Numerical Methods in Engineering
MSJ 1533

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Meshless Element Free Galerkin (EFG)

Meshless Element Free Galerkin (EFG)

Finite difference (FD) is the simplest method for the solution of boundary value problems. However it suffers the problem of low accuracy solutions

For the problem of boundary value problems in two-dimension and three-dimension when the geometry of the problems are complex with large deformation of structures, **Finite Element method (FEM)** will suffers the problem of ill condition element shape function and fail to produce reliable solutions. It require remeshing process.

Meshless Element Free Galerkin (MEFG) Method which does not depend on the element and mesh. It did not facing the same problem induced by large deformation of materials.

Meshless Element Free Galerkin (EFG)

Moving least-Square (MLS) approximants

MLS approximation $u^h(x)$ of the function u(x) is posed as polynomial of order m with non-constant coefficients

$$u^h(x)=a_0(x)+a_1(x)x \qquad \text{(linear basis)}$$
$$u^h(x)=a_0(x)+a_1(x)x+a_2(x)x^2 \qquad \text{(quadratic basis)}$$

$$u^h(x,\bar{x})=\sum_{j=0}^{m} p_j(x)\cdot a_j(\bar{x})=\mathbf{P}^T(x)\cdot\mathbf{a}(\bar{x})$$

$\bar{x}$ is node value

where p(x) is complete polynomial of order m

$$\mathbf{a}^T(x)=[a_0(x) \quad a_1(x) \quad a_2(x) \quad \cdots \quad a_m(x)]$$

The unknown parameters $a_j(x)$ at any given point are determined by minimizing the difference between the local approximations at that point and the nodal parameters, u_j.

$$J=\sum_{i=1}^{n} w(x-x_i)\cdot\left(u^h(x_i,x)-u_i\right)^2=\sum_{i=1}^{n} w(x-x_i)\cdot\left(\mathbf{P}^T(x_i)\cdot\mathbf{a}(x)-u_i\right)^2$$

where local approximation $u^h(x_i,x)=\sum_{j=0}^{m} p_j(x_i)\cdot a_j(x)=\mathbf{P}^T(x_i)\cdot\mathbf{a}(x)$

Meshless Element Free Galerkin (EFG)

Moving least-Square (MLS) approximants

$$J = \sum_{i=1}^{n} w(x-x_i)\cdot\left(\mathbf{P}^T(x_i)\cdot\mathbf{a}(x)-u_i\right)\cdot\left(\mathbf{P}^T(x_i)\cdot\mathbf{a}(x)-u_i\right)$$
$$= \sum_{i=1}^{n} w(x-x_i)\cdot\left[\mathbf{P}^T(x_i)\mathbf{a}(x)\cdot\mathbf{P}^T(x_i)\mathbf{a}(x)-2u_i\mathbf{P}^T(x_i)\mathbf{a}(x)+u_i^2\right]$$
$$= \sum_{i=1}^{n} w(x-x_i)\cdot\left[\mathbf{a}^T(x)\mathbf{P}(x_i)\mathbf{P}^T(x_i)\mathbf{a}(x)-2u_i\mathbf{P}^T(x_i)\mathbf{a}(x)+u_i^2\right]$$

Minimize the value of J with respect to $\mathbf{a}(x)$, we get

$$\frac{d}{d\{\mathbf{a}\}}J = \frac{d}{d\{\mathbf{a}\}}\sum_{i=1}^{n} w(x-x_i)\cdot\left[\mathbf{a}^T(x)\mathbf{P}(x_i)\mathbf{P}^T(x_i)\mathbf{a}(x)-2u_i\mathbf{P}^T(x_i)\mathbf{a}(x)+u_i^2\right]$$
$$= \sum_{i=1}^{n} w(x-x_i)\cdot\left[\mathbf{P}(x_i)\mathbf{P}^T(x_i)\mathbf{a}(x)+\left[\mathbf{P}(x_i)\mathbf{P}^T(x_i)\right]^T\cdot\mathbf{a}(x)-2u_i\left[\mathbf{P}^T(x_i)\right]^T\right]$$
$$= \sum_{i=1}^{n} w(x-x_i)\cdot\left[2\mathbf{P}(x_i)\mathbf{P}^T(x_i)\mathbf{a}(x)-2u_i\mathbf{P}(x_i)\right]$$

y (scalar or a vector)	$\partial y/\partial\mathbf{x}$
A$\mathbf{x}$	A
$\mathbf{x}^T$A	A
$\mathbf{x}^T\mathbf{x}$	2$\mathbf{x}$
$\mathbf{x}^T$A$\mathbf{x}$	A$\mathbf{x}$+A$^T\mathbf{x}$

given $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\ \mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

We get

$$\frac{\partial}{\partial\mathbf{x}}A\mathbf{x} = \frac{\partial}{\partial\mathbf{x}}\begin{bmatrix}(x_1+2x_2)\\ (3x_1+4x_2)\end{bmatrix} = \begin{bmatrix}\partial(x_1+2x_2)/\partial x_1 & \partial(x_1+2x_2)/\partial x_2\\ \partial(3x_1+4x_2)/\partial x_1 & \partial(3x_1+4x_2)/\partial x_2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = A$$

$$\frac{\partial}{\partial\mathbf{x}}\mathbf{x}^T A = \frac{\partial}{\partial\mathbf{x}}\left[(x_1+3x_2)\quad(2x_1+4x_2)\right] = \begin{bmatrix}\partial(x_1+3x_2)/\partial x_1 & \partial(2x_1+4x_2)/\partial x_1\\ \partial(x_1+3x_2)/\partial x_2 & \partial(2x_1+4x_2)/\partial x_2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = A$$

Using the Taylor's series expansion, we get

$$G(Y+H) \approx G(Y)+G'(Y)H+\dots,$$

where $H=(h_1,h_2,\dots,h_n)^T$ and $G'(Y)$ is the $n\times n$ Jacobian matrix $J(Y)$:

$$\mathbf{J}(\mathbf{Y}) = \mathbf{G}'(\mathbf{Y}) = \frac{\partial\mathbf{G}}{\partial\mathbf{Y}} = \begin{bmatrix}\partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \cdots & \partial g_1/\partial y_n\\ \partial g_2/\partial y_1 & \partial g_2/\partial y_2 & \cdots & \partial g_2/\partial y_n\\ \vdots & \vdots & \ddots & \vdots\\ \partial g_n/\partial y_1 & \partial g_n/\partial y_2 & \cdots & \partial g_n/\partial y_n\end{bmatrix}$$

$$(\mathbf{x}+\mathbf{h})^T\mathbf{A} \approx (\mathbf{x})^T\mathbf{A} + \mathbf{h}^T\mathbf{G}'(\mathbf{x})+\dots$$
$$\mathbf{G}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}$$

Meshless Element Free Galerkin (EFG)

Moving least-Square (MLS) approximants

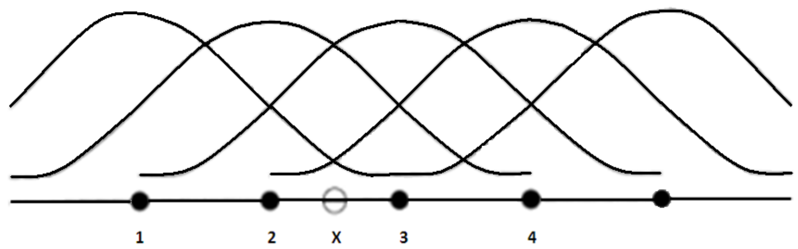
Let $\frac{d}{d\{\mathbf{a}\}} J = 0$, we get

$$\sum_{i=1}^{n} w(x - x_i) \cdot 2\mathbf{P}(x_i)\mathbf{P}^T(x_i)\mathbf{a}(x) = \sum_{i=1}^{n} w(x - x_i) \cdot 2u_i\mathbf{P}(x_i)$$

$$\sum_{i=1}^{n} w(x - x_i) \cdot \mathbf{P}(x_i)\mathbf{P}^T(x_i)\mathbf{a}(x) = \sum_{i=1}^{n} w(x - x_i) \cdot \mathbf{P}(x_i)u_i$$

$$\Downarrow$$

$$\mathbf{A}_{2\times2}(x)\mathbf{a}_{2\times1}(x) = \mathbf{B}_{2\times n}(x)\mathbf{u}_{n\times1}$$



***w*(*x*–*x*$_i$)** : weight function, Overlapping domain Influence and local node numbering at point *x*

$$\mathbf{A}_{2\times2}(x) = \sum_{i=1}^{n} w(x - x_i)\mathbf{P}(x_i)\mathbf{P}^T(x_i)$$

$$= w(x - x_1)\begin{bmatrix} 1 & x_1 \\ x_1 & x_1^2 \end{bmatrix} + w(x - x_2)\begin{bmatrix} 1 & x_2 \\ x_2 & x_2^2 \end{bmatrix} + \cdots + w(x - x_n)\begin{bmatrix} 1 & x_n \\ x_n & x_n^2 \end{bmatrix}$$

$$\mathbf{B}_{2\times n}(x) = \left[w(x - x_1)\mathbf{P}(x_1) \quad \cdots \quad w(x - x_n)\mathbf{P}(x_n) \right] = \left[w(x - x_1)\begin{bmatrix} 1 \\ x_1 \end{bmatrix} \quad \cdots \quad w(x - x_n)\begin{bmatrix} 1 \\ x_n \end{bmatrix} \right]$$

We pre-multiply both of equation, we get

$$\mathbf{A}_{2\times2}^{-1}(x)\mathbf{A}_{2\times2}(x)\mathbf{a}_{2\times1}(x) = \mathbf{A}_{2\times2}^{-1}(x)\mathbf{B}_{2\times n}(x)\mathbf{u}_{n\times1}$$

$$\mathbf{a}_{2\times1}(x) = \mathbf{A}_{2\times2}^{-1}(x)\mathbf{B}_{2\times n}(x)\mathbf{u}_{n\times1}$$

Meshless Element Free Galerkin (EFG)

Moving least-Square (MLS) approximants

By substituting into local approximation, we get

$$u^h(x_i, x) = \mathbf{P}^T_{1\times2}(x_i)\cdot \mathbf{a}_{2\times1}(x)$$

$$u^h(x) = \sum_{i=1}^{n} \phi_i(x) u_i = \mathbf{\Phi}(x)\mathbf{u} \qquad \text{(a)}$$

In contrast to FEM, shape functions In (a) are only approximants and not Interpolant, $u_i \neq u(\mathbf{x}_i)$. Special techniques Are needed to treat displacement BC.

where the shape functions, $\Phi_i(x)$ is defined by

$$\underbrace{\Phi_i(x)}_{\text{scalar}} = \mathbf{P}^T_{1\times2}\mathbf{A}^{-1}_{2\times2}\mathbf{B}_{2\times i} = \mathbf{P}^T\mathbf{A}^{-1}\underbrace{\mathbf{B}_i}_{\text{Column-}i}$$

The spatial derivative of shape functions, $\Phi_i(x)$ is obtained by

$$\Phi_{i,x} = \left(\mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_i\right)_x = \mathbf{P}^T_x\mathbf{A}^{-1}\mathbf{B}_i + \mathbf{P}^T\left(\mathbf{A}^{-1}\right)_x\mathbf{B}_i + \mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_{i,x}$$

Where,

$$\mathbf{B}_{i,x}(x) = \frac{dw(x-x_i)}{dx}\mathbf{P}(x_i)$$

And $\mathbf{A}^{-1}_x$ is computed by using matrix property as below:

$$\mathbf{A}^{-1}_x = -\mathbf{A}^{-1}\mathbf{A}_x\mathbf{A}^{-1}$$

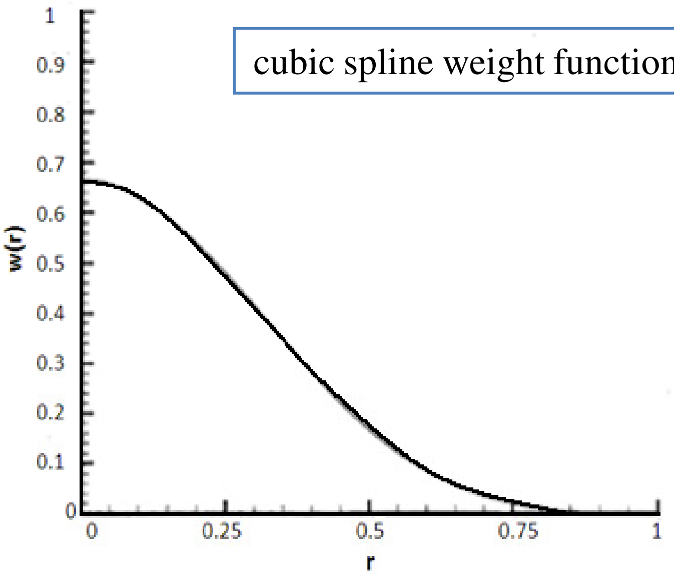
Where,

$$\mathbf{A}_x = \frac{dw(x-x_1)}{dx}\begin{bmatrix}1 & x_1\\ x_1 & x_1^2\end{bmatrix} + \frac{dw(x-x_2)}{dx}\begin{bmatrix}1 & x_2\\ x_2 & x_2^2\end{bmatrix} + \cdots + \frac{dw(x-x_n)}{dx}\begin{bmatrix}1 & x_n\\ x_n & x_n^2\end{bmatrix}$$

Meshless Element Free Galerkin (EFG)

Moving least-Square (MLS) approximants

The **weight function**, **window function** or **kernel** should be non-zero only over a small neighborhood of x_i, is called the domain of influence of node, in order to generate a set of sparse discrete equations



cubic spline weight function (symmetry on r) over normalized radius r

$$w(x-x_I)=w(r)=\begin{cases} \tfrac{2}{3}-4r^2+4r^3 & ,r\le \tfrac{1}{2} \\ \tfrac{4}{3}-4r+4r^2-\tfrac{4}{3}r^3 & ,\tfrac{1}{2}\le r\le 1 \\ 0 & ,r\ge 1 \end{cases}$$

$$\int_0^1 w(r)dr = \tfrac{1}{4},\ w(-r)=w(r), w'(-r)=w'(r)$$

$$w(r=0)=\tfrac{2}{3},\ w(r=\tfrac{1}{2})=\tfrac{1}{6},\ w(r\ge 1)=0$$

$$w'(r=0)=0,\ w'(r=\tfrac{1}{2})=-\tfrac{1}{dm_I}, w'(r=-\tfrac{1}{2})=\tfrac{1}{dm_I}, w'(r\ge 1)=0$$

Where $r=d_I/dm_I$ where dm_I is the size of the domain of influence of the I^{th} node.

$$r=r_I=\frac{|x-x_I|}{dm_I}$$

Using chain rule, we get

$$sign(x)=\begin{cases} -1 & ,x<0 \\ 0 & ,x=0 \\ 1 & ,x>0 \end{cases}$$

$$w_I'(r)=\frac{dw_I}{dx}=\frac{dw_I}{dr}\frac{dr}{dx}=\frac{1}{dm_I}\begin{cases} (-8r+12r^2)sign(x-x_I) & ,r\le \tfrac{1}{2} \\ (-4+8r-4r^2)sign(x-x_I) & ,\tfrac{1}{2}\le r\le 1 \\ 0 & ,r>1 \end{cases}$$

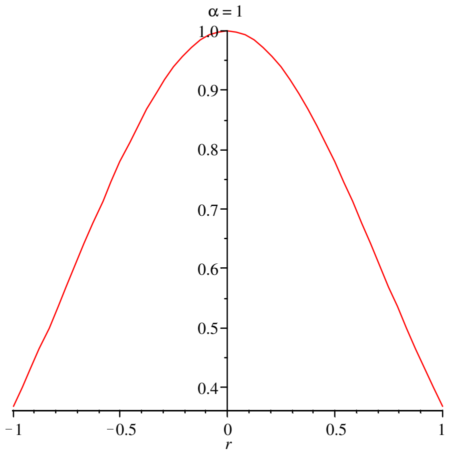
Meshless Element Free Galerkin (EFG)

Moving least-Square (MLS) approximants

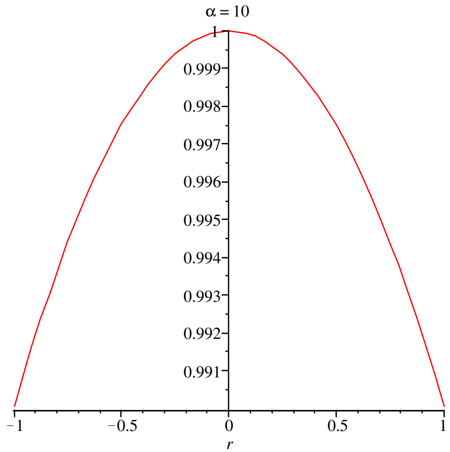
Other **weight function**, **window function** or **kernel** is given as below:

exponential weight function (symmetry on *r*) over normalized radius *r*

$$w(x-x_I)=w(r)=\begin{cases} e^{-\left(\frac{r}{\alpha}\right)^2} & ,|r|\le 1 \\ 0 & ,|r|>1 \end{cases}$$



$$\int_{r=-1}^{1} w(r)dr = 1.493648266$$



$$\int_{r=-1}^{1} w(r)dr = 1.993353286$$

$$r = r_I = \frac{|x-x_I|}{dm_I}$$

1-D ordinary differential equation - EFG

Consider the following one-dimensional problem on the domain $0 \le x \le 1$

$$Eu_{xx} + b = 0$$

where $u(x)$ is the displacement, E is the Young's modulus and b is body force per unit volume. Their boundaries can be written as:

$$Eu_x\Big|_{x=\Gamma_t} = \bar{t} \quad (Natural\, boundary)$$

$$u\Big|_{x=\Gamma_u} = \bar{u} \quad (Essential\, boundary)$$

Let the function $I(x,\lambda)$ is given as:

$$I(u,\lambda) = \int_0^1 \left[\tfrac{E}{2}(u_x)^2 - bu \right] dx - \bar{t}u\Big|_{x=\Gamma_t} - \lambda(u - \bar{u})\Big|_{x=\Gamma u}$$

We minimize the function $I(x,\lambda)$ by using calculus of variations. We get

$$\delta I(u,\lambda) = 0 = \int_0^1 \left[Eu_x \delta u_x - b\delta u \right] dx - \bar{t}\,\delta u\Big|_{x=\Gamma_t} - \lambda \delta u\Big|_{x=\Gamma u} - \delta\lambda(u - \bar{u})\Big|_{x=\Gamma u}$$

The approximate solution u, test function δu are constructed as below:

$$u(x) = \sum_{i=1}^{n} \phi_I(x) u_i = \mathbf{\Phi}(x)\mathbf{u}$$

$$\delta u(x) = \mathbf{\Phi}(x)\delta u = \mathbf{\Phi}(x)\boldsymbol{\psi}$$

$$u_x = \left(\tfrac{d}{dx} \mathbf{\Phi} \right)\mathbf{u} = \mathbf{\Phi}_x \mathbf{u}$$

$$\delta u_x = \left(\tfrac{d}{dx} \mathbf{\Phi}(x) \right)\boldsymbol{\psi} = \mathbf{\Phi}_x \boldsymbol{\psi}$$

$$I(u) = \int_0^1 \left[\tfrac{E}{2}(u_x)^2 - bu \right] dx - \bar{t}u\Big|_{x=\Gamma_t}$$

$$\delta I(u) = 0 = \int_0^1 Eu_x(\delta u)_x\, dx - \int_0^1 b\,\delta u\, dx - \bar{t}\,\delta u(\Gamma_t)$$

$$0 = \delta u(Eu_x)\Big|_{\Gamma_u}^{\Gamma_t} - \int_0^1 \delta u(Eu_{xx})dx - \int_0^1 b\,\delta u\, dx - \bar{t}\,\delta u(\Gamma_t)$$

$$0 = (\bar{t})\delta u - \delta u(Eu_x)\Big|_{\Gamma_u} - \int_0^1 \delta u(Eu_{xx} + b)dx - \bar{t}\,\delta u(\Gamma_t)$$

$$\boxed{\lambda = -Eu_x\Big|_{x=\Gamma_u}}$$

1-D ordinary differential equation - EFG

$$\delta I(u,\lambda)=0=\int_0^1\left[Eu_x\delta u_x-b\,\delta u\right]dx-\bar{t}\,\delta u\Big|_{x=\Gamma_t}-\lambda\delta u\Big|_{x=\Gamma u}-\delta\lambda(u-\bar{u})\Big|_{x=\Gamma u}$$

$$\int_0^1\boldsymbol{\psi}^T\boldsymbol{\Phi}_x^T E\boldsymbol{\Phi}_x\mathbf{u}dx-\int_0^1\boldsymbol{\psi}^T\boldsymbol{\Phi}^T b dx-\boldsymbol{\psi}^T\boldsymbol{\Phi}^T\bar{t}\Big|_{\Gamma_t}-\delta\lambda(\boldsymbol{\Phi}(x)\mathbf{u}-\bar{u})\Big|_{\Gamma_u}-\boldsymbol{\psi}^T\boldsymbol{\Phi}^T\lambda\Big|_{\Gamma_u}=0$$

where, $\boldsymbol{\psi}^{T}$ is arbitrary and can be assumed as [1 0 0…0], [0 1 0…0], …, [0 0 0…1] (set as constant)

$$\boldsymbol{\psi}^T\int_0^1\boldsymbol{\Phi}_x^T E\boldsymbol{\Phi}_x\mathbf{u}dx-\boldsymbol{\psi}^T\boldsymbol{\Phi}^T\lambda\Big|_{\Gamma_u}=\boldsymbol{\psi}^T\int_0^1\boldsymbol{\Phi}^T b dx+\boldsymbol{\psi}^T\boldsymbol{\Phi}^T\bar{t}\Big|_{\Gamma_t}$$

$$\delta\lambda(\boldsymbol{\Phi}(x)\mathbf{u}-\bar{u})\Big|_{\Gamma_u}=0\ \rightarrow -\boldsymbol{\Phi}(x)\mathbf{u}\Big|_{x=\Gamma_u}=-\bar{u}\Big|_{x=\Gamma_u}$$

Let

$$K_{IJ}=\int_0^1\Phi_{I,x}^T E\Phi_{J,x}dx\ ,\ \mathbf{G}=-\Phi^T(x)\Big|_{x=\Gamma_u}=-\begin{bmatrix}\Phi_1(\Gamma_u)\\ \vdots\\ \Phi_n(\Gamma_u)\end{bmatrix},\ \mathbf{f}=\begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix},\ f_I=\Phi_I(x)\bar{t}\Big|_{x=\Gamma_t}+\int_0^1\Phi_I(x)b dx,\ \mathbf{q}=-\bar{u}\Big|_{x=\Gamma_u}$$

We get

$$\mathbf{Ku}+\mathbf{G}\lambda=\mathbf{f}\ ,\ \mathbf{G}^T\mathbf{u}=\mathbf{q}$$

$$\begin{bmatrix}\mathbf{K} & \mathbf{G}\\ \mathbf{G}^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \lambda\end{bmatrix}=\begin{bmatrix}\mathbf{f}\\ \mathbf{q}\end{bmatrix}$$

If $\Gamma_u = x_1$

$$\mathbf{u}=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$$

1-D ordinary differential equation - EFG

Gauss Quadrature:

$$\int_{-1}^{1} f(t)dt = 2f(t=0)$$ 1-points – exact at $2n$-1=1 order of polynomial

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$ 2-points – exact at $2n$-1=3 order of polynomial

$$\int_{-1}^{1} f(t)dt = \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right)$$ 3-points – exact at $2n$-1=5 order of polynomial

$$\int_a^b f(x)$$

$$\downarrow \quad x = \frac{(b-a)t + (a+b)}{2}$$

$$\int_{-1}^{1} f\left(\frac{(b-a)t + (a+b)}{2}\right)\frac{(b-a)}{2}dt$$

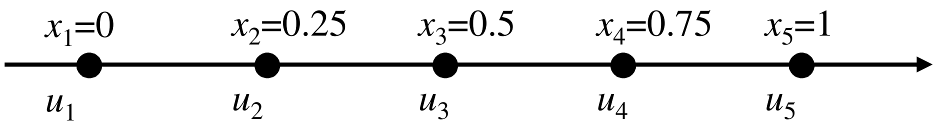
Example: solve the below for x=0(0.25)1

$Eu_{xx} + x = 0$, $\quad 0<x<1$

$u(0) = 0 = \bar{u}$ (essential boundary, $x=\Gamma_u=0$)

$u_x(1) = 0 = \bar{t}$ (natural boundary, $x=\Gamma_t=1$), Exact solution: $u(x)=[½x-x^3/6]/E$

$d_{max}=2$, $c_I=\Delta x=0.25$, $d_{mI}=d_{max}\times c_I=0.5$



$$\mathbf{A}(x) = w(x-x_1)\begin{bmatrix} 1 & x_1 \\ x_1 & x_1^2 \end{bmatrix} + \cdots + w(x-x_5)\begin{bmatrix} 1 & x_5 \\ x_5 & x_5^2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{11} = w(x-0) + \cdots + w(x-1),\ A_{12} = 0 + 0.25w(x-0.25) + \cdots + 1w(x-1)$$

$$A_{21} = A_{12},\ A_{22} = 0 + 0.25^2 w(x-0.25) + \cdots + 1^2 w(x-1)$$

$$\mathbf{P}^T(x) = [1 \quad x],\ \mathbf{P}_x^T(x) = [0 \quad 1]$$

$$\mathbf{B}_i(x) = w(x-x_i)\mathbf{P}(x_i) = w(x-x_i)\begin{bmatrix} 1 \\ x_i \end{bmatrix}$$

For explanation purpose, we use one integration with 1-points Gauss Quadrature

$$K_{IJ} = \int_0^1 \Phi_{I,x}^T E \Phi_{J,x} dx = \int_0^1 \alpha_{IJ}(x) dx$$

$$K_{IJ} = \frac{1}{2}\int_{-1}^{1} \alpha_{IJ}\left(\frac{t+1}{2}\right)dt = \frac{1}{2}\left(2\alpha_{IJ}\left(\frac{0+1}{2}\right)\right)$$

1-D ordinary differential equation - EFG

$$\mathbf{Ku}+\mathbf{G}\lambda=\mathbf{f}\ ,\ \mathbf{G}^T\mathbf{u}=\mathbf{q}$$

$$\begin{bmatrix}\mathbf{K} & \mathbf{G}\\ \mathbf{G}^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \lambda\end{bmatrix}=\begin{bmatrix}\mathbf{f}\\ \mathbf{q}\end{bmatrix}$$

$$K_{IJ}=\int_0^1\Phi_{I,x}^T E\Phi_{J,x}dx\ ,\ \mathbf{G}=-\Phi^T(x)\Big|_{x=\Gamma_u}=-\begin{bmatrix}\Phi_1(\Gamma_u)\\ \vdots\\ \Phi_N(\Gamma_u)\end{bmatrix},\ \mathbf{f}=\begin{bmatrix}f_1\\ \vdots\\ f_N\end{bmatrix},\ f_I=\Phi_I(x)\bar{t}\Big|_{x=\Gamma_t}+\int_0^1\Phi_I(x)b\,dx,\ \mathbf{q}=-\bar{u}\Big|_{x=\Gamma_u}$$

$$\delta u(\Gamma_u)=0=\Psi(0)$$

$$\mathbf{G}=-\Phi^T(x)\Big|_{x=\Gamma_u}=-\begin{bmatrix}\Phi_1(0)\\ \vdots\\ \Phi_5(0)\end{bmatrix}$$

$$\mathbf{q}=-\bar{u}\Big|_{x=\Gamma_u}=0$$

$$w(x_0-x_0)=w(r=0)=2/3,\ w(x_0-x_1)=w(r=1/2)=1/6$$
$$w(x_0-x_2)=w(r=1)=0,\ w(x_0-x_3)=w(r=1\tfrac{1}{2})=0,\cdots$$
$$w(-r)=w(r)$$

$$\Phi_i(x)=\mathbf{P}_{1\times2}^T\mathbf{A}_{2\times2}^{-1}\mathbf{B}_{2\times i}\ ,$$

$$\Phi_i(0)=\mathbf{P}^T(0)\{\mathbf{A}(0)\}^{-1}\mathbf{B}_i(0),\mathbf{P}^T(0)=[1\quad 0],\mathbf{P}^T(0.25)=[1\quad 0.25]$$

$$\mathbf{B}(x)=[w(x-x_1)\mathbf{P}(x_1)\ \ \cdots\ \ w(x-x_5)\mathbf{P}(x_5)]$$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$\mathbf{A}(0)=w(0-x_1)\begin{bmatrix}1 & x_1\\ x_1 & x_1^2\end{bmatrix}+\cdots+w(0-x_5)\begin{bmatrix}1 & x_5\\ x_5 & x_5^2\end{bmatrix}$$

$$=w(0-0)\begin{bmatrix}1 & 0\\ 0 & 0^2\end{bmatrix}+w(0-0.25)\begin{bmatrix}1 & 0.25\\ 0.25 & 0.25^2\end{bmatrix}+w(0-0.5)[\ \ ]+w(0-0.75)[\ \ ]+w(0-1)[\ \ ]$$

$$=2/3\begin{bmatrix}1 & 0\\ 0 & 0^2\end{bmatrix}+1/6\begin{bmatrix}1 & 0.25\\ 0.25 & 0.25^2\end{bmatrix}+0[\ \ ]+0[\ \ ]+0[\ \ ]=\begin{bmatrix}5/6 & 1/24\\ 1/24 & 1/96\end{bmatrix}\to[\mathbf{A}(0)]^{-1}=\begin{bmatrix}1.5 & -6\\ -6 & 120\end{bmatrix}$$

$$\mathbf{B}_1(0)=w(0-0)\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2/3\\ 0\end{bmatrix},\ \mathbf{B}_2(0)=w(0-0.25)\begin{bmatrix}1\\ 0.25\end{bmatrix}=\begin{bmatrix}1/6\\ 1/24\end{bmatrix},\ \mathbf{B}_3(0)=w(0-0.5)\begin{bmatrix}1\\ 0.5\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}=\mathbf{B}_4(0)=\mathbf{B}_5(0)\ \Longrightarrow\ \Phi_3(0)=\Phi_4(0)=\Phi_5(0)=0$$

1-D ordinary differential equation - EFG

$$\mathbf{G} = -\begin{bmatrix}\Phi_1(0)\\ \vdots \\ \Phi_5(0)\end{bmatrix},\ \Phi_i(x) = \mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_i\ ,\ \Phi_i(0) = \mathbf{P}^T(0)\{\mathbf{A}(0)\}^{-1}\mathbf{B}_i(0)$$

$$\Phi_1(0) = [1\quad 0]\begin{bmatrix}1.5 & -6\\ -6 & 120\end{bmatrix}\begin{bmatrix}2/3\\ 0\end{bmatrix} = 1,\ \Phi_2(0) = [1\quad 0]\begin{bmatrix}1.5 & -6\\ -6 & 120\end{bmatrix}\begin{bmatrix}1/6\\ 1/24\end{bmatrix} = 0,\ \Phi_3(0) = \Phi_4(0) = \Phi_5(0) = 0$$

$$\mathbf{G} = -\begin{bmatrix}\Phi_1(0)\\ \vdots \\ \Phi_5(0)\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix}f_1\\ \vdots \\ f_N\end{bmatrix},\ f_I = \Phi_I(x)\bar{t}\Big|_{x=\Gamma_t} + \int_0^1 \Phi_I(x)b\,dx = \int_0^1 \Phi_I(x)x\,dx$$

$\Phi_i(x) = \mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_i$ Use 1-point Gauss Quadrature: $x = \dfrac{t+1}{2}$ ⇨

$$f_I = \int_{-1}^1 \Phi_I\left(\frac{t+1}{2}\right)\frac{t+1}{2}\,\tfrac{1}{2}\,dt$$

$$= \frac{0+1}{2}\Phi_I\left(\frac{0+1}{2}\right)$$

$$\mathbf{A}\left(\frac{0+1}{2}\right) = w(1/2 - x_1)\begin{bmatrix}1 & x_1\\ x_1 & x_1^2\end{bmatrix} + \cdots + w(1/2 - x_5)\begin{bmatrix}1 & x_5\\ x_5 & x_5^2\end{bmatrix}$$

$$= w(r=1)\begin{bmatrix}1 & 0\\ 0 & 0^2\end{bmatrix} + w(r=0.5)\begin{bmatrix}1 & 0.25\\ 0.25 & 0.25^2\end{bmatrix} + w(r=0)\begin{bmatrix}1 & 0.5\\ 0.5 & 0.5^2\end{bmatrix} + w(r=-0.5)\begin{bmatrix}1 & 0.75\\ 0.75 & 0.75^2\end{bmatrix} + w(r=-1)\begin{bmatrix}1 & 1\\ 1 & 1^2\end{bmatrix}$$

$$= 0[\ \ ] + 1/6\begin{bmatrix}1 & 0.25\\ 0.25 & 0.25^2\end{bmatrix} + 2/3\begin{bmatrix}1 & 0.5\\ 0.5 & 0.5^2\end{bmatrix} + 1/6\begin{bmatrix}1 & 0.75\\ 0.75 & 0.75^2\end{bmatrix} + 0[\ \ ] = \begin{bmatrix}1 & 0.5\\ 0.5 & 0.2708\end{bmatrix} \rightarrow [\mathbf{A}(1/2)]^{-1} = \begin{bmatrix}13.0002 & -24.0004\\ -24.0004 & 48.0008\end{bmatrix}$$

1-D ordinary differential equation - EFG

$$\Phi_i(\tfrac{1}{2}) = \mathbf{P}(\tfrac{1}{2})^T[\mathbf{A}(\tfrac{1}{2})]^{-1}\mathbf{B}_i(\tfrac{1}{2}) = [1 \quad \tfrac{1}{2}]\begin{bmatrix} 13.0002 & -24.0004 \\ -24.0004 & 48.0008 \end{bmatrix} w(\tfrac{1}{2}-x_i)\begin{bmatrix} 1 \\ x_i \end{bmatrix}$$

$$\Phi_1(\tfrac{1}{2}) = 0,\ \Phi_2(\tfrac{1}{2}) = 0.16667,\ \Phi_3(\tfrac{1}{2}) = 0.66667,\ \Phi_4(\tfrac{1}{2}) = 0.16667,\ \Phi_5(\tfrac{1}{2}) = 0$$

$$f_1 = 0,\ f_2 = 0.08333,\ f_3 = 0.33333,\ f_4 = 0.08333,\ f_5 = 0$$

$$f_I = \int_{-1}^{1}\Phi_I\left(\frac{t+1}{2}\right)\frac{t+1}{2}\tfrac{1}{2}dt = \tfrac{1}{2}\Phi_I(\tfrac{1}{2})$$

$$\mathbf{f} = \begin{bmatrix} 0 \\ 0.08333 \\ 0.33333 \\ 0.08333 \\ 0 \end{bmatrix}$$

$$K_{IJ} = \int_0^1 \Phi_{I,x}^T E \Phi_{J,x} dx = E\,\tfrac{1}{2}\int_{-1}^{1}\Phi_{I,x}^T\left(\frac{t+1}{2}\right)\Phi_{J,x}\left(\frac{t+1}{2}\right)dt = E\Phi_{I,x}^T(\tfrac{1}{2})\Phi_{J,x}(\tfrac{1}{2})$$

$$\Phi_{i,x} = \left(\mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_i\right)_x = \mathbf{P}_x^T\mathbf{A}^{-1}\mathbf{B}_i + \mathbf{P}^T\left(\mathbf{A}^{-1}\right)_x\mathbf{B}_i + \mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_{i,x}$$

$$\mathbf{P}_x^T(x) = [0 \quad 1]$$

$$\mathbf{A}_x^{-1} = \left(\mathbf{A}^{-1}\right)_x = -\mathbf{A}^{-1}\mathbf{A}_x\mathbf{A}^{-1} \rightarrow \mathbf{A}_x^{-1}(\tfrac{1}{2}) = -\mathbf{A}^{-1}(\tfrac{1}{2})\mathbf{A}_x(\tfrac{1}{2})\mathbf{A}^{-1}(\tfrac{1}{2})$$

$$w'(\tfrac{1}{2}-x_1) = w'(r=1) = 0,\ w'(\tfrac{1}{2}-x_2) = w'(r=\tfrac{1}{2}) = -2$$
$$w'(\tfrac{1}{2}-x_3) = w'(r=0) = 0,\ w'(\tfrac{1}{2}-x_4) = w'(r=-\tfrac{1}{2}) = 2$$
$$w'(\tfrac{1}{2}-x_5) = w'(r=-1) = 0$$

$$\mathbf{A}_x = w'(x-x_1)\begin{bmatrix} 1 & x_1 \\ x_1 & x_1^2 \end{bmatrix} + w'(x-x_2)\begin{bmatrix} 1 & x_2 \\ x_2 & x_2^2 \end{bmatrix} + \cdots + w'(x-x_5)\begin{bmatrix} 1 & x_5 \\ x_5 & x_5^2 \end{bmatrix}$$

$$\mathbf{A}_x(\tfrac{1}{2}) = w'(\tfrac{1}{2}-0)\begin{bmatrix} 1 & 0 \\ 0 & 0^2 \end{bmatrix} + w'(\tfrac{1}{2}-0.25)\begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.25^2 \end{bmatrix} + \cdots + w'(\tfrac{1}{2}-1)\begin{bmatrix} 1 & 1 \\ 1_5 & 1^2 \end{bmatrix}$$
$$= 0[\ ] - 2\begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.25^2 \end{bmatrix} + 0[\ ] + 2\begin{bmatrix} 1 & 0.75 \\ 0.75 & 0.75^2 \end{bmatrix} + 0[\ ] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{B}_i(x) = w(x-x_i)\begin{bmatrix} 1 \\ x_i \end{bmatrix},\ \mathbf{B}_{i,x}(x) = w'(x-x_i)\begin{bmatrix} 1 \\ x_i \end{bmatrix}$$

$$\mathbf{A}(\tfrac{1}{2}) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.2708 \end{bmatrix} \rightarrow [\mathbf{A}(\tfrac{1}{2})]^{-1} = \begin{bmatrix} 13.0002 & -24.0004 \\ -24.0004 & 48.0008 \end{bmatrix}$$

$$\mathbf{A}_x^{-1}(\tfrac{1}{2}) = -\mathbf{A}^{-1}(\tfrac{1}{2})\mathbf{A}_x(\tfrac{1}{2})\mathbf{A}^{-1}(\tfrac{1}{2}) = \begin{bmatrix} 48 & -48 \\ -48 & 0 \end{bmatrix}$$

Use Ms Excel or calculator for matrix multiplication!

1-D ordinary differential equation - EFG

$$\Phi_{i,x}(1/2) = \mathbf{P}_x^T \mathbf{A}^{-1}\mathbf{B}_i\big|_{x=1/2} + \mathbf{P}^T\left(\mathbf{A}^{-1}\right)_x \mathbf{B}_i\big|_{x=1/2} + \mathbf{P}^T\mathbf{A}^{-1}\mathbf{B}_{i,x}\big|_{x=1/2}$$

$$\Phi_{i,x}(1/2) = [0 \quad 1]\begin{bmatrix} 13.0002 & -24.0004 \\ -24.0004 & 48.0008 \end{bmatrix}\mathbf{B}_i + [1 \quad 1/2]\begin{bmatrix} 48 & -48 \\ -48 & 0 \end{bmatrix}\mathbf{B}_i + [1 \quad 1/2]\begin{bmatrix} 13.0002 & -24.0004 \\ -24.0004 & 48.0008 \end{bmatrix}\mathbf{B}_{i,x}$$

$$K_{IJ} = \int_0^1 \Phi_{I,x}^T E \Phi_{J,x} dx = E\,1/2 \int_{-1}^1 \Phi_{I,x}^T\left(\frac{t+1}{2}\right)\Phi_{J,x}\left(\frac{t+1}{2}\right)dt = E\Phi_{I,x}^T(1/2)\Phi_{J,x}(1/2)$$

$$K_{11} = E\Phi_{1,x}^T(1/2)\Phi_{1,x}(1/2) = 0, \cdots, K_{22} = E\Phi_{2,x}^T(1/2)\Phi_{2,x}(1/2) = 4E, \cdots$$

$$\Phi_{1,x}(1/2) = 0,\ \Phi_{2,x}(1/2) = -2,\ \Phi_{3,x}(1/2) = 0,\ \Phi_{4,x}(1/2) = 2,\ \Phi_{5,x}(1/2) = 0$$

$$\mathbf{K} = E\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{q} = [0] \qquad \mathbf{G} = -\begin{bmatrix} \Phi_1(0) \\ \vdots \\ \Phi_5(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} \vdots \\ \int_{-1}^1 \Phi_I\left(\frac{t+1}{2}\right)\frac{t+1}{2}\frac{1}{2}dt \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0.08333 \\ 0.33333 \\ 0.08333 \\ 0 \end{bmatrix}$$

$$\mathbf{K}\mathbf{u} + \mathbf{G}\lambda = \mathbf{f}\ ,\ \mathbf{G}^T\mathbf{u} = \mathbf{q}$$

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{q} \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} Eu_1 \\ Eu_2 \\ Eu_3 \\ Eu_4 \\ Eu_5 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0.08333 \\ 0.33333 \\ 0.08333 \\ 0 \\ 0 \end{bmatrix}$$

$\Uparrow$

ill-condition of matrix due to
1-Point Gauss Quadrature integration with one integration

1-D ordinary differential equation - EFG

$$f_I = \int_{-1}^{1} \Phi_I\left(\frac{t+1}{2}\right)\frac{t+1}{2}\tfrac{1}{2}\,dt$$

$$= \frac{-\frac{1}{\sqrt{3}}+1}{4}\Phi_I\left(\frac{-\frac{1}{\sqrt{3}}+1}{2}\right) + \frac{\frac{1}{\sqrt{3}}+1}{4}\Phi_I\left(\frac{\frac{1}{\sqrt{3}}+1}{2}\right)$$

2-Points Gauss Quadrature in one integration

e.g.

$$\begin{bmatrix} 3.2525 & -1.4051 & -1.8478 & 0 & 0 & -1 \\ -1.4051 & 0.6070 & 0.7983 & 0 & 0 & 0 \\ -1.8478 & 0.7983 & 2.1102 & 0.8218 & -1.8451 & 0 \\ 0 & 0 & 0.8218 & 0.6369 & -1.4299 & 0 \\ 0 & 0 & -1.8451 & -1.4299 & 3.2106 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Eu_1 \\ Eu_2 \\ Eu_3 \\ Eu_4 \\ Eu_5 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0.02697 \\ 0.06806 \\ 0.05007 \\ 0.25419 \\ 0.1005 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} Eu_1 \\ Eu_2 \\ Eu_3 \\ Eu_4 \\ Eu_5 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -3070 \\ 2334.5 \\ 2672.3 \\ 2531.8 \\ 0.11059 \end{bmatrix}$$

Solution is not acceptable due to
2-Points Gauss Quadrature integration in one integration

Please note that real solution is obtained after applying interpolant.

$$u_{real}(x) = \sum_{i=1}^{n} \phi_i(x)u_i = \mathbf{\Phi}(x)\mathbf{u}$$

For actual calculation, we use sub-intervals integration with Gauss Quadrature

$$K_{IJ} = \int_0^1 \Phi_{I,x}^T E \Phi_{J,x}\,dx = \int_0^1 \alpha_{IJ}(x)\,dx = \int_{x_1}^{x_2} \alpha_{IJ}(x)\,dx + \int_{x_2}^{x_3} + \cdots$$

$$\int_{x_1}^{x_2} = \frac{x_2 - x_1}{2}\int_{-1}^{1} \alpha_{IJ}\left(\frac{(x_2-x_1)t + (x_2+x_1)}{2}\right)dt = \frac{x_2-x_1}{2}\left(\alpha_{IJ}\left(\frac{(x_2-x_1)\left(-\frac{1}{\sqrt{3}}\right)+(x_2+x_1)}{2}\right) + \alpha_{IJ}\left(\frac{(x_2-x_1)\left(\frac{1}{\sqrt{3}}\right)+(x_2+x_1)}{2}\right)\right)$$

1-D ordinary differential equation - EFG

For sub-interval integration with 2-p Gauss Quadrature

$$\mathbf{Ku}+\mathbf{G}\lambda=\mathbf{f}\ ,\ \mathbf{G}^T\mathbf{u}=\mathbf{q}$$

$$\begin{bmatrix}\mathbf{K} & \mathbf{G}\\ \mathbf{G}^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \lambda\end{bmatrix}=\begin{bmatrix}\mathbf{f}\\ \mathbf{q}\end{bmatrix}$$

$$\begin{bmatrix}3.0777 & -2.1766 & -0.8733 & -0.0278 & 0 & -1\\ -2.1766 & 3.4514 & -0.4137 & -0.8333 & -0.0278 & 0\\ -0.8733 & -0.4137 & 2.5739 & -0.4137 & -0.8733 & 0\\ -0.0278 & -0.8333 & -0.4137 & 3.4514 & -2.1766 & 0\\ 0 & -0.0278 & -0.8733 & -2.1766 & 3.0777 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}Eu_1\\ Eu_2\\ Eu_3\\ Eu_4\\ Eu_5\\ \lambda\end{bmatrix}=\begin{bmatrix}0.0156\\ 0.06255\\ 0.12475\\ 0.16711\\ 0.12999\\ 0\end{bmatrix}$$

$$\begin{bmatrix}Eu_1\\ Eu_2\\ Eu_3\\ Eu_4\\ Eu_5\\ \lambda\end{bmatrix}=\begin{bmatrix}0\\ 0.12518\\ 0.23262\\ 0.31679\\ 0.33341\\ -0.5\end{bmatrix}$$

Solution is acceptable!

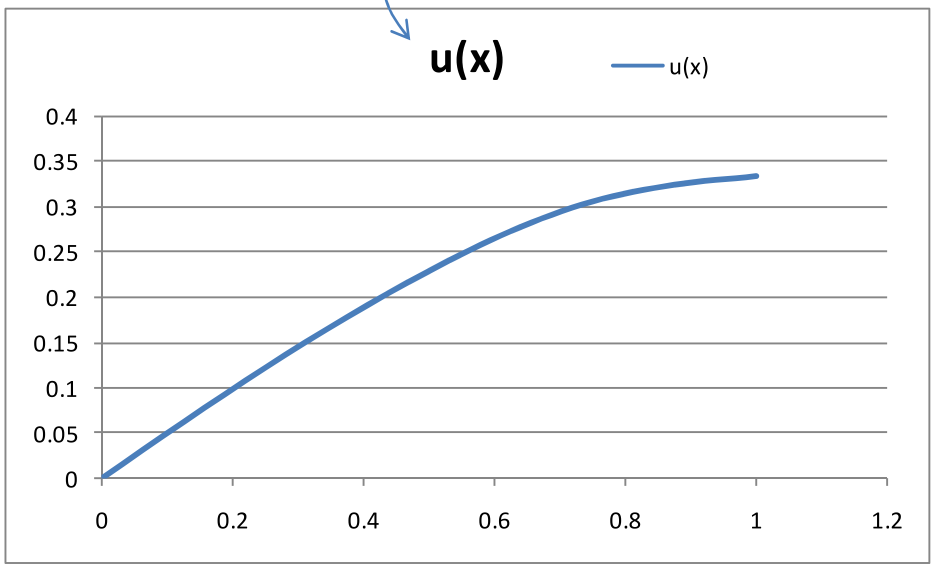
Please note that real solution is obtained after applying interpolant.

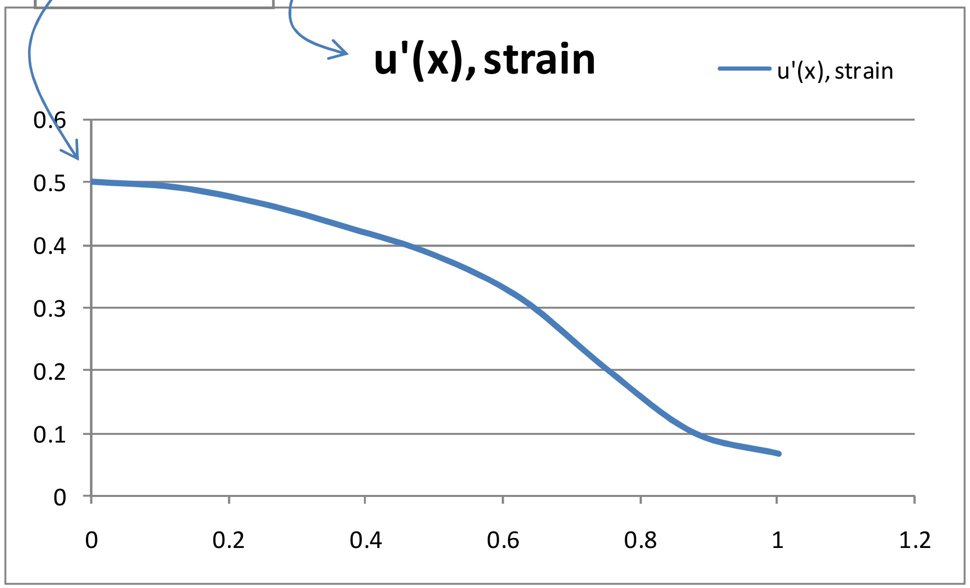
$$u_{real}(x)=\sum_{i=1}^{n}\phi_i(x)u_i=\mathbf{\Phi}(x)\mathbf{u}$$

$$u(x)=\sum_{i=1}^{n}\phi_I(x)u_i=\mathbf{\Phi}(x)\mathbf{u}$$

$$\lambda=-Eu_x\big|_{x=\Gamma_u}$$

$$u_x=\left(\frac{d}{dx}\mathbf{\Phi}\right)\mathbf{u}=\mathbf{\Phi}_{,x}\mathbf{u}$$

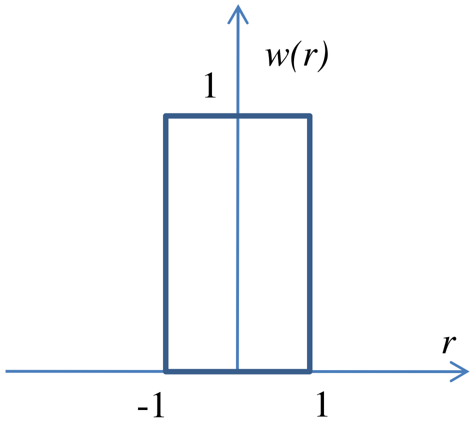




1-D ordinary differential equation - EFG

The **weight function**, **window function** or **kernel** should be non-zero only over a small neighborhood of x_i, is called the domain of influence of node, in order to generate a set of sparse discrete equations

Constant with compact support weight function (symmetry on r) over normalized radius r



$$w(x-x_I)=w(r)=\begin{cases}1, & r\le 1\\ 0, & r>1\end{cases}$$

$$r=r_I=\frac{|x-x_I|}{x_2-x_1}$$

$$\frac{d}{dx}w(x-x_I)=w'(r)=\begin{cases}0, & r\le 1\\ 0, & r>1\end{cases}$$