QUESTION 1 (6 MARKS)

By using only the **identities of hyperbolic functions**, prove that

$$\sinh(3x) = 4\sinh^3 x + 3\sinh x.$$

(6 marks)

QUESTION 2 (5 MARKS)

Find $\dfrac{dy}{dx}$ for the following equation

$$e^x \tanh(x+y) + \sinh^{-1}(\tan(2y)) = 0.$$

(5 marks)

QUESTION 3 (6 MARKS)

Solve the following integral

$$\int \frac{1}{x\sqrt{4x^4 - 1}}\, dx.$$

(6 marks)

QUESTION 4 (7 MARKS)

Determine whether the following integrals converge or diverge.

a) $\displaystyle\int_0^\infty xe^{-3x}\, dx.$

(3 marks)

b) $\displaystyle\int_1^3 \frac{1}{(x-2)^2}\, dx.$

(4 marks)

QUESTION 5 (8 MARKS)

Find the first five terms of the Maclaurin series expansion for $f(x) = e^x$.
(3 marks)

i. Hence, show the Maclaurin series expansion for $g(x) = e^{2x^3}$ is given by

$$e^{2x^3} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{3n}.$$

(2 marks)

ii. From (i), evaluate the following integral

$$\int_0^1 \left(e^{2x^3} - 2x^3 - 1 \right) \, dx.$$

(3 marks)

QUESTION 6 (8 MARKS)

Let π be the plane that contains the points $A(-3, -2, 5), B(0, 1, 7)$ and $C(4, -1, 2)$.

i. Find the angle of BCA.

(3 marks)

ii. Find the equation of the plane π.

(3 marks)

iii. Find the shortest distance between a point $Q(5, 8, 10)$ and the plane π.
(2 marks)

QUESTION 7 (20 MARKS)

a) Given matrix

$$\mathbf{P} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

i. Find an inverse matrix of $\mathbf{P}$ using elementary row operations (ERO).

(7 marks)

ii. Hence, solve the following system of linear equations by using result in (i)

$$\begin{aligned} -x - y + z &= 5, \\ y + z &= -3, \\ x + z &= 1. \end{aligned}$$

(4 marks)

b) Determine the eigenvalues for the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Hence, find the eigenvector for the largest eigenvalue.

(9 marks)

QUESTION 8 (20 MARKS)

a) Show the Cartesian equation from the polar equation $r = 8\sin\theta$ is a circle of radius 4 and centred at (0,4).

(4 marks)

b) Given the polar equation $r^2 = 4\cos 2\theta$.

i. Test the symmetries of the above polar equation.

(6 marks)

ii. Copy and complete the following table:

θ°	0	10	20	30	40	45
r				$\pm\sqrt{2}$		

Hence, sketch the graph for $r^2 = 4\cos 2\theta$.
(Use the polar grid provided).

(4 marks)

iii. Sketch the graph $\tan\theta = \dfrac{1}{\sqrt{3}}$ on the same diagram.

(2 marks)

iv. Find the points of intersection between the curves $r^2 = 4\cos 2\theta$ and $\tan\theta = \dfrac{1}{\sqrt{3}}$.

(4 marks)

QUESTION 9 (20 MARKS)

a) Given $u = 2 - i$ and $v = i - 3$. Express $\dfrac{u}{u^2 + v}$ in the form $a + ib$. Hence, determine the modulus and argument of $\dfrac{u}{u^2 + v}$.

(5 marks)

b) Given a complex number $z = 8i$.

i. Express z in polar form.

(2 marks)

ii. Find all possible values of $z^{\frac{1}{3}}$ and sketch them on an Argand diagram.

(6 marks)

c) Use De Moivre's Theorem to show

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

Hence, obtain all solutions of x for the following equation

$$3x - 4x^3 = 1.$$

(7 marks)

Trigonometric
$\cos^2 x + \sin^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$\cot^2 x + 1 = \text{cosec}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$
$2 \sin x \sin y = -\cos(x + y) + \cos(x - y)$
$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

Logarithm
$a^x = e^{x \ln a}$
$\log_a x = \dfrac{\log_b x}{\log_b a}$

Hyperbolic
$\sinh x = \dfrac{e^x - e^{-x}}{2}$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$
$\cosh^2 x - \sinh^2 x = 1$
$1 - \tanh^2 x = \text{sech}^2 x$
$\coth^2 x - 1 = \text{cosech}^2 x$
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sinh 2x = 2 \sinh x \cosh x$
$\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$
$\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Inverse Hyperbolic
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}),\ -\infty < x < \infty$
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),\ x \geq 1$
$\tanh^{-1} x = \dfrac{1}{2} \ln \left(\dfrac{1 + x}{1 - x} \right),\ -1 < x < 1$

Differentiations	**Integrations**
$\frac{d}{dx}[k] = 0$, k constant.	$\int k\,dx = kx + C$, k constant.
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.
$\frac{d}{dx}[e^x] = e^x$.	$\int e^x dx = e^x + C$.
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{x}$.	$\int \frac{dx}{x} = \ln\|x\| + C$.
$\frac{d}{dx}[\cos x] = -\sin x$.	$\int \sin x\,dx = -\cos x + C$.
$\frac{d}{dx}[\sin x] = \cos x$.	$\int \cos x\,dx = \sin x + C$.
$\frac{d}{dx}[\tan x] = \sec^2 x$.	$\int \sec^2 x\,dx = \tan x + C$.
$\frac{d}{dx}[\cot x] = -\mathrm{cosec}^2 x$.	$\int \mathrm{cosec}^2 x\,dx = -\cot x + C$.
$\frac{d}{dx}[\sec x] = \sec x \tan x$.	$\int \sec x \tan x\,dx = \sec x + C$.
$\frac{d}{dx}[\mathrm{cosec}\, x] = -\mathrm{cosec}\, x \cot x$.	$\int \mathrm{cosec}\, x \cot x\,dx = -\mathrm{cosec}\, x + C$.
$\frac{d}{dx}[\cosh x] = \sinh x$.	$\int \sinh x\,dx = \cosh x + C$.
$\frac{d}{dx}[\sinh x] = \cosh x$.	$\int \cosh x\,dx = \sinh x + C$.
$\frac{d}{dx}[\tanh x] = \mathrm{sech}^2 x$.	$\int \mathrm{sech}^2 x\,dx = \tanh x + C$.
$\frac{d}{dx}[\coth x] = -\mathrm{cosech}^2 x$.	$\int \mathrm{cosech}^2 x\,dx = -\coth x + C$.
$\frac{d}{dx}[\mathrm{sech}\, x] = -\mathrm{sech}\, x \tanh x$.	$\int \mathrm{sech}\, x \tanh x\,dx = -\mathrm{sech}\, x + C$.
$\frac{d}{dx}[\mathrm{cosech}\, x] = -\mathrm{cosech}\, x \coth x$.	$\int \mathrm{cosech}\, x \coth x\,dx = -\mathrm{cosech}\, x + C$.

Differentiations of Inverse Functions
$\dfrac{d}{dx}[\sin^{-1} x] = \dfrac{1}{\sqrt{1-x^2}},\ \lvert x\rvert < 1.$
$\dfrac{d}{dx}[\cos^{-1} x] = \dfrac{-1}{\sqrt{1-x^2}},\ \lvert x\rvert < 1.$
$\dfrac{d}{dx}[\tan^{-1} x] = \dfrac{1}{1+x^2}.$
$\dfrac{d}{dx}[\cot^{-1} x] = \dfrac{-1}{1+x^2}.$
$\dfrac{d}{dx}[\sec^{-1} x] = \dfrac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1.$
$\dfrac{d}{dx}[\mathrm{cosec}^{-1} x] = \dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1.$
$\dfrac{d}{dx}[\sinh^{-1} x] = \dfrac{1}{\sqrt{x^2+1}}.$
$\dfrac{d}{dx}[\cosh^{-1} x] = \dfrac{1}{\sqrt{x^2-1}},\ \lvert x\rvert > 1.$
$\dfrac{d}{dx}[\tanh^{-1} x] = \dfrac{1}{1-x^2},\ \lvert x\rvert < 1.$
$\dfrac{d}{dx}[\coth^{-1} x] = \dfrac{1}{1-x^2},\ \lvert x\rvert > 1.$
$\dfrac{d}{dx}[\mathrm{sech}^{-1} x] = \dfrac{-1}{x\sqrt{1-x^2}},\ 0 < x < 1.$
$\dfrac{d}{dx}[\mathrm{cosech}^{-1} x] = \dfrac{-1}{\lvert x\rvert\sqrt{1+x^2}},\ x \neq 0.$

Integrations Resulting in Inverse Functions
$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C.$
$\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C.$
$\displaystyle\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \sec^{-1}(x) + C.$
$\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}(x) + C.$
$\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + C,\ x > 0.$
$\displaystyle\int \frac{dx}{1-x^2} = \tanh^{-1} x + C,\ \lvert x\rvert < 1.$
$\displaystyle\int \frac{dx}{x^2-1} = \coth^{-1} x + C,\ \lvert x\rvert > 1.$
$\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\mathrm{sech}^{-1}(x) + C,\ \lvert x\rvert < 1.$
$\displaystyle\int \frac{dx}{\lvert x\rvert\sqrt{1+x^2}} = -\mathrm{cosech}^{-1} x + C,\ x \neq 0.$

(Detach this sheet and attach to your answer booklet)

NAME: ....

LECTURER'S NAME: ....

SECTION: ....