ix = \frac{e^{ix} + e^{-ix}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x
\]

\[
\sin ix = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{-x} - e^x}{2} = \sinh x
\]

so that

\[
\tan ix = i \tanh x.
\]

Using these results, we can evaluate functions such as \(\sin z\), \(\cos z\), \(\tan z\), \(\sinh z\), \(\cosh z\) and \(\tanh z\).

For example, to evaluate

\[
\cos z = \cos(x + iy)
\]

we use the identity

\[
\cos(A + B) = \cos A \cos B - \sin A \sin B
\]

and obtain

\[
\cos z = \cos x \cos iy - \sin x \sin iy
\]

Using results in (3), this gives

\[
\cos z = \cos x \cosh y - i \sin x \sinh y.
\]
Example: Find the values of

a) $\sinh(3 + 4i)$

b) $\tan\left(\frac{\pi}{4} - 3i\right)$

c) $\sin\left(\frac{\pi}{4}(1 + i)\right)$

(Ans: a) -6.548-7.619i b) 0.005-1.0i c) 0.9366+0.6142i)
Pascal’s Triangle

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
1 & 10 & 45 & 120 & 200 & 252 & 200 & 120 & 45 & 10 & 1 \\
\end{array}
\]

\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,\]
\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,\]
\[(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,\]
\[(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,\]
\[(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.\]

In general:

\[
(x + y)^n = c_1 x^n y^0 + c_2 x^{n-1} y^1 + \ldots + c_{n+1} x^0 y^n
\]