Numerical Methods in Engineering
MSJ 1533

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Finite Element Method (FEM)

Finite Element Method (FEM)

Comparison with the finite difference method (FDM)

The finite difference method (FDM) is an alternative way of approximating solutions of PDEs.
The differences between FEM and FDM are:

- The finite difference method is an approximation to the differential equation; the finite element method is an approximation to its solution.
- The most attractive feature of the FEM is its ability to handle complex geometries (and boundaries) with relative ease. While FDM in its basic form is restricted to handle rectangular shapes and simple alterations.
- The most attractive feature of finite differences is that it can be very easy to implement.
- The quality of the approximation between grid points is poor in FDM comparing to FEM.
- The quality of a FEM approximation is often higher than in the corresponding FDM approach, but this is extremely problem dependent and several examples to the contrary can be provided.

Generally, FEM is the method of choice in all types of analysis in **structural mechanics** while computational fluid dynamics (CFD) tends to use FDM or other methods (e.g., finite volume method). CFD problems usually require discretization of the problem into a **large number** of cells/gridpoints (millions and more), therefore cost of the solution favors **simpler**, **lower order approximation** within each cell.

Finite Element Method

Steady-state 2-D heat conduction

The heat flow through the wall of a heated room on a winter day is an example of conduction. In a thermally isotropic medium, Fourier's law for 2-D heat flow is:

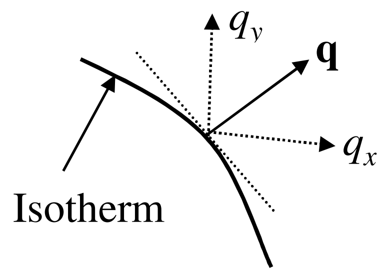
Index, not partial derivative!

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}$$

$T=T(x,y)$=temperature, q_x and q_y are components of heat flux (W/m^2), k is thermal conductivity (W/m. °C). (1W=1J/s=1Nm/s). Minus sign: heat is transferred in direction of decreasing temperature. k is material property.

$\mathbf{q}=q_x\mathbf{i}+q_y\mathbf{j}$, resultant heat flux (at right angles to an isotherm or a line of constant temperature.

$\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ are temperature gradients along x and y.



Constitutive relation- contains a material property.

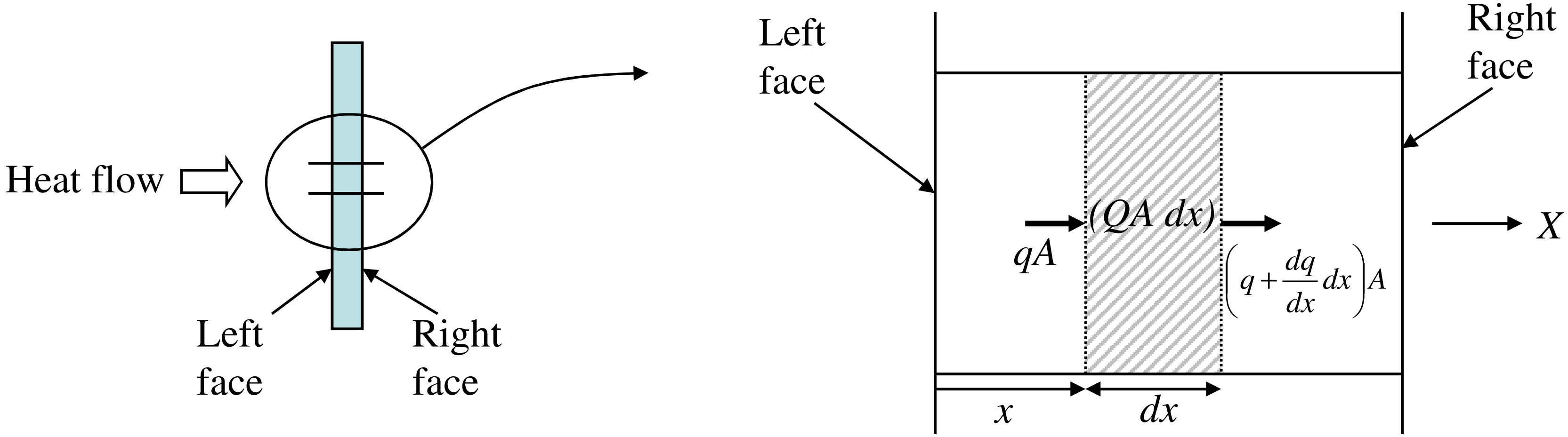
Convection-the flow of heat through a gas or a liquid

$q=h(T_s\text{-}T_\infty)$, q is convective heat flux (W/m^2), h is convection heat transfer coefficient or film coef (W/m$^2\cdot$°C), T_s and T_∞ are surface and fluid temperature.

Finite Element Method

Steady-state 1-D heat conduction

Governing equation (heat conduction in plane wall with uniform heat generation)



Let A =area normal to direction of heat flow,

Q(W/m^3) =internal heat generated per unit volume.

Heat rate (heat flux × area) enter the control volume **+** heat rate generated **=** Heat rate leaving control volume.

$$qA + QAdx = \left(q + \frac{dq}{dx}dx\right)A \xrightarrow{\text{simplify}} Q = \frac{dq}{dx}$$

$$q = -k\frac{small - big}{dx} = +ve$$

+ve=heat flux same direction with x-axis

Substitute Fourier's law $q = -k\dfrac{dT}{dx}$ $\Rightarrow$ $\dfrac{d}{dx}\left(k\dfrac{dT}{dx}\right) + Q = 0$

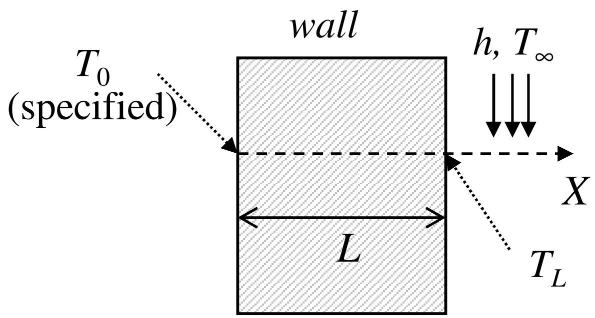
Q is called source when +ve (heat is generated) and is called sink when –ve (heat is consumed)

Here, Q is referred as source.

Finite Element Method

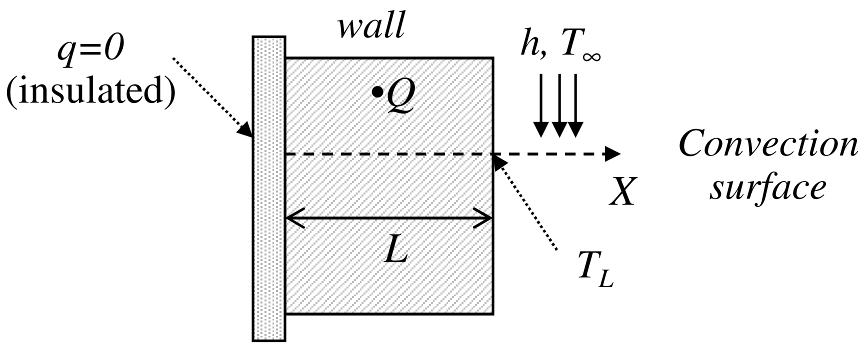
Steady-state 1-D heat conduction, Boundary conditions

Specified temperature



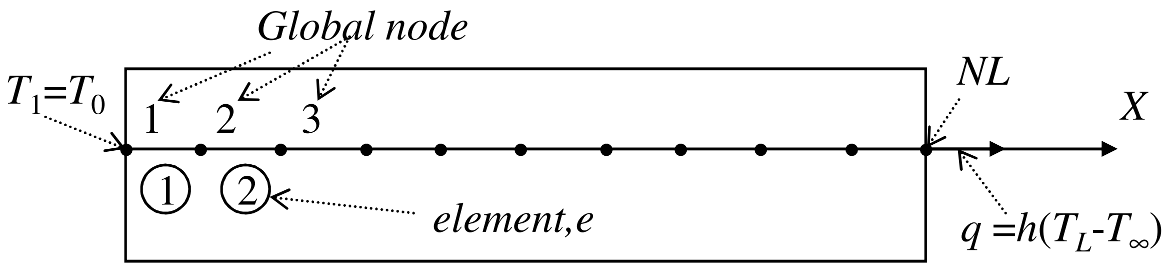
Wall of tank contain hot liquid at T_0,
airstream of T_∞ passed on outside,
maintain T_L at boundary.
$T|_{x=0} = T_0$, $q|_{x=L} = h(T_L - T_\infty)$. [note: $T_L > T_\infty$]

Specified heat flux



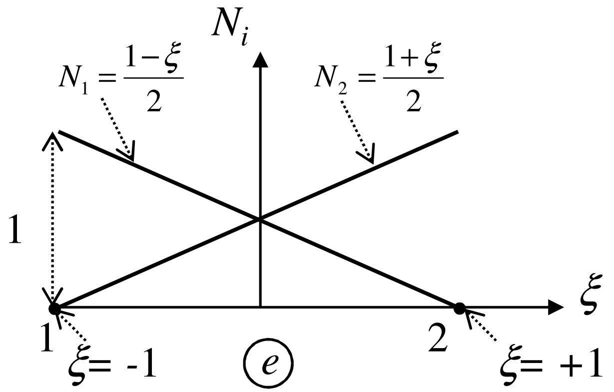
A wall where the inside surface is insulated
And outside is convection surface.
$q|_{x=0} = 0$, $q|_{x=L} = h(T_L - T_\infty)$.

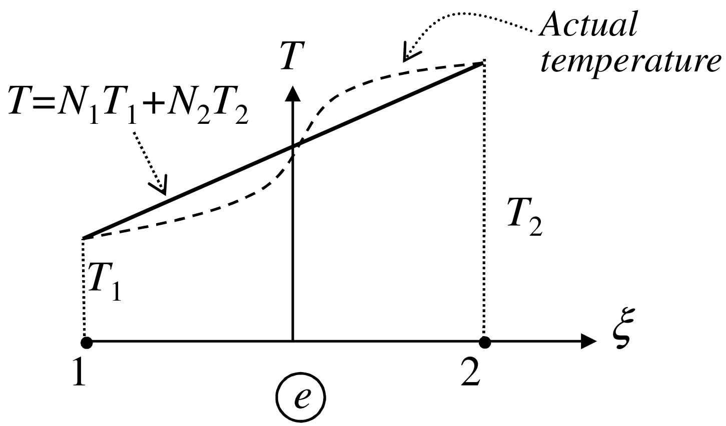
1-D element : two-node element with linear shape functions



Finite Element Method

1-D element





$$T(\xi) = N_1T_1 + N_2T_2 = \mathbf{N}\mathbf{T}^e$$

where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{T}^e=[T_1, T_2]^{\mathrm{T}}$.

Please note $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1,\ d\xi = \frac{2}{x_2 - x_1}dx = \frac{2}{l_e}dx.$

$$x = N_1x_1 + N_2x_2$$

$$x = \frac{(1-\xi)}{2}x_1 + \frac{(1+\xi)}{2}x_2$$

Use chain rule, $\frac{dT}{dx} = \frac{dT}{d\xi} \cdot \frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi} \cdot \mathbf{T}^e = \frac{1}{x_2 - x_1}[-1,1]\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e.$

where $\mathbf{B}_T = \frac{d}{dx}\mathbf{N} = \frac{1}{x_2 - x_1}[-1,\ 1] = \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}$

$$\int_e f dx = \int_{-1}^{1} f\ J d\xi\ ,\quad J = \frac{l_e}{2} = Jacobian$$

Finite Element Method

Galerkin's approach for heat conduction

Problem: $$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0 \qquad T|_{x=0}=T_0,\quad q|_{x=L}=h(T_L\text{-}T_\infty).$$

Assume: $$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q\right]dx=0$$

$\phi(x)$ constructed from same basis function of T, with $\phi(0)=0$. ϕ as a virtual temperature change that is consistent with boundary conditions.

Weighted-Residual Method

First term use integration by part: $$\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du \quad \Longrightarrow \quad \boxed{\phi k\frac{dT}{dx}\Big|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q dx = 0}$$

Now, $$\phi k\frac{dT}{dx}\Big|_0^L = \phi(L)k(L)\frac{dT}{dx}(L) - \phi(0)k(0)\frac{dT}{dx}(0)$$

Since, $q=-k\dfrac{dT}{dx}$ So, $\phi(0)=0$, $q(L)=-k(L)[dT(L)/dx]=h(T_L\text{-}T_\infty)$, we get $\phi k\dfrac{dT}{dx}\Big|_0^L = -\phi(L)h(T_L-T_\infty).$

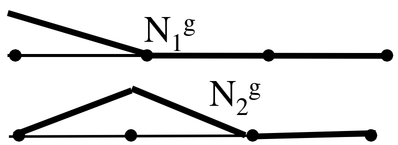
Finally, we get $$\boxed{-\phi(L)h(T_L-T_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q dx = 0} \Longleftarrow$$

Weak form – reduced (weakened) continuity of T

A global virtual-temperature vector is denoted: $\boldsymbol{\psi}=[\psi_1, \psi_2, \ldots, \psi_{NL}]^T$, or element-wise: $\boldsymbol{\psi}^e=[\psi_i, \psi_{i+1}]^T$.

The test function within each element is interpolated as: (global nodes) $\phi=\mathbf{N}\boldsymbol{\psi}$, or element-wise $\phi^e=\mathbf{N}^e\boldsymbol{\psi}^e$.

$$\frac{d\phi^e}{dx} = \frac{d}{dx}\phi^e = \frac{d}{dx}\left(\mathbf{N}^e\boldsymbol{\psi}^e\right) = \left(\frac{d\mathbf{N}^e}{d\xi}\frac{d\xi}{dx}\right)\cdot\boldsymbol{\psi}^e = \mathbf{B}_T\boldsymbol{\psi}^e$$



Finite Element Method

Galerkin's approach for heat conduction

<u>Some matrix concept</u>: $(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$.
Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$=row vector, $\mathbf{AB}^T$=scalar $\rightarrow$ $\mathbf{AB}^T=(\mathbf{AB}^T)^T$.
$\mathbf{AB}^T\mathbf{CD}^T=(\mathbf{AB}^T)^T\mathbf{CD}^T=\mathbf{B}^{TT}\mathbf{A}^T\mathbf{CD}^T=\mathbf{B}(\mathbf{A}^T\mathbf{C})\mathbf{D}^T$=scalar

$N_i(x_j)=\delta_{ij}$ (Kronecker delta function,global)

We get,

$$-\phi(L)h(T_L-T_\infty)-\int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx+\int_0^L \phi Q dx=0=-(\mathbf{N}(L)\boldsymbol{\psi})h(T_L-T_\infty)-\sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}}k\frac{d\phi}{dx}\frac{dT}{dx}dx+\sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}}\phi Q dx$$

$$=-\psi_{NL}h(T_L-T_\infty)-\sum_e\frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dT^e}{dx}d\xi+\sum_e\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi=0$$

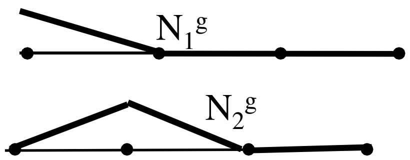
$$d\xi=\frac{2}{l_e}dx$$

Note that, $\frac{d\phi^e}{dx}\frac{dT^e}{dx}=(\mathbf{B}_T\boldsymbol{\psi}^e)(\mathbf{B}_T\mathbf{T}^e)=(\mathbf{B}_T\boldsymbol{\psi}^e)^T(\mathbf{B}_T\mathbf{T}^e)=\boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{T}^e$ and, $\mathbf{N}^e\boldsymbol{\psi}^e$=scalar=$(\mathbf{N}^e\boldsymbol{\psi}^e)^T=\boldsymbol{\psi}^T\mathbf{N}^T$.

$$0=-\psi_{NL}h(T_L-T_\infty)-\sum_e\frac{k_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{T}^e d\xi+\sum_e\frac{Q_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T\mathbf{N}^T d\xi$$

$$\mathbf{k}_T=k_e\int\mathbf{B}_T^T\mathbf{B}_T dx=k_e\int\tfrac{d}{dx}(\mathbf{N}^T)\tfrac{d}{dx}(\mathbf{N})dx$$

$$0=-\psi_{NL}h(T_L-T_\infty)-\sum_e\boldsymbol{\psi}^T\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi\mathbf{T}^e+\sum_e\boldsymbol{\psi}^T\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi$$



Note that: $\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{1}{l_e^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\int_{-1}^{1}d\xi=\frac{2}{l_e^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$, $\int_{-1}^{1}\mathbf{N}^T d\xi=\int_{-1}^{1}\begin{Bmatrix}(1-\xi)/2\\(1+\xi)/2\end{Bmatrix}d\xi=\begin{bmatrix}1\\1\end{bmatrix}$.

Finally, $$0=-\psi_{NL}h(T_L-T_\infty)-\sum_e\boldsymbol{\psi}^T\mathbf{k}_T\mathbf{T}^e+\sum_e\boldsymbol{\psi}^T\mathbf{r}_Q$$ where, $\mathbf{k}_T=\frac{k_e}{l_e}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$, $\mathbf{r}_Q=\frac{Q_e l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}$.

Finite Element Method

Galerkin's approach for heat conduction

Some matrix concept: **AMC+ANC=(AM+AN)C=A(M+N)C.**

Let: $\mathbf{k}_T^{e=i} = \begin{bmatrix} k_{i,i} & k_{i,i+1} \\ k_{i+1,i} & k_{i+1,i+1} \end{bmatrix}, \quad \mathbf{r}_Q^{e=i} = \begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}.$

Please note: k_T are symmetry! $\rightarrow k_{i,j}=k_{j,i}$

2 elements example:

$$\boldsymbol{\psi}^T\mathbf{k}_T^{e=1}\mathbf{T}^{e=1} + \boldsymbol{\psi}^T\mathbf{k}_T^{e=2}\mathbf{T}^{e=2} = [\psi_1 \quad \psi_2]\begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} k_{2,2} & k_{2,3} \\ k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_2 \\ T_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & 2k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \boldsymbol{\Psi}^T\mathbf{K}_T\mathbf{T}.$$

We also get:

$$\boldsymbol{\psi}^T\mathbf{r}_Q^{e=1} + \boldsymbol{\psi}^T\mathbf{r}_Q^{e=2} = [\psi_1 \quad \psi_2]\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} r_2 \\ r_3 \end{bmatrix} = [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} + [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} 0 \\ r_2 \\ r_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} r_1 \\ 2r_2 \\ r_3 \end{bmatrix} = \boldsymbol{\Psi}^T\mathbf{R} = [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}.$$

Finite Element Method

Galerkin's approach for heat conduction

Finally, we get:

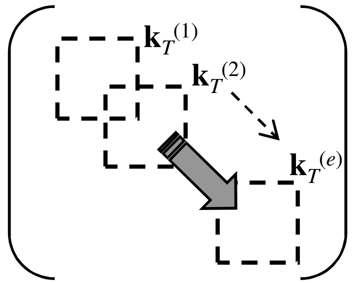
$$0 = -\psi_{NL} h(T_L - T_\infty) - \sum_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{T}^e + \sum_e \boldsymbol{\psi}^T \mathbf{r}_Q$$

$$0 = -\psi_{NL} h T_L + \psi_{NL} h T_\infty - \left(\boldsymbol{\psi}_{e=1}^T \mathbf{k}_T^{e=1} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \cdots + \boldsymbol{\psi}_{e=NL-1}^T \mathbf{k}_T^{e=NL-1} \begin{bmatrix} T_{NL-1} \\ T_{NL} \end{bmatrix} \right)$$

$$+ \left(\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \cdots + \begin{bmatrix} \psi_{NL-1} & \psi_{NL} \end{bmatrix} \begin{bmatrix} r_{NL-1} \\ r_{NL} \end{bmatrix} \right)$$

$$0 = -\psi_{NL} h T_L + \psi_{NL} h T_\infty - \boldsymbol{\Psi}^T \mathbf{K}_T \mathbf{T} + \boldsymbol{\Psi}^T \mathbf{R}.$$

$\mathbf{K}_T =$



The global matrices $\mathbf{K}_T$ and $\mathbf{R}$ are assembled from element matrices $\mathbf{k}_T$ and $\mathbf{r}_Q$.

Now, let $\boldsymbol{\Psi}^T = [\psi_1, \psi_2, \ldots, \psi_{NL}] = [0,1,0,\ldots,0]$, and $T_1 = T_0$, we get

$$-0 + 0 - \begin{bmatrix} K_{21} & K_{22} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NL} \end{bmatrix} + R_2 = 0 \rightarrow \begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{NL} \end{bmatrix} = R_2 - K_{21} T_0.$$

Continue the process, finally let $\boldsymbol{\Psi}^T = [\psi_1, \psi_2, \ldots, \psi_{NL}] = [0,0,\ldots,1]$, we get ($T_L = T_{NL}$)

$$-1 \cdot h T_L + 1 \cdot h T_\infty - \begin{bmatrix} K_{NL,1} & K_{NL,2} & \cdots & K_{NL,NL} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NL} \end{bmatrix} + R_{NL} = 0 \rightarrow \begin{bmatrix} K_{NL,2} & K_{NL,3} & \cdots & (K_{NL,NL} + h) \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{NL} \end{bmatrix} = (R_{NL} + h T_\infty) - K_{NL,1} T_0.$$

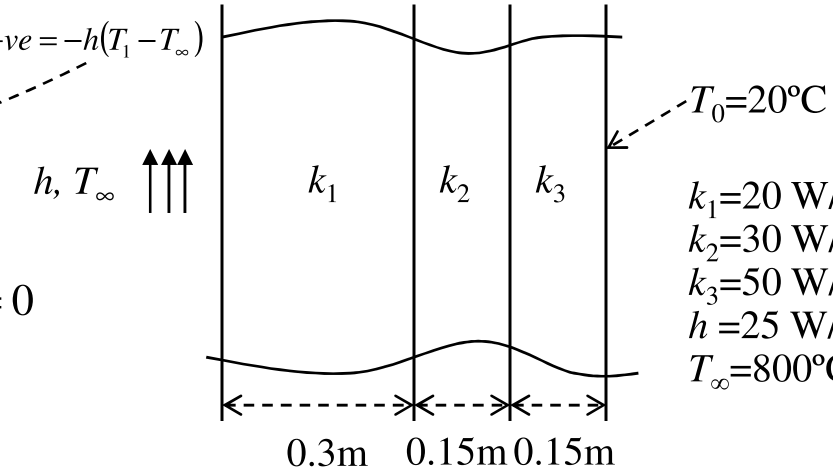
Finally, the compact form is given:

$$\begin{bmatrix} K_{2,2} & K_{2,3} & \cdots & K_{2,NL} \\ K_{3,2} & K_{3,3} & \cdots & K_{3,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,2} & K_{NL,3} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{NL} \end{Bmatrix} = \begin{Bmatrix} R_2 \\ R_3 \\ \vdots \\ R_{NL}+hT_\infty \end{Bmatrix} - \begin{Bmatrix} K_{2,1}T_0 \\ K_{3,1}T_0 \\ \vdots \\ K_{NL,1}T_0 \end{Bmatrix}.$$

Try insulation at $x=L$, $\phi(L)=0$
Try $Q=2$

Problem: A composite wall consists of 3 materials. The outer temperature is T_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with T_∞=800ºC and h=25 W/m^2·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k\left.\frac{\partial T}{\partial x}\right|_{x_1} = -k\frac{T_{1.5}-T_1}{\Delta x} \approx -k\frac{400-500}{+0.1} = +ve = -h(T_1-T_\infty)$$



k_1=20 W/m·ºC
k_2=30 W/m·ºC
k_3=50 W/m·ºC
h =25 W/m^2·ºC
T_∞=800ºC

Solution: we use 3 elements of linear element.

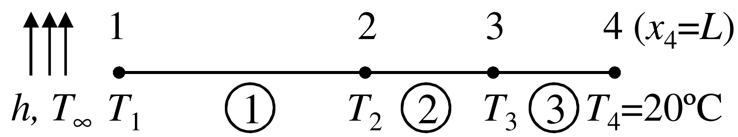
B.C.: $T_4 = T_0$=20, $q|_{x=0} = -h(T_1-T_\infty)$. [$T_\infty > T_1$] We get

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q\right]dx = 0 \rightarrow \phi k\left.\frac{dT}{dx}\right|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q dx = 0$$

$$\left.\phi k\frac{dT}{dx}\right|_0^L = \phi(L)k(L)\frac{dT}{dx}(L) - \phi(0)k(0)\frac{dT}{dx}(0)$$

So, let $\boldsymbol{\phi(L)=0}$, $q(0) = -k(0)[dT(0)/dx] = -h(T_1-T_\infty)$, we get

$$\left.\phi k\frac{dT}{dx}\right|_0^L = -\phi(0)h(T_1-T_\infty).$$



3 elements of linear FE

Finite Element Method

Galerkin's approach for heat conduction

Let ϕ=**N**Ψ, we get

$$0=-\psi_1 h(T_1-T_\infty)-\sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dT^e}{dx}d\xi+\sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi$$

Finally,

$$\boxed{0=-\psi_1 h(T_1-T_\infty)-\sum_e \boldsymbol{\psi}^T\mathbf{k}_T\mathbf{T}^e+\sum_e \boldsymbol{\psi}^T\mathbf{r}_Q} \Rightarrow \boxed{0=-\psi_1 hT_1+\psi_1 hT_\infty-\boldsymbol{\Psi}^T\mathbf{K}_T\mathbf{T}+\boldsymbol{\Psi}^T\mathbf{R}.}$$

Now, let $\boldsymbol{\Psi}^T=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[1,0,0,0]$, and $T_4=T_0$, we get

$$-h(T_1-T_\infty)-\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix}+R_1=0\rightarrow\begin{bmatrix}(K_{11}+h) & K_{12} & K_{13}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}=R_1+hT_\infty-K_{14}T_0$$

let $\boldsymbol{\Psi}^T=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[0,1,0,0]$, we get

$$-0+0-\begin{bmatrix}K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix}+R_2=0\rightarrow\begin{bmatrix}K_{2,1} & K_{2,2} & K_{2,3}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}=R_2-K_{2,4}T_0.$$

Finally, let $\boldsymbol{\Psi}^T=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[0,0,1,0]$, we get

$$-0+0-\begin{bmatrix}K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix}+R_3=0\rightarrow\begin{bmatrix}K_{3,1} & K_{3,2} & K_{3,3}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}=R_3-K_{3,4}T_0.$$

Finite Element Method

Galerkin's approach for heat conduction

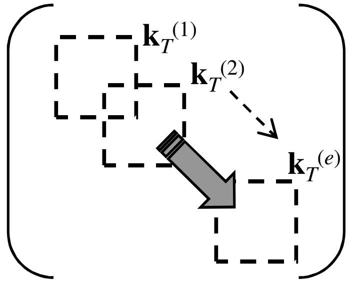
Finally, we get

$$\begin{bmatrix} (K_{11}+h) & K_{1,2} & K_{1,3} \\ K_{2,1} & K_{2,2} & K_{2,3} \\ K_{3,1} & K_{3,2} & K_{3,3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_1 + hT_\infty \\ R_2 \\ R_3 \end{bmatrix} - \begin{bmatrix} K_{1,4}T_0 \\ K_{2,4}T_0 \\ K_{3,4}T_0 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots \ (a)$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{1\,2}, \quad \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\,3}, \quad \mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{3\,4}$$

K$_T$ =



The global $\mathbf{K}_T = \Sigma\ \mathbf{k}_T$ is obtained

$$\mathbf{K}_T = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, we get $\mathbf{r}_Q=[0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0]^T$.

Given T_0=20ºC, T_∞=800ºC and h=25 W/m^2·ºC, eq. (a) becomes

$$66.7\begin{bmatrix} 1.375 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 8 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0+25(800) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -5(66.7)(20) \end{bmatrix} = \begin{bmatrix} 20{,}000 \\ 0 \\ 6670 \end{bmatrix}$$

This linear system can be solved using Thomas algorithm and we get $[T_1, T_2, T_3]$=**[304.6, 119.0, 57.1] ºC**

$$\mathbf{Ax}=\mathbf{b},\ \ \mathbf{A}=\mathbf{LU} \rightarrow \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & \cdots & 0 & c_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ c_2 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & \alpha_n \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

The whole Thomas algorithm can be summarized :

1. $\alpha_1 = d_1$
2. $\alpha_i = d_i - c_i\beta_{i-1}$, i=2,3,…,n
3. $\beta_i = e_i/\alpha_i$, i=1,2,…,n-1.
4. $w_1 = b_1/\alpha_1$
5. $w_i = (b_i - c_i w_{i-1})/\alpha_i$, i=2,3,…,n.
6. $x_n = w_n$
7. $x_i = w_i - \beta_i x_{i+1}$, i=n-1, n-2,…,1.

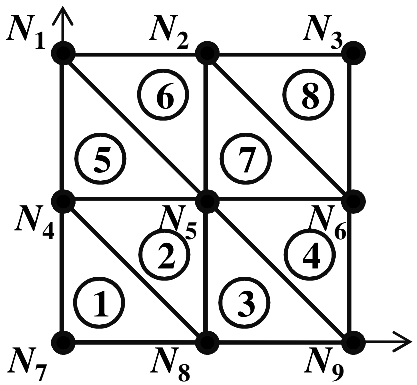
Finite Element Method

Galerkin's approach for heat conduction

Preprocessing

Preprocessing of the problem includes one or more of the following tasks:

- Read geometry and material data (E), and boundary and initial conditions of the problem.
- Mesh generation.
- Generation of node numbers.
- Generation of coordinates and connectivity.



element	1	2	3	← local
1	7	8	4	↑
2	8	5	4	
3	8	9	5	
4	9	6	5	Global
5	4	5	1	
6	5	2	1	
7	5	6	2	↓
8	6	3	2	

Linear triangular element

Processing of FEM

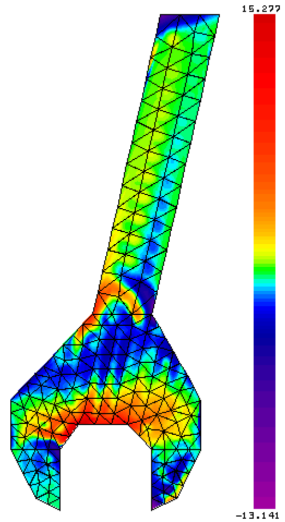
Processing of the FEM includes one or more of the following tasks:

- Calculate element matrices.
- Assemble element equations.
- Solve the system of equations.

Finite Element Method

Galerkin's approach for heat conduction

Postprocessing



Contour plot for stress

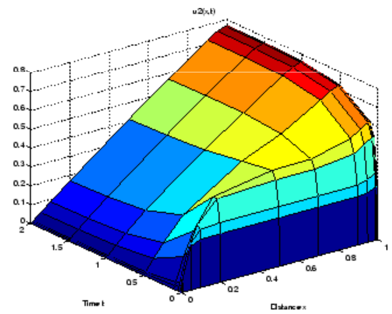
Postprocessing of the FEM includes one or more of the following tasks:

- Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
- Interpretation of the results to check whether the solution makes sense (based on physical Process and experience when other solutions are not available.
- Tabular and/or graphical presentation of the results. Contour plotting uses $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$

Interpolation of temperature within each element is given

$$T(\xi) = N_1 T_1 + N_2 T_2 = \mathbf{N}\mathbf{T}^e$$

where $N_1 = (1-\xi)/2$, $N_2 = (1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N} = [N_1, N_2]$, $\mathbf{T}^e = [T_1, T_2]^{\mathrm{T}}$.


Contour plot for $u_2(x,t)$

The derivative of the solution is obtained by differentiation

Use chain rule, $$\frac{dT}{dx} = \frac{dT}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e = \frac{1}{l_e}[-1, 1]\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e.$$

For element 1, we get $$\frac{dT^{e=1}}{dx} = \mathbf{B}_T\mathbf{T}^{e=1} = \frac{1}{l_e}[-1, 1]\mathbf{T}^{e=1} = \frac{1}{0.3}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}304.6 \\ 119.0\end{bmatrix} = -618.67$$

For element 2, we get $$\frac{dT^{e=2}}{dx} = \mathbf{B}_T\mathbf{T}^{e=2} = \frac{1}{l_e}[-1, 1]\mathbf{T}^{e=2} = \frac{1}{0.15}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}119.0 \\ 57.1\end{bmatrix} = -412.67$$

Note that the derivative above is discontinuous, for any order element, at the nodes connecting the different elements because the continuity of the derivative of FE solution at the connecting nodes is not imposed.

Finite Element Method

Galerkin's method with penalty approach

Calculus of variations

Let $F(x,u,u')$ with fixed value of independent variable x, F depends on u and u'. The change εv in u, where ε is a constant, v is a function, is called **variation** of u (denoted by δu):

$\delta u = \varepsilon v$. →(**variation** of u), operator δ is called **variational operator**.

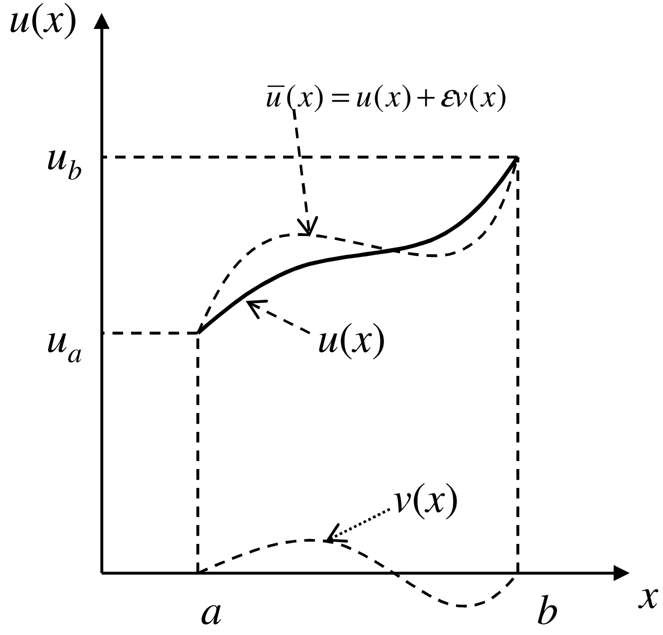
The variation δu represents an admissible change in function $u(x)$ at fixed value of x.

Expand in powers of ε gives ($[u+\varepsilon v]$, $[u'+\varepsilon v']$ are dependent functions)

$$\Delta F = F(x,u+\varepsilon v,u'+\varepsilon v') - F(x,u,u') = F(x,u,u') + \varepsilon v\frac{\partial F}{\partial u} + \varepsilon v'\frac{\partial F}{\partial u'} +$$

$$\frac{(\varepsilon v)^2}{2!}\frac{\partial^2 F}{\partial u^2} + \frac{(\varepsilon v)(\varepsilon v')}{2!}\frac{\partial^2 F}{\partial u \partial u'} + \frac{(\varepsilon v')^2}{2!}\frac{\partial^2 F}{\partial u'^2} + \cdots - F(x,u,u')$$

$$= \varepsilon v\frac{\partial F}{\partial u} + \varepsilon v'\frac{\partial F}{\partial u'} + O(\varepsilon^2). \quad where \lim_{\varepsilon\to 0} O(\varepsilon^2) = 0.$$



The **first variation** of F is

$$\delta F = \varepsilon\left[\lim_{\varepsilon\to 0}\frac{F(x,u+\varepsilon v,u'+\varepsilon v') - F(x,u,u')}{\varepsilon}\right] = \varepsilon\left[\lim_{\varepsilon\to 0}\frac{\Delta F}{\varepsilon}\right]$$

$$= \varepsilon\left[\frac{d}{d\varepsilon}\big(F(u+\varepsilon v)\big)\right]_{\varepsilon=0} = \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'.$$

Analogy for total differential of F with fixed x, dx=0

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'.$$

Finite Element Method

Galerkin's method with penalty approach

Calculus of variations

Let $F=F(x,y,u,v,u_x,v_x,u_y,v_y)$, where $u=u(x,y)$ and $v=v(x,y)$ are dependent variables,

The first variation of F is $\delta F=\delta_u F+\delta_v F$, where $\delta_u F=\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u_x}\delta u_x+\frac{\partial F}{\partial u_y}\delta u_y$, $\delta_v F=\frac{\partial F}{\partial v}\delta v+\frac{\partial F}{\partial v_x}\delta v_x+\frac{\partial F}{\partial v_y}\delta v_y$.

If $F_1=F_1(u)$ and $F_2=F_2(u)$, then

$$\delta(F_1 \pm F_2)=\delta F_1 \pm \delta F_2,\quad \delta(F_1\cdot F_2)=(\delta F_1)F_2+F_1(\delta F_2)$$

$$\delta\left(\frac{F_1}{F_2}\right)=\frac{(\delta F_1)F_2-F_1(\delta F_2)}{F_2^2},\quad \delta(F_1)^n=n(F_1)^{n-1}\delta F_1$$

$$\frac{d}{dx}(\delta u)=\frac{d}{dx}(\varepsilon v)=\varepsilon\frac{dv}{dx}=\delta\left(\frac{du}{dx}\right)$$

$$\delta\int_a^b u(x)dx=\int_a^b \delta u(x)dx,\quad \textit{where } a,b \textit{ are fixed.}$$

Some examples

$$I(u)=\int_a^b F(x,u,u')dx \rightarrow \delta I(u)=\delta\int_a^b F(x,u,u')dx=\int_a^b \delta F(x,u,u')dx=\int_a^b\left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u'}\delta u'\right)dx$$

$$I(u)=\int_a^b\left(p(x)\frac{du}{dx}+q(x)u^2\right)dx+Pu(a) \rightarrow \delta I(u)=\int_a^b\left(p(x)\frac{d\delta u}{dx}+2q(x)u\delta u\right)dx+P\delta u(a)$$

$$I(u,v)=\int_\Omega\left(p(x,y)\frac{du}{dx}\frac{dv}{dx}+q(x,y)v\right)dxdy+\int_\Gamma Quds \rightarrow \delta I=\int_\Omega\left(p(x,y)\left(\frac{d\delta u}{dx}\frac{dv}{dx}+\frac{du}{dx}\frac{d\delta v}{dx}\right)+q(x,y)\delta v\right)dxdy+\int_\Gamma Q\delta uds$$

→functions of position, p and q, do not undergo variation since not functions of dependent variables.

Finite Element Method

Galerkin's method with penalty approach

Euler equations

Find a function $u=u(x)$ such that $u(a)=u_a$, $u(b)=u_b$,
and $I(u)=\int_a^b F(x,u(x),u'(x))dx$
is extremum.

The second variation $\delta^2 I(u)$ of functional $I(u)$ is given, $\delta^2 I(u)=\frac{\varepsilon^2}{2}\left[\frac{d^2}{d\varepsilon^2}I(u+\varepsilon v)\right]_{\varepsilon=0}$
sufficient condition for $I(u)$ relative minimum (max) is $\delta^2 I(u)$ is greater (less) than zero.

We get,

$$0=\delta I(u)=\varepsilon\left.\frac{dI(u+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0}=\int_a^b \delta F dx=\int_a^b\left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u'}\delta u'\right)dx$$

$$=\varepsilon\int_a^b\left(\frac{\partial F}{\partial u}v+\frac{\partial F}{\partial u'}v'\right)dx$$

Integration by part: $\int_{x=a}^{x=b} udv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} vdu$

$$\rightarrow 0=\int_a^b v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left.\left(\frac{\partial F}{\partial u'}v\right)\right|_a^b$$

Without constraint on u', the boundary term vanished if v zero at $x=a$ & b.

Finally, we get **Euler equation**

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0\ ,\ a<x<b.$$

Finite Element Method

Galerkin's method with penalty approach

Natural and essential boundary conditions

Find the extremum of $I(u)$ subject to no end conditions [the set of v is arbitrary even at end point, i.e., $v(a)\neq 0$ and $v(b)\neq 0$], the functional has the form

$$I(u)=\int_a^b F\big(x,u(x),u'(x)\big)dx-Q_a u(a)-Q_b u(b)$$

where Q_a and Q_b are known values.

We get,

$$0=\delta I(u)=\int_a^b \varepsilon v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left(\frac{\partial F}{\partial u'}\varepsilon v\right)\Bigg|_a^b-Q_a\varepsilon v(a)-Q_b\varepsilon v(b)$$

To eliminate boundary conditions, let $\left(-\frac{\partial F}{\partial u'}-Q_a\right)v\Big|_{x=a}=0,\ \left(\frac{\partial F}{\partial u'}-Q_b\right)v\Big|_{x=b}=0.$

And we get,

(1) $v(a)=0, v(b)=0,$

(2) $v(a)=0,\ \frac{\partial F}{\partial u'}\Big|_{x=b}-Q_b=0,$

(3) $-\frac{\partial F}{\partial u'}\Big|_a-Q_a=0,\ v(b)=0,$

(4) $-\frac{\partial F}{\partial u'}\Big|_a-Q_a=0,\ \frac{\partial F}{\partial u'}\Big|_{x=b}-Q_b=0.$

u is fixed at $x=0, L$

Note that $v=0$ at end point is equivalent to requirement that u is specified (some value) at that point.

Essential boundary conditions: v (and its derivatives) to vanish at boundary. E.g. $v=0$ on boundary.

Natural boundary conditions: coefficient of v (and its derivatives) is specified some value.

e.g. $\frac{\partial F}{\partial u'}=Q$ on boundary.

Finite Element Method

Galerkin's method with penalty approach

$$\frac{\partial F}{\partial T} = -Q,\ \frac{\partial F}{\partial T'} = kT'$$

Examples , let total potential energy is

$$I(u) = \int_0^L \left[\frac{A}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{h}{2}[u(L)]^2$$

Use principle of minimum total potential energy, we need to find the minimum with $\delta I(u)=0$,

$$\delta I(u) = \int_0^L \left(A\frac{du}{dx}\frac{d\delta u}{dx} - f\delta u \right) dx + hu(L)\delta u(L) = \int_0^L \left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right]\delta u\, dx + \left[A\frac{du}{dx}\delta u \right]_0^L + hu(L)\delta u(L)$$

$$0 = \int_0^L \delta u\left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[A\frac{du}{dx} + hu(L) \right]_{x=L} - \delta u(0)\left[A\frac{du}{dx} \right]_{x=0}$$

Let δu is arbitrary in $0<x\le L$ but with **$\delta u(0)=0$**, the above boundary term vanishes and we get

Euler equation $-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f = 0, \quad 0 < x < L$ **Natural boundary condition** $A\frac{du}{dx} + hu(L)\Big|_{x=L} = 0$

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0 = -\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q \right]$$

Based on the above example, the problem of

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0 \qquad T|_{x=0} = T_0,\quad q|_{x=L} = h(T_L - T_\infty).$$

Is equivalent to the minimizing the functional with **$\delta T(0)=0$** or T is fixed at $x=0$.

$$I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\, dx + \frac{h}{2}(T_L - T_\infty)^2$$

$$I(u) = \int_0^L \left[\frac{A}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{h}{2}[u(L)]^2 + \frac{h}{2}[u(0)]^2 \Longrightarrow 0 = \delta I(u) = \int_0^L \delta u\left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[A\frac{du}{dx} + hu(L) \right]_{x=L} - \delta u(0)\left[A\frac{du}{dx} - hu(0) \right]_{x=0}$$

with **$\delta u(0)\neq 0$**, **$\delta u(L)\neq 0$** ,and $A\frac{du}{dx} + hu(L)\Big|_{x=L} = 0 \qquad A\frac{du}{dx} - hu(0)\Big|_{x=0} = 0$

Finite Element Method

Galerkin's method with penalty approach

Problem: **minimize** the quadratic function

$f(x,y)=4x^2-3y^2+2xy+6x-3y+5$

subject to the **constraint** $G(x,y)=2x+3y=0$

Lagrange Multiplier Method. The modified functional is

$$F(x,y)=f(x,y)+\lambda G(x,y)$$

$$\frac{\partial F}{\partial x}=8x+2y+6+2\lambda=0$$

$$\frac{\partial F}{\partial y}=-6y+2x-3+3\lambda=0$$

$$\frac{\partial F}{\partial \lambda}=2x+3y=0$$

Solve 3 algebraic equations, we get x = -3, y = 2, λ = 7.

Penalty Function Method. The modified functional is

$$F(x,y)=f(x,y)+\frac{\gamma}{2}G^2(x,y)$$

$$\frac{\partial F}{\partial x}=8x+2y+6+2\gamma(2x+3y)=0$$

$$\frac{\partial F}{\partial y}=-6y+2x-3+3\gamma(2x+3y)=0$$

In the limit $\gamma \to \infty$, the results approaches the exact solution

γ	x	y	$G(x,y)$
0	-0.5769	-0.6923	-3.2308
1	1.5	-3	-6
10	-3.6702	2.7447	0.8936
100	-3.0537	2.0596	0.0716
1000	-3.0053	2.0058	0.0068
10000	-3.0005	2.0006	0.0008
∞	-3	2	0

Finite Element Method

Galerkin's method with penalty approach

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0$$

$$T|_{x=0}=T_0,\quad q|_{x=L}=h(T_L\text{-}T_\infty).$$

equivalent →

$$I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx-\int_0^L QT\,dx+\frac{h}{2}(T_L-T_\infty)^2$$

Let

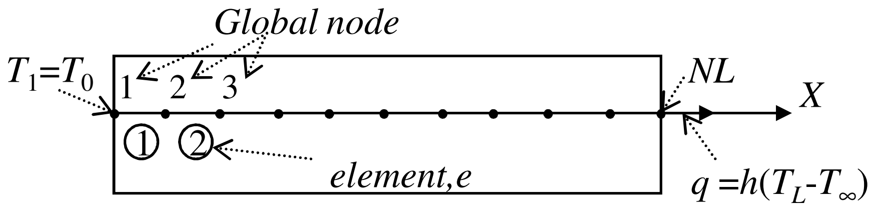
$$I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx-\int_0^L QT\,dx+\frac{h}{2}(T_L-T_\infty)^2+\frac{\gamma}{2}(T_1-T_0)^2$$

Use global node: $T=N_1^gT_1+\cdots+N_{NL}^gT_{NL}=\mathbf{NT}$

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i\frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^gT_i\right)dx+\frac{h}{2}(T_{NL}-T_\infty)^2+\frac{\gamma}{2}(T_1-T_0)^2$$

The minimization of energy is equivalent to $\frac{\partial I(T)}{\partial T_i}=0,\quad i=1,2,\cdots,NL.$

For i=1, involve e=1 only



$$\frac{\partial I(T)}{\partial T_1}=\int_0^L k\left(\sum_i T_i\frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\Bigg|_{e=1}-\int_0^L QN_1^g dx\Big|_{e=1}+\gamma(T_1-T_0)$$

$$0=\int_{e=1}k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1}k_{e=1}\left(\frac{dN_2}{dx}T_2\right)\frac{dN_1}{dx}dx-r_1^{e=1}+\gamma(T_1-T_0)$$

$$=k_{11}^{e=1}T_1+k_{12}^{e=1}T_2-r_1^{e=1}+\gamma(T_1-T_0)=\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\end{bmatrix}-r_1^{e=1}+\gamma(T_1-T_0).$$

Local node $\rightarrow N_i$

$$\mathbf{k}_T=k_e\int\mathbf{B}_T^T\mathbf{B}_T dx=k_e\int\tfrac{d}{dx}\left(\mathbf{N}^T\right)\tfrac{d}{dx}(\mathbf{N})dx$$

Finite Element Method

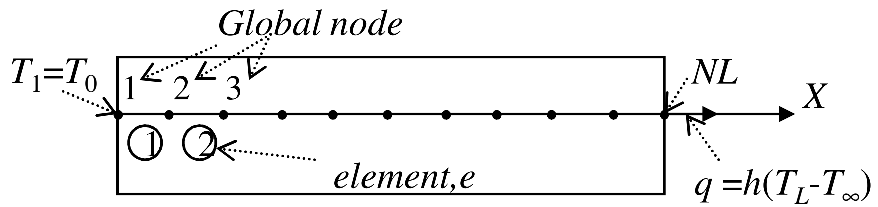
Galerkin's method with penalty approach

For i=2, involve e= 1& 2.

$$\frac{\partial I(T)}{\partial T_2} = \int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\Bigg|_{e=1,2} - \int_0^L QN_2^g\, dx\Bigg|_{e=1,2} + 0$$

$$0 = \int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1 + \frac{dN_2}{dx}T_2\right)\frac{dN_2}{dx}dx + \int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_2 + \frac{dN_2}{dx}T_3\right)\frac{dN_1}{dx}dx - \int_{e=1} QN_2 dx - \int_{e=2} QN_1 dx$$

$$0 = \begin{bmatrix} k_{21} & k_{22}\end{bmatrix}^{e=1}\begin{bmatrix} T_1 \\ T_2\end{bmatrix} + \begin{bmatrix} k_{11} & k_{12}\end{bmatrix}^{e=2}\begin{bmatrix} T_2 \\ T_3\end{bmatrix} - r_2^{e=1} - r_1^{e=2}$$



For i=NL, involve e= NL-1.

$$\frac{\partial I(T)}{\partial T_{NL}} = \int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_{NL}^g}{dx}dx\Bigg|_{e=NL-1} - \int_0^L QN_{NL}^g\, dx\Bigg|_{e=NL-1} + h(T_{NL} - T_\infty)$$

$$0 = \int_{e=NL-1} k_{e=NL-1}\left(\frac{dN_1}{dx}T_{NL-1} + \frac{dN_2}{dx}T_{NL}\right)\frac{dN_2}{dx}dx - \int_{e=NL-1} QN_2 dx + h(T_{NL} - T_\infty)$$

$$0 = \begin{bmatrix} k_{21} & k_{22}\end{bmatrix}^{e=NL-1}\begin{bmatrix} T_{NL-1} \\ T_{NL}\end{bmatrix} - r_2^{e=NL-1} + h(T_{NL} - T_\infty).$$

Finite Element Method

Galerkin's method with penalty approach

Combining all *NL* equations, we finally get

$$\begin{bmatrix} (K_{11}+\gamma) & K_{12} & \cdots & K_{1,NL} \\ K_{21} & K_{22} & \cdots & K_{2,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,1} & K_{NL,2} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NL} \end{bmatrix} = \begin{bmatrix} (R_1+\gamma T_0) \\ R_2 \\ \vdots \\ (R_{NL}+hT_\infty) \end{bmatrix}$$

In the limit $\gamma \rightarrow \infty$, the results approaches the exact solution. A simple scheme suggests that $\gamma = \max |K_{ij}| \times 10^4$

Note: The minimization of the functional $I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_L - T_\infty)^2$

$$\delta I(T) = 0 = \int_0^L k\left(\frac{dT}{dx}\right)\delta\left(\frac{dT}{dx}\right)dx - \int_0^L Q\,\delta T\,dx + h(T_L - T_\infty)\delta(T_L)$$

$$0 = \int_0^L k\,\frac{dT}{dx}\frac{d\delta T}{dx}\,dx - \int_0^L Q\,\delta T\,dx + h(T_L - T_\infty)\delta(T_L)$$

Set the arbitrary function $\boldsymbol{\delta T \rightarrow \phi}$, where $\boldsymbol{\phi(0)=0}$ or $\delta(T_0=0)=0$.

$$0 = \int_0^L k\,\frac{dT}{dx}\frac{d\phi}{dx}\,dx - \int_0^L Q\phi\,dx + h(T_L - T_\infty)\phi(L).$$

Finite Element Method

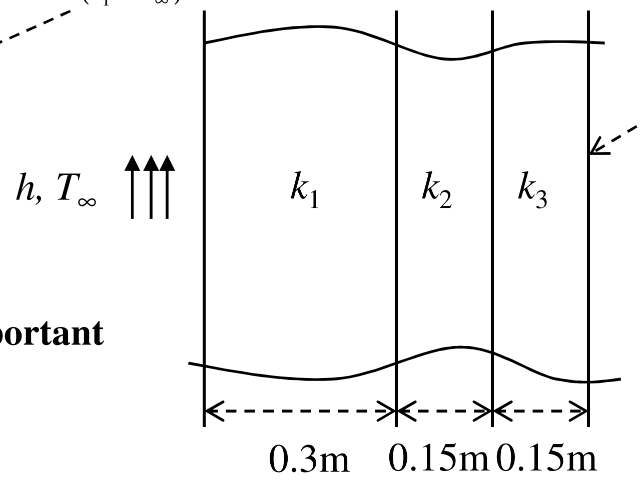
Galerkin's method with penalty approach

Problem: A composite wall consists of 3 materials. The outer temperature is T_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with T_∞=800ºC and h=25 W/m²·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k \left.\frac{\partial T}{\partial x}\right|_{x_1} = -k\frac{T_{1.1}-T_1}{\Delta x} \approx -k\frac{400-500}{+0.1} = +ve = -h(T_1 - T_\infty)$$

Solution: we use 3 elements of linear element.

B.C.: $T_4 = T_0$=20, $q|_{x=0} = -h(T_1 - T_\infty)$. We get



k_1=20 W/m·ºC
k_2=30 W/m·ºC
k_3=50 W/m·ºC
h =25 W/m²·ºC
T_∞=800ºC

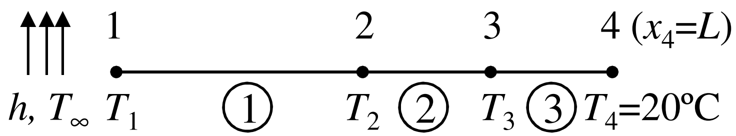
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0 \qquad T|_{x=L} = T_0=20, \quad q|_{x=0} = -h(T_1 - T_\infty).$$

⇓ equivalent

Sign not important

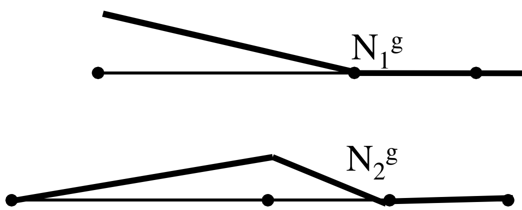
$$I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2$$

Now, let
$$I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2 + \frac{\gamma}{2}(T_4 - T_0)^2$$



3 elements of linear FE

Use global node: $T = N_1^g T_1 + \cdots + N_4^g T_4$



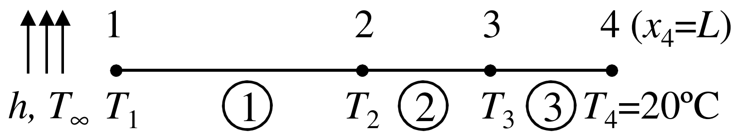
$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g T_i\right)dx + \frac{h}{2}(T_\infty - T_1)^2 + \frac{\gamma}{2}(T_4 - T_0)^2$$

The minimization of energy is equivalent to $\dfrac{\partial I(T)}{\partial T_i} = 0, \quad i = 1,2,3,4.$

Finite Element Method

Galerkin's method with penalty approach

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}\left(T_\infty-T_1\right)^2+\frac{\gamma}{2}\left(T_4-T_0\right)^2$$



3 elements of linear FE

For i=1, involve e=1 only

$$\frac{\partial I(T)}{\partial T_1}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_1^g dx\bigg|_{e=1}+h\left(T_\infty-T_1\right)(-1)$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_2}{dx}T_2\right)\frac{dN_1}{dx}dx-r_1^{e=1}+h\left(T_1-T_\infty\right)$$

Same final form

$$=k_{11}^{e=1}T_1+k_{12}^{e=1}T_2-r_1^{e=1}+h\left(T_1-T_\infty\right)=\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\end{bmatrix}-r_1^{e=1}+h\left(T_1-T_\infty\right).$$

For i=2, involve e= 1& 2.

$$\frac{\partial I(T)}{\partial T_2}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\bigg|_{e=1,2}-\int_0^L QN_2^g\, dx\bigg|_{e=1,2}+0$$

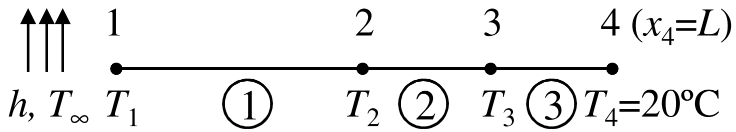
$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1+\frac{dN_2}{dx}T_2\right)\frac{dN_2}{dx}dx+\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_2+\frac{dN_2}{dx}T_3\right)\frac{dN_1}{dx}dx-\int_{e=1} QN_2dx-\int_{e=2} QN_1dx$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=2}\begin{bmatrix}T_2\\T_3\end{bmatrix}-r_2^{e=1}-r_1^{e=2}$$

Finite Element Method

Galerkin's method with penalty approach

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}\left(T_\infty-T_1\right)^2+\frac{\gamma}{2}\left(T_4-T_0\right)^2$$



3 elements of linear FE

For i=3, involve e= 2 & 3.

$$\frac{\partial I(T)}{\partial T_3}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_3^g}{dx}dx\bigg|_{e=2,3}-\int_0^L QN_3^g\,dx\bigg|_{e=2,3}+0$$

$$0=\int_{e=2}k_{e=2}\left(\frac{dN_1}{dx}T_2+\frac{dN_2}{dx}T_3\right)\frac{dN_2}{dx}dx+\int_{e=3}k_{e=3}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4\right)\frac{dN_1}{dx}dx-\int_{e=2}QN_2dx-\int_{e=3}QN_1dx$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=2}\begin{bmatrix}T_2\\T_3\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=3}\begin{bmatrix}T_3\\T_4\end{bmatrix}-r_2^{e=2}-r_1^{e=3}$$

For i=4, involve e= 3.

$$\frac{\partial I(T)}{\partial T_4}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_4^g}{dx}dx\bigg|_{e=3}-\int_0^L QN_4^g\,dx\bigg|_{e=3}+\gamma\left(T_4-T_0\right)$$

$$0=\int_{e=3}k_{e=3}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4\right)\frac{dN_2}{dx}dx-\int_{e=3}QN_2dx+\gamma\left(T_4-T_0\right)$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=3}\begin{bmatrix}T_3\\T_4\end{bmatrix}-r_2^{e=3}+\gamma\left(T_4-T_0\right).$$

Finite Element Method

Galerkin's method with penalty approach

Combining all 4 equations, we finally get

$$\begin{bmatrix} (K_{11}+h) & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & (K_{44}+\gamma) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} (R_1 + hT_\infty) \\ R_2 \\ R_3 \\ (R_4 + \gamma T_0) \end{bmatrix}$$

and let γ= max $|K_{ij}|\times 10^4 = 66.7\times 8\times 10^4$

We get

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 80{,}005 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 25\times 800 \\ 0 \\ 0 \\ 10{,}672\times 10^4 \end{bmatrix}$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global $\mathbf{K}_T = \Sigma\, \mathbf{k}_T$ is obtained

$$\mathbf{K}_T = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, we get $\mathbf{r}_Q=[0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0]^T$.

Given T_0=20ºC, T_∞=800ºC and h=25 W/m^2·ºC,

a tridiagonal linear system can be solved using Thomas algorithm and we get

$[T_1, T_2, T_3, T_4]$=**[304.6, 119.0, 57.1, 20.0] ºC**

$$\mathbf{Ax}=\mathbf{b},\ \mathbf{A}=\mathbf{LU}\rightarrow \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & \cdots & 0 & c_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ c_2 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & \alpha_n \end{pmatrix}\cdot\begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

The whole Thomas algorithm can be summarized :

1. $\alpha_1=d_1$
2. $\alpha_i=d_i-c_i\beta_{i-1}$, i=2,3,…,n
3. $\beta_i=e_i/\alpha_i$, i=1,2,…,n-1.
4. $w_1=b_1/\alpha_1$
5. $w_i=(b_i-c_iw_{i-1})/\alpha_i$, i=2,3,…,n.
6. $x_n=w_n$
7. $x_i=w_i-\beta_ix_{i+1}$, i=n-1, n-2,…,1.

Finite Element Method

1D element generation (quadratic shape function)

Lagrange's interpolation

$$P(x)=\sum_{1}^{N}L_i(x)f_i,\ L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{N}\frac{(x-x_j)}{(x_i-x_j)}$$

$$N_1(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

Now we require that our function $u(x)$ be approximated locally by the quadratic function

$$u(\xi)=c_1+c_2\xi+c_3\xi^2$$

Our node points are defined at $\xi_{1,2,3}=-1,0,1$ and we require that

$$u_1=c_1-c_2+c_3$$

$$u_2=c_1$$

$$u_3=c_1+c_2+c_3$$

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}1&-1&1\\1&0&0\\1&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

$$\mathbf{c}=\mathbf{A}\mathbf{u}$$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0&1&0\\-0.5&0&0.5\\0.5&-1&0.5\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

$$u(\xi)=(u_2)+(-0.5u_1+0.5u_3)\xi+(0.5u_1-u_2+0.5u_3)\xi^2$$

$$=(-0.5\xi+0.5\xi^2)u_1+(1-\xi^2)u_2+(0.5\xi+0.5\xi^2)u_3$$

The temperature field within the element is written in terms of the nodal temperature as

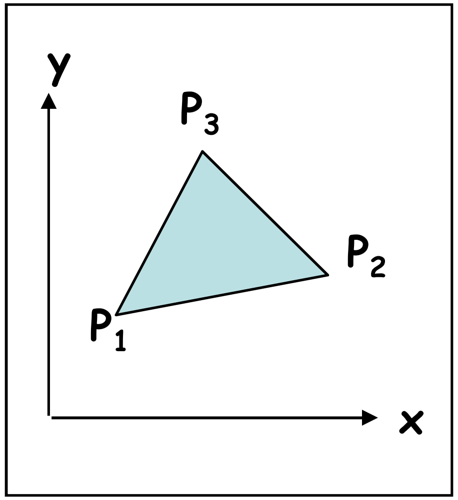
$$T(\xi)=N_1T_1+N_2T_2+N_3T_3=\mathbf{N}\mathbf{T}^e$$

Where $N_1(\xi)=-\tfrac{1}{2}\xi(1-\xi)$, $N_2(\xi)=(1+\xi)(1-\xi)$, $N_3(\xi)=\tfrac{1}{2}\xi(1+\xi)$, ξ varies from – 1 to +1, $\mathbf{N}=[N_1, N_2, N_3]$, $\mathbf{T}^e=[T_1, T_2, T_3]^T$.

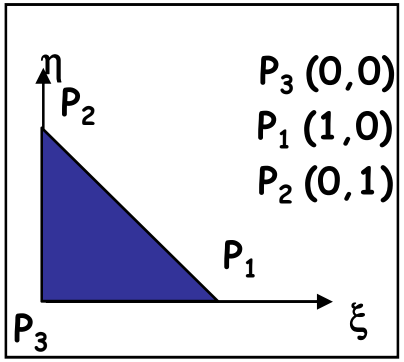
Finite Element Method

2-D element generation – triangle element

Any triangle with corners $P_i(x_i,y_i)$, i=1,2,3 can be transformed into a rectangular, equilateral triangle with



before



after

$$x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta$$

$$y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta$$

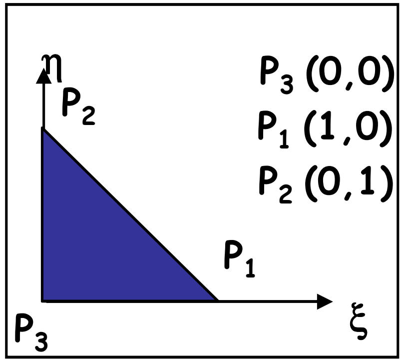
using counterclockwise numbering. Note that if η=0, then these equations are equivalent to the 1-D tranformations. We seek to approximate a function by the linear form

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta$$

we proceed in the same way as in the 1-D case

Finite Element Method

2-D element generation – triangle element



... and we obtain

$$u_3 = u(0,0) = c_1$$

$$u_1 = u(1,0) = c_1 + c_2$$

$$u_2 = u(0,1) = c_1 + c_3$$

... and we obtain the coefficients as a function of the values at the grid nodes by matrix inversion

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{c} = \mathbf{A}\mathbf{u}$$

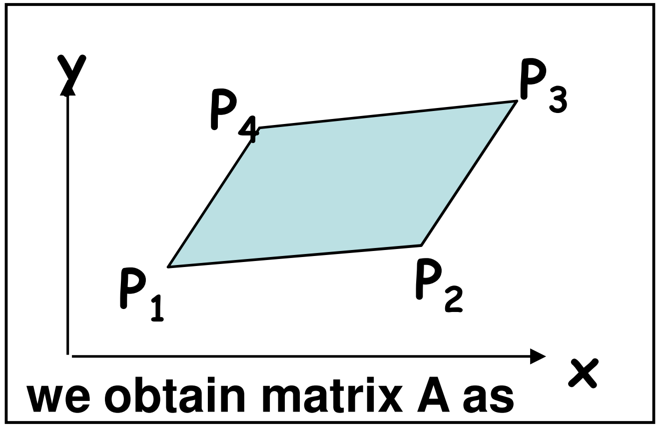
$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta = (u_3) + (u_1 - u_3)\xi + (u_2 - u_3)\eta$$

$$= (\xi)u_1 + (\eta)u_2 + (1 - \xi - \eta)u_3 = N_1u_1 + N_2u_2 + N_3u_3$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

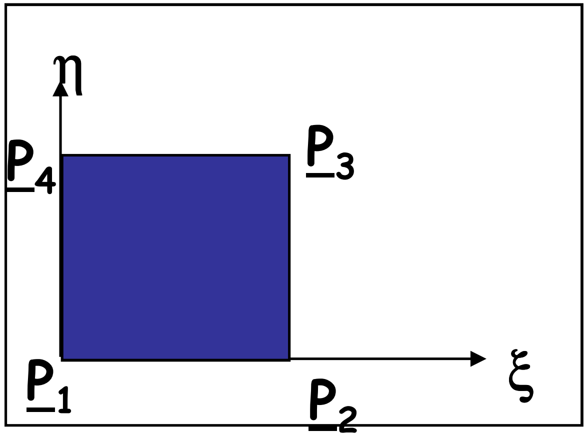
Finite Element Method

2-D element generation – linear rectangle element



we obtain matrix A as

before



after

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

C=AU=AP

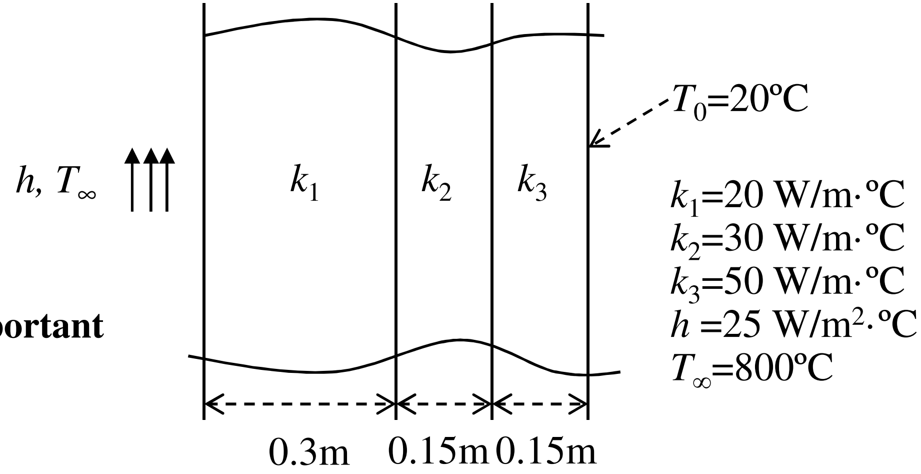
$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta$$

$$N_1(\xi,\eta) = (1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \xi(1-\eta)$$
$$N_3(\xi,\eta) = \xi\eta$$
$$N_4(\xi,\eta) = (1-\xi)\eta$$

and the basis functions

Galerkin's method with penalty approach– quadratic shape functions

Problem: A composite wall consists of 3 materials. The outer temperature is T_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with T_∞=800ºC and h=25 W/m^2·ºC. Determine the temperature distribution in the wall using quadratic shape functions.



Solution: we use 3 elements of quadratic shape functions.

B.C.: $T_7 = T_0$=20, $q|_{x=0} = -h(T_1 - T_\infty)$. We get

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0$$

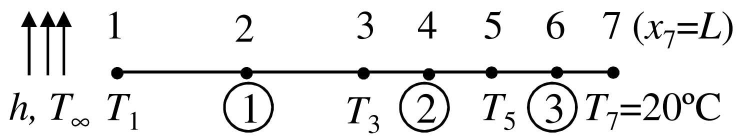
$$T|_{x=L} = T_0=20, \quad q|_{x=0} = -h(T_1 - T_\infty).$$

⇓ equivalent

Sign not important

$$I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2$$

Now, let $$I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2 + \frac{\gamma}{2}(T_7 - T_0)^2$$



3 elements of quadratic FE

Use global node: $T = N_1^g T_1 + \cdots + N_7^g T_7$

$$I(T) = \int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g T_i\right)dx + \frac{h}{2}(T_\infty - T_1)^2 + \frac{\gamma}{2}(T_7 - T_0)^2$$

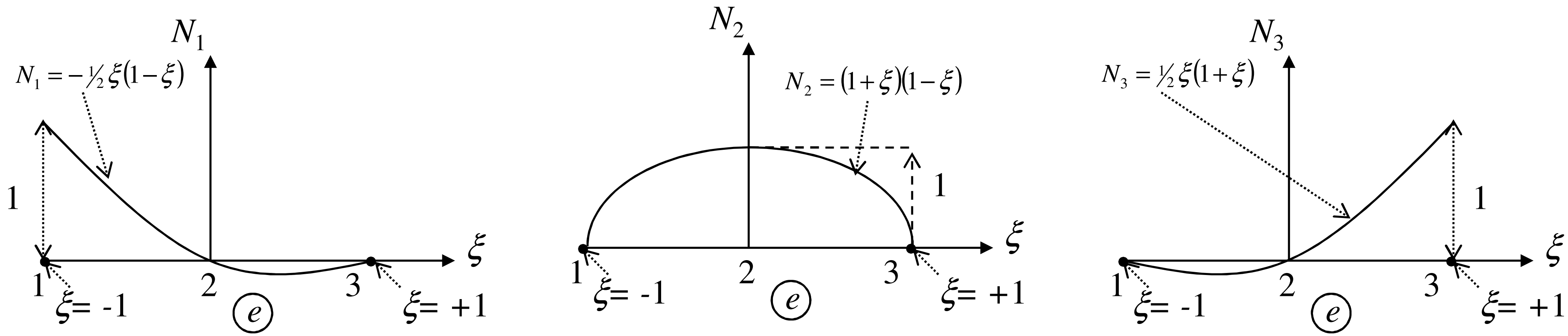
The minimization of energy is equivalent to $\dfrac{\partial I(T)}{\partial T_i} = 0\,, \quad i = 1,\cdots,7.$

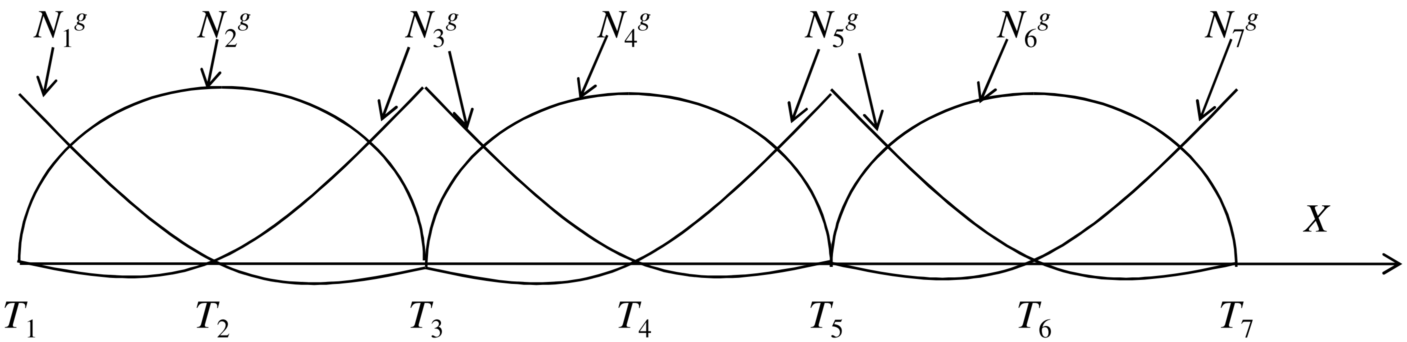
Galerkin's method with penalty approach– quadratic shape functions

The temperature field within the element is written in terms of the nodal temperature as

$$\boxed{T(\xi)=N_1T_1+N_2T_2+N_3T_3=\mathbf{N}\mathbf{T}^e}$$

Where $N_1(\xi)= -\tfrac{1}{2}\xi(1-\xi)$, $N_2(\xi)=(1+\xi)(1-\xi)$, $N_3(\xi)=\tfrac{1}{2}\xi(1+\xi)$, ξ varies from – 1 to +1, $\mathbf{N}=[N_1, N_2, N_3]$, $\mathbf{T}^e=[T_1, T_2, T_3]^T$.





Please note

$$\xi=\frac{2(x-x_2)}{x_3-x_1},\quad d\xi=\frac{2}{x_3-x_1}dx=\frac{2}{l_e}dx.$$

$$\int_{-1}^{1}\mathbf{N}^T d\xi=\int_{-1}^{1}\begin{bmatrix}-\tfrac{1}{2}\xi(1-\xi)\\(1+\xi)(1-\xi)\\\tfrac{1}{2}\xi(1+\xi)\end{bmatrix}d\xi=\begin{bmatrix}1/3\\4/3\\1/3\end{bmatrix}$$

Use chain rule, $\dfrac{dT}{dx}=\dfrac{dT}{d\xi}\cdot\dfrac{d\xi}{dx}=\dfrac{2}{x_3-x_1}\dfrac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e=\dfrac{2}{x_3-x_1}[-\tfrac{1-2\xi}{2},-2\xi,\tfrac{1+2\xi}{2}]\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e.$

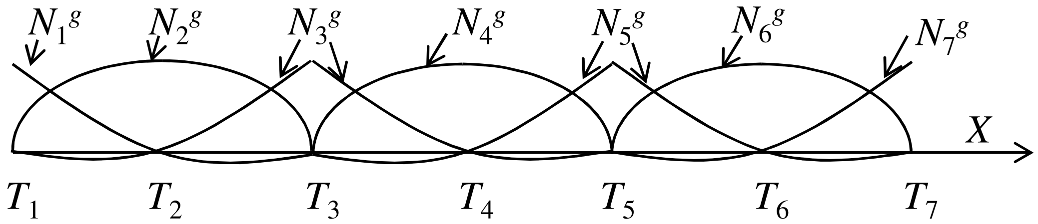
$$\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{2}{3l_e^2}\begin{bmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{bmatrix}$$

$$\mathbf{B}_T=\frac{d}{dx}\mathbf{N}=\frac{2}{x_3-x_1}[-\tfrac{1-2\xi}{2},-2\xi,\tfrac{1+2\xi}{2}]=\frac{2}{l_e}[-\tfrac{1-2\xi}{2},-2\xi,\tfrac{1+2\xi}{2}]$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}(T_\infty-T_1)^2+\frac{\gamma}{2}(T_7-T_0)^2$$

Use global node: $T = N_1^g T_1 + \cdots + N_7^g T_7$



For i=1, involve e=1 only

$$\frac{\partial I(T)}{\partial T_1}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\Bigg|_{e=1}-\int_0^L QN_1^g dx\Bigg|_{e=1}+h(T_\infty-T_1)(-1)$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_2}{dx}T_2\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_3}{dx}T_3\right)\frac{dN_1}{dx}dx-r_1^{e=1}+h(T_1-T_\infty)$$

$$=k_{11}^{e=1}T_1+k_{12}^{e=1}T_2+k_{13}^{e=1}T_3-r_1^{e=1}+h(T_1-T_\infty)=\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}-r_1^{e=1}+h(T_1-T_\infty).$$

For i=2, involve e= 1.

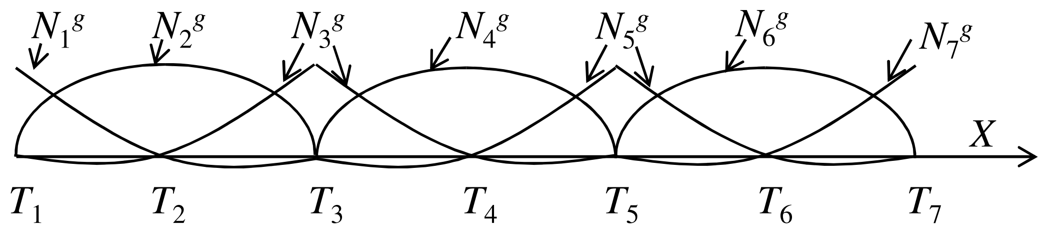
$$\frac{\partial I(T)}{\partial T_2}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\Bigg|_{e=1}-\int_0^L QN_2^g\, dx\Bigg|_{e=1}+0$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1+\frac{dN_2}{dx}T_2+\frac{dN_3}{dx}T_3\right)\frac{dN_2}{dx}dx-\int_{e=1} QN_2 dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}-r_2^{e=1}$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}\left(T_\infty-T_1\right)^2+\frac{\gamma}{2}\left(T_7-T_0\right)^2$$



Use global node: $T=N_1^g T_1+\cdots+N_7^g T_7$

For *i*=3, involve *e*= 1& 2.

$$\frac{\partial I(T)}{\partial T_3}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_3^g}{dx}dx\bigg|_{e=1,2}-\int_0^L QN_3^g\,dx\bigg|_{e=1,2}+0$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1+\frac{dN_2}{dx}T_2+\frac{dN_3}{dx}T_3\right)\frac{dN_3}{dx}dx+\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4+\frac{dN_3}{dx}T_5\right)\frac{dN_1}{dx}dx-\int_{e=1} QN_3dx-\int_{e=2} QN_1dx$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=2}\begin{bmatrix}T_3\\T_4\\T_5\end{bmatrix}-r_3^{e=1}-r_1^{e=2}$$

For *i*=4, involve *e*= 2.

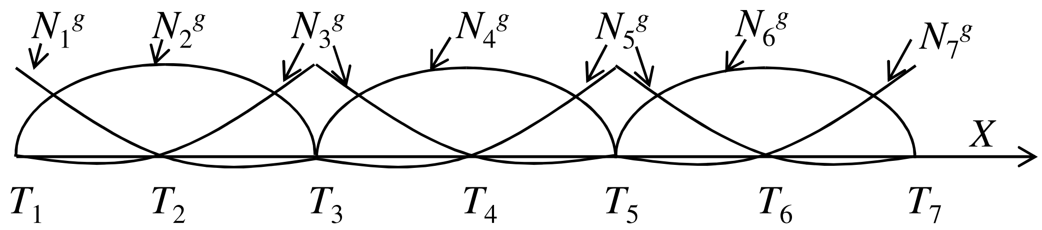
$$\frac{\partial I(T)}{\partial T_4}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_4^g}{dx}dx\bigg|_{e=2}-\int_0^L QN_4^g\,dx\bigg|_{e=2}+0$$

$$0=\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4+\frac{dN_3}{dx}T_5\right)\frac{dN_2}{dx}dx-\int_{e=2} QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=2}\begin{bmatrix}T_3\\T_4\\T_5\end{bmatrix}-r_2^{e=2}$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}(T_\infty-T_1)^2+\frac{\gamma}{2}(T_7-T_0)^2$$



Use global node: $T = N_1^g T_1 + \cdots + N_7^g T_7$

For *i*=5, involve *e*=2 & 3.

$$\frac{\partial I(T)}{\partial T_5}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_5^g}{dx}dx\Big|_{e=2,3}-\int_0^L QN_5^g\,dx\Big|_{e=2,3}+0$$

$$0=\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4+\frac{dN_3}{dx}T_5\right)\frac{dN_3}{dx}dx+\int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}T_5+\frac{dN_2}{dx}T_6+\frac{dN_3}{dx}T_7\right)\frac{dN_1}{dx}dx-\int_{e=2} QN_3dx-\int_{e=3} QN_1dx$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=2}\begin{bmatrix}T_3\\T_4\\T_5\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=3}\begin{bmatrix}T_5\\T_6\\T_7\end{bmatrix}-r_3^{e=2}-r_1^{e=3}$$

For *i*=6, involve *e*= 3.

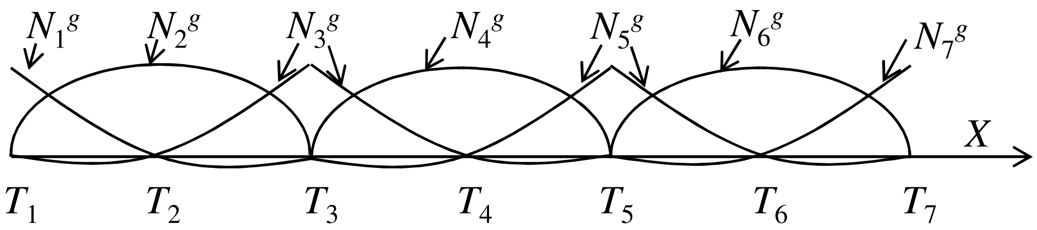
$$\frac{\partial I(T)}{\partial T_6}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_6^g}{dx}dx\Big|_{e=3}-\int_0^L QN_6^g\,dx\Big|_{e=3}+0$$

$$0=\int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}T_5+\frac{dN_2}{dx}T_6+\frac{dN_3}{dx}T_7\right)\frac{dN_2}{dx}dx-\int_{e=3} QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=3}\begin{bmatrix}T_5\\T_6\\T_7\end{bmatrix}-r_2^{e=3}$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}(T_\infty-T_1)^2+\frac{\gamma}{2}(T_7-T_0)^2$$



Use global node: $T=N_1^g T_1+\cdots+N_7^g T_7$

For i=7, involve e= 3.

$$\frac{\partial I(T)}{\partial T_7}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_7^g}{dx}dx\bigg|_{e=3}-\int_0^L QN_7^g\,dx\bigg|_{e=3}+\gamma(T_7-T_0)$$

$$0=\int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}T_5+\frac{dN_2}{dx}T_6+\frac{dN_3}{dx}T_7\right)\frac{dN_3}{dx}dx-\int_{e=3} QN_3dx+\gamma(T_7-T_0)$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=3}\begin{bmatrix}T_5\\T_6\\T_7\end{bmatrix}-r_3^{e=3}+\gamma(T_7-T_0)$$

$$\mathbf{k}_T=k_e\int \mathbf{B}_T^T\mathbf{B}_T dx=\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{k_e}{3l_e}\begin{bmatrix}7 & -8 & 1\\-8 & 16 & -8\\1 & -8 & 7\end{bmatrix}$$

Combining all 7 equations, we finally get

$$\begin{bmatrix}(K_{11}+h) & K_{12} & K_{13} & 0 & 0 & 0 & 0\\K_{21} & K_{22} & K_{23} & 0 & 0 & 0 & 0\\K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & 0 & 0\\0 & 0 & K_{43} & K_{44} & K_{45} & 0 & 0\\0 & 0 & K_{53} & K_{54} & K_{55} & K_{56} & K_{57}\\0 & 0 & 0 & 0 & K_{65} & K_{66} & K_{67}\\0 & 0 & 0 & 0 & K_{75} & K_{76} & (K_{77}+\gamma)\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\\T_5\\T_6\\T_7\end{bmatrix}=\begin{bmatrix}(R_1+hT_\infty)\\R_2\\R_3\\R_4\\R_5\\R_6\\(R_7+\gamma T_0)\end{bmatrix}$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)}=\frac{20}{0.9}\begin{bmatrix}7 & -8 & 1\\-8 & 16 & -8\\1 & -8 & 7\end{bmatrix},\ \mathbf{k}_T^{(2)}=\frac{30}{0.45}\begin{bmatrix}7 & -8 & 1\\-8 & 16 & -8\\1 & -8 & 7\end{bmatrix},\ \mathbf{k}_T^{(3)}=\frac{50}{0.45}\begin{bmatrix}7 & -8 & 1\\-8 & 16 & -8\\1 & -8 & 7\end{bmatrix}$$

Galerkin's method with penalty approach– quadratic shape functions

let γ= max $|K_{ij}|\times 10^4$=80×(200/9)×10^4

$$\frac{200}{9}\begin{bmatrix} 8.125 & -8 & 1 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 \\ 1 & -8 & 28 & -24 & 1 & 0 & 0 \\ 0 & 0 & -24 & 48 & -24 & 0 & 0 \\ 0 & 0 & 3 & -24 & 56 & -40 & 5 \\ 0 & 0 & 0 & 0 & -40 & 80 & -40 \\ 0 & 0 & 0 & 0 & 5 & -40 & 800{,}035 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} (0+20{,}000) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 355.556\times 10^6 \end{bmatrix}$$

Since no heat generation Q occurs in this problem,

we get $\mathbf{r}_Q=[0\ 0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0\ 0\ 0\ 0\ 0]^T$.

Given T_0=20ºC, T_∞=800ºC and h=25 W/m$^2\cdot$ºC,

this linear system can be solved and we get

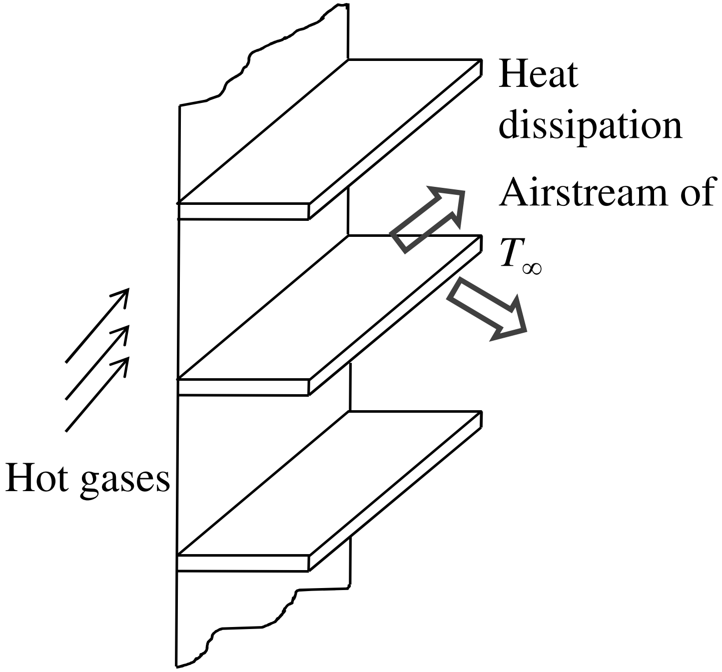
$[T_1, T_2, T_3, T_4, T_5, T_6, T_7]$=**[304.76, 211.91, 119.05, 88.10, 57.14, 38.57, 20.00] ºC**

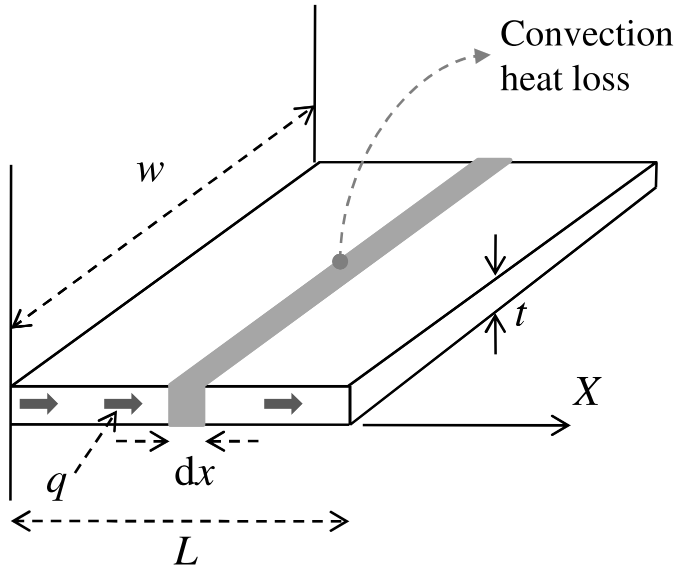
Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

A fin is an extended surface that is added onto a structure to increase the rate of heat removal.

$$P = 2(w+t)$$
$$A_c = wt$$
$$\frac{P}{A_c} \approx \frac{2}{t}$$





The governing equation derived from conduction equation with heat source: $\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0$

The convection heat loss in fin can be considered as a –ve heat source: $Q = -\frac{(Pdx)h(T-T_\infty)}{A_c dx} = -\frac{Ph}{A_c}(T-T_\infty)$

where P = perimeter of fin, A_c=area of cross section.

Finally, we get $$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{Ph}{A_c}(T-T_\infty) = 0$$

Let the case where the base of fin is held at T_0 and the tip of the fin is insulated (heat going out of the tip is negligible), the boundary conditions: $T = T_0$ [x=0] , $q = 0$ [x=L].

Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

Let $\phi(x)$ be any function satisfying $\phi(0)$=0 using the same basis as T, we get $\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{Ph}{A_c}(T-T_\infty)\right]dx=0$

Integrating the first term by parts, we have

$$\boxed{\phi k\frac{dT}{dx}\bigg|_0^L-\int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx-\frac{Ph}{A_c}\int_0^L \phi T dx+\frac{Ph}{A_c}T_\infty\int_0^L \phi\, dx=0}$$

use $\boxed{q=-k\frac{dT}{dx}}$

Using $\phi(0)$=0, $k(L)[dT(L)/dx]$=0, and the isoparametric relations

$$dx=\frac{l_e}{2}d\xi,\quad T=\mathbf{N}\mathbf{T}^e,\quad \phi=\mathbf{N}\mathbf{\Psi},\quad \frac{dT}{dx}=\left(\frac{d}{dx}\mathbf{N}\right)\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e,\quad \frac{d\phi}{dx}=\left(\frac{d}{dx}\mathbf{N}\right)\mathbf{\Psi}=\mathbf{B}_T\mathbf{\Psi}$$

We get $$-\sum_e \boldsymbol{\psi}^T\left[\frac{k_e l_e}{2}\int_{-1}^1 \mathbf{B}_T^T\mathbf{B}_T d\xi\right]\mathbf{T}^e-\frac{Ph}{A_c}\sum_e \boldsymbol{\psi}^T\left(\tfrac{l_e}{2}\int_{-1}^1 \mathbf{N}^T\mathbf{N}d\xi\right)\mathbf{T}^e+\frac{PhT_\infty}{A_c}\sum_e \boldsymbol{\psi}^T\left(\frac{l_e}{2}\int_{-1}^1 \mathbf{N}^T d\xi\right)=0 \qquad \text{.. ….. }(a)$$

We define $$\mathbf{h}_T=\frac{Ph}{A_c}\left(\frac{l_e}{2}\int_{-1}^1 \mathbf{N}^T\mathbf{N}d\xi\right)=\frac{Ph}{A_c}\frac{l_e}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\ \rightarrow\ \frac{P}{A_c}\approx\frac{2}{t}\ \rightarrow\ \mathbf{h}_T=\frac{hl_e}{3t}\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

and $$\mathbf{r}_\infty=\frac{Ph}{A_c}T_\infty\left(\frac{l_e}{2}\int_{-1}^1 \mathbf{N}^T d\xi\right)\ \rightarrow\ \frac{P}{A_c}\approx\frac{2}{t}\ \rightarrow\ \frac{PhT_\infty}{A_c}\frac{l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}\approx\frac{hT_\infty l_e}{t}\begin{bmatrix}1\\1\end{bmatrix}$$

Eq. (a) reduces to $$\boxed{-\sum_e \boldsymbol{\psi}^T(\mathbf{k}_T+\mathbf{h}_T)\mathbf{T}^e+\sum_e \boldsymbol{\psi}^T\mathbf{r}_\infty=0}\ \rightarrow\ \boxed{\boxed{-\mathbf{\Psi}^T(\mathbf{K}_T+\mathbf{H}_T)\mathbf{T}+\mathbf{\Psi}^T\mathbf{R}_\infty=0}}$$

Where the above hold for all $\mathbf{\Psi}$ satisfying ψ_1=0. Let $\mathbf{\Psi}^T$=[ψ_1, ψ_2, …, ψ_{NL}]. We now generate (NL-1) equations by letting $\mathbf{\Psi}^T$=[0, 1, …, 0] … $\mathbf{\Psi}^T$=[0, 0, …, 1], and denoting $K_{ij}=(K_T+H_T)_{ij}$, we obtain

$$\begin{bmatrix}K_{22}&K_{23}&\cdots&K_{2,NL}\\K_{32}&K_{33}&\cdots&K_{3,NL}\\\vdots&\vdots&\ddots&\vdots\\K_{NL,2}&K_{NL,3}&\cdots&K_{NL,NL}\end{bmatrix}\cdot\begin{bmatrix}T_2\\T_3\\\vdots\\T_{NL}\end{bmatrix}=\begin{bmatrix}\\\mathbf{R}_\infty\\\\\end{bmatrix}-\begin{bmatrix}K_{21}T_0\\K_{31}T_0\\\vdots\\K_{NL,1}T_0\end{bmatrix}$$

where we let $T_1=T_0$.

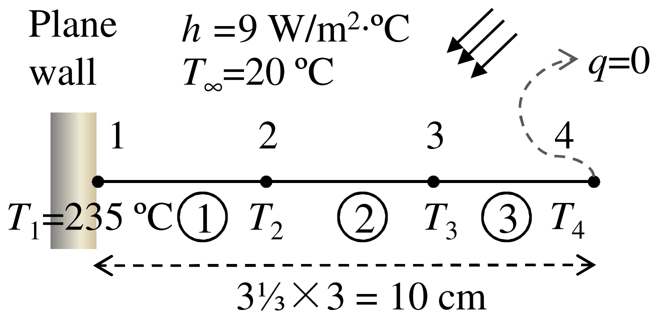
Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

Problem: A metallic fin, with thermal conductivity k=360 W/m·ºC, 0.1 cm thick, and 10 cm long, extends from a plane wall whose temperature is 235 ºC. Determine the temperature distribution and amount of heat Transferred from the fin to the air at 20 ºC with h=9 W/m^{2}·ºC. Take the width of fin to be 1 m.

Solution:

Let $\phi(x)$ be any function satisfying $\phi(0)=0$ and $P/A_c \approx 2/t$, we get



t=0.1 cm, w=1m, k=360 W/m·ºC

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{Ph}{A_c}(T-T_\infty)\right]dx = 0 = \int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{2h}{t}(T-T_\infty)\right]dx$$

$$\phi k\frac{dT}{dx}\bigg|_0^L - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx - \frac{2h}{t}\int_0^L \phi T dx + \frac{2h}{t}T_\infty\int_0^L \phi\, dx = 0$$

Using $\phi(0)=0$, $k(L)[dT(L)/dx]=0$, and the isoparametric relations

$$dx=\frac{l_e}{2}d\xi,\quad T=\mathbf{N}\mathbf{T}^e,\quad \phi=\mathbf{N}\mathbf{\Psi},\quad \frac{dT}{dx}=\left(\frac{d}{dx}\mathbf{N}\right)\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e,\quad \frac{d\phi}{dx}=\left(\frac{d}{dx}\mathbf{N}\right)\mathbf{\Psi}=\mathbf{B}_T\mathbf{\Psi}$$

We get
$$-\sum_e \boldsymbol{\psi}^T\left[\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi\right]\mathbf{T}^e - \frac{2h}{t}\sum_e \boldsymbol{\psi}^T\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T\mathbf{N}d\xi\right)\mathbf{T}^e + \frac{2hT_\infty}{t}\sum_e \boldsymbol{\psi}^T\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi\right)=0$$

We define
$$\mathbf{h}_T=\frac{2h}{t}\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T\mathbf{N}d\xi\right)=\frac{2h}{t}\frac{l_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}=\frac{hl_e}{3t}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\qquad \mathbf{r}_\infty=\frac{2h}{t}T_\infty\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi\right)=\frac{2hT_\infty}{t}\frac{l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}=\frac{hT_\infty l_e}{t}\begin{bmatrix}1\\1\end{bmatrix}$$

Finally, we get
$$\boxed{-\sum_e \boldsymbol{\psi}^T(\mathbf{k}_T+\mathbf{h}_T)\mathbf{T}^e+\sum_e \boldsymbol{\psi}^T\mathbf{r}_\infty=0} \rightarrow \boxed{-\mathbf{\Psi}^T(\mathbf{K}_T+\mathbf{H}_T)\mathbf{T}+\mathbf{\Psi}^T\mathbf{R}_\infty=0}$$

The element conductivity matrices are $\mathbf{k}_T=\frac{k_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$
and we get
$$\mathbf{k}_T^{(1)}=\frac{360}{3.33\times10^{-2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\mathbf{k}_T^{(2)}=\mathbf{k}_T^{(3)}$$

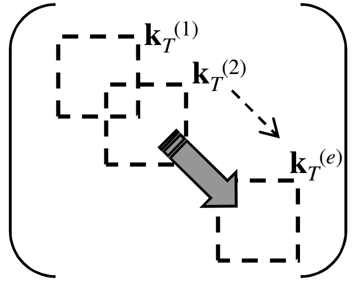
Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

The global $\mathbf{K}_T=\Sigma\, \mathbf{k}_T$ is obtained

$$\mathbf{K}_T = \frac{360}{3.33\times10^{-2}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = [\mathbf{K}_{ij}]$$

$\mathbf{K}_T =$



Now, we calculate for

$$\mathbf{h}_T = \frac{hl_e}{3t}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \mathbf{h}_T^{(1)} = \frac{9\times3.33\times10^{-2}}{3\times10^{-3}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \mathbf{h}_T^{(2)} = \mathbf{h}_T^{(3)} \rightarrow \mathbf{H}_T = 99.9\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [\mathbf{H}_{ij}]$$

and

$$\mathbf{r}_\infty = \frac{hT_\infty l_e}{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \mathbf{r}_\infty^{(1)} = \frac{9\times20\times3.33\times10^{-2}}{10^{-3}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{r}_\infty^{(2)} = \mathbf{r}_\infty^{(3)} \rightarrow \mathbf{R}_\infty = 5994\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{-\mathbf{\Psi}^T\left(\mathbf{K}_T + \mathbf{H}_T\right)\mathbf{T} + \mathbf{\Psi}^T\mathbf{R}_\infty = 0}$$

letting $\mathbf{\Psi}^T$=[0, 1, 0, 0] and $T_1=T_0$, we get

$$-\left([K_{21} \quad K_{22} \quad K_{23} \quad K_{24}] + [H_{21} \quad H_{22} \quad H_{23} \quad H_{24}]\right)\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_2 = 0 \rightarrow \left([K_{22} \quad K_{23} \quad K_{24}] + [H_{22} \quad H_{23} \quad H_{24}]\right)\cdot\begin{bmatrix} T_2 \\ T_3 \\ T_4 \end{bmatrix} = R_2 - (K_{21} + H_{21})T_0$$

letting $\mathbf{\Psi}^T$=[0, 0, 1, 0] and we get

$$-\left([K_{31} \quad K_{32} \quad K_{33} \quad K_{34}] + [H_{31} \quad H_{32} \quad H_{33} \quad H_{34}]\right)\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_3 = 0 \rightarrow \left([K_{32} \quad K_{33} \quad K_{34}] + [H_{32} \quad H_{33} \quad H_{34}]\right)\cdot\begin{bmatrix} T_2 \\ T_3 \\ T_4 \end{bmatrix} = R_3 - (K_{31} + H_{31})T_0$$

Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

$$\boxed{-\boldsymbol{\Psi}^T(\mathbf{K}_T+\mathbf{H}_T)\mathbf{T}+\boldsymbol{\Psi}^T\mathbf{R}_\infty=0}$$

letting $\boldsymbol{\Psi}^T$=[0, 0, 0, 1] and we get

$$-([K_{41}\quad K_{42}\quad K_{43}\quad K_{44}]+[H_{41}\quad H_{42}\quad H_{43}\quad H_{44}])\cdot\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix}+R_4=0\;\rightarrow\;([K_{42}\quad K_{43}\quad K_{44}]+[H_{42}\quad H_{43}\quad H_{44}])\cdot\begin{bmatrix}T_2\\T_3\\T_4\end{bmatrix}=R_4-(K_{41}+H_{41})T_0$$

Finally, we get

$$\left(\frac{360}{3.33\times10^{-2}}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}+99.9\begin{bmatrix}4&1&0\\1&4&1\\0&1&2\end{bmatrix}\right)\cdot\begin{bmatrix}T_2\\T_3\\T_4\end{bmatrix}=5994\begin{bmatrix}2\\2\\1\end{bmatrix}-\begin{bmatrix}-10711\times235\\0\\0\end{bmatrix}$$

Since this is tridiagonal linear system, it can be solved using Thomas algorithm, we get

[T_2, T_3, T_4]=**[209.8, 195.2, 190.5]ºC**.

$$q=-k\frac{\partial T}{\partial x}=h(T_s-T_\infty)\;\rightarrow\;k\Delta T\approx h(T_s-T_\infty)\Delta x$$

vertical horizontal

The total heat loss in the fin can be computed as $H=\Sigma_e H_e$

The loss H_e in each element is

$$H_e=h(T_{av}-T_\infty)A_s$$

$$\boxed{k\Delta T=(\text{W/m}\cdot°\text{C})\cdot°\text{C}=\text{W/m}}$$

where (area of surface=element length×2[width+thick]), (thick is ignorable)

A_s=2×(1×0.0333)m^2, and T_{av} is the average temperature within the element.

We get H_e^1=9([235+209.8]/2-20)×A_s= 121.3, H_e^2=9([209.8+195.2]/2-20)×A_s=109.4,

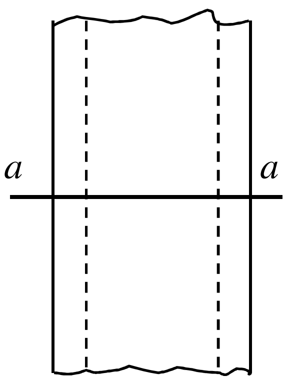
H_e^3=9([195.2+190.5]/2-20)×A_s= 103.6.

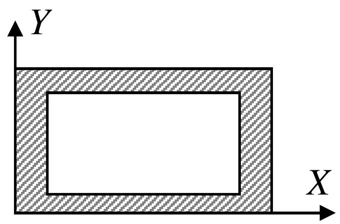
Finally, we get H_{loss}=121.3+109.4+103.6=334.3 W/m.

Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

A chimney of rectangular cross section. Once the temperature distribution is known, the heat flux can be determined from Fourier's law.
Consider a differential control volume, let control volume has a constant thickness τ in the z direction.
The heat generation Q is denoted by Q (W/m^3). Since
The heat rate (=heat flux × area) entering the control volume plus the heat rate generated equals the heat rate coming out, we have





Section a - a

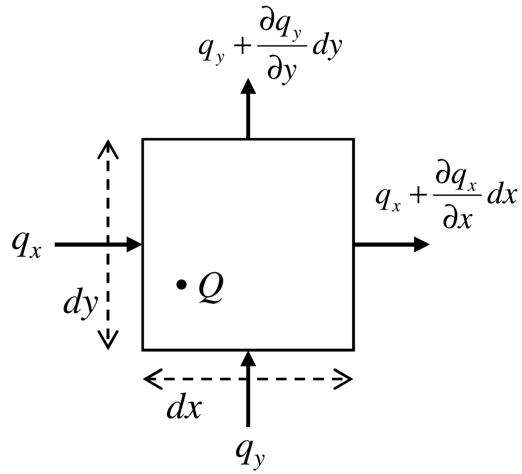
$$q_x \tau dy + q_y \tau dx + Q \tau dx dy = \left(q_x + \frac{\partial q_x}{\partial x} dx \right) \tau dy + \left(q_y + \frac{\partial q_y}{\partial y} dy \right) \tau dx$$

After simplification, we get

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - Q = 0$$

Substituting for q_x=-$k\partial T/\partial x$ and q_y=-$k\partial T/\partial y$ into above, we get heat diffusion equation:

$$\boxed{\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + Q = 0}$$



A differential control volume for heat transfer

Integration by parts – higher dimension
Ω=open bounded subset of R^n, Γ=piecewise smooth boundary, **n** outward unit vector normal to Γ

$$\int_\Omega \nabla u \cdot \mathbf{v} d\Omega = \int_\Gamma u\mathbf{v} \cdot \mathbf{n} d\Gamma - \int_\Omega u \nabla \cdot \mathbf{v} d\Omega$$

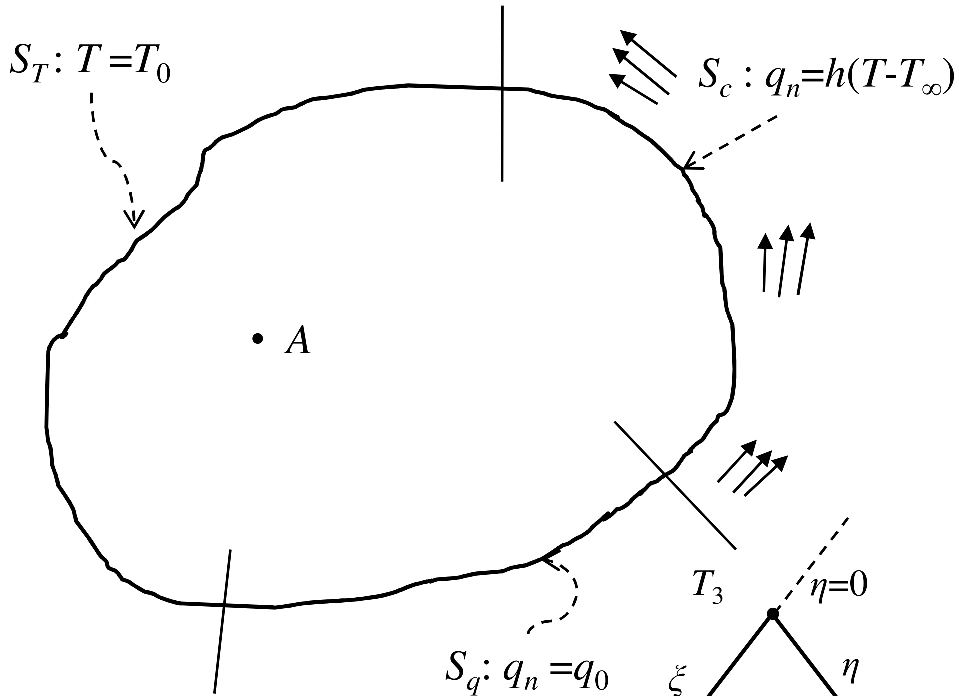
$$\int_\Omega \nabla u \cdot \nabla v d\Omega = \int_\Gamma u \nabla v \cdot \mathbf{n} d\Gamma - \int_\Omega u \nabla \cdot \nabla v d\Omega = \int_\Gamma u \nabla v \cdot \mathbf{n} d\Gamma - \int_\Omega u \nabla^2 v d\Omega$$

Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

3 types of boundary conditions:

- Specified temperature $T=T_0$ on S_T.
- Specified heat flux $q_n=q_0$ on S_q,
- Convection $q_n=h(T-T_\infty)$ on S_c.



Triangular element:

The temperature field within an element is

$T=N_1T_1+N_2T_2+N_3T_3$

or $T=\mathbf{N}\mathbf{T}^e$

where $\mathbf{N}=[\xi, \eta, 1-\xi-\eta]$ are element shape function

$\mathbf{T}^e=[T_1,T_2,T_3]^T$.

We have

$$x=N_1x_1+N_2x_2+N_3x_3=\xi x_1+\eta x_2+(1-\xi-\eta)x_3$$

$$y=N_1y_1+N_2y_2+N_3y_3=\xi y_1+\eta y_2+(1-\xi-\eta)y_3$$

Using chain rule, we get

$$\frac{\partial T}{\partial \xi}=\frac{\partial T}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial \xi},\quad \frac{\partial T}{\partial \eta}=\frac{\partial T}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial \eta}$$

or

$$\begin{bmatrix} T_\xi \\ T_\eta \end{bmatrix}=\begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix}\cdot\begin{bmatrix} T_x \\ T_y \end{bmatrix}=\begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}\cdot\begin{bmatrix} T_x \\ T_y \end{bmatrix}=\mathbf{J}\begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

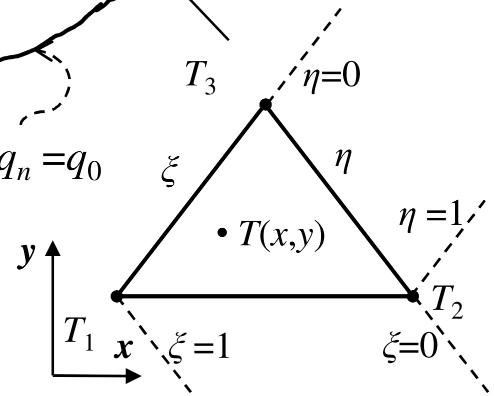
where

$x_{ij}=x_i-x_j$, $y_{ij}=y_i-y_j$, and $|\det \mathbf{J}|=2A_e$,

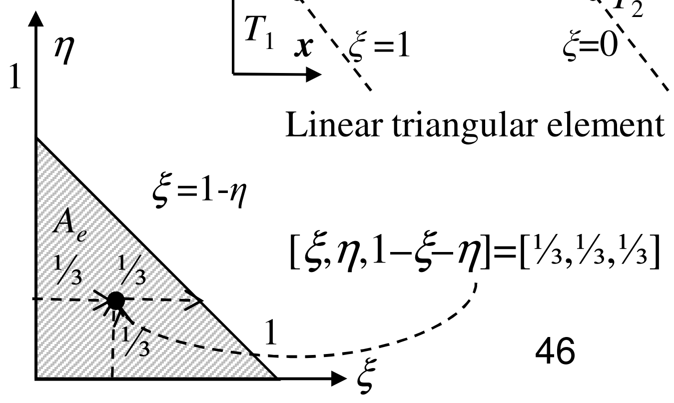
where A_e is the area of the triangular.

$\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$

$$\mathbf{r}_{13}\times\mathbf{r}_{23}=|\mathbf{r}_{13}|\cdot|\mathbf{r}_{23}|\sin\theta\cdot(\pm\mathbf{k})=2A_e(\pm\mathbf{k})$$
$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{13} & y_{13} & 0 \\ x_{23} & y_{23} & 0 \end{vmatrix}=\begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}\mathbf{k}=(\det\mathbf{J})\mathbf{k}$$



Linear triangular element



Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

Using the inverse of matrix 2×2, we get

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \mathbf{J}^{-1}\begin{bmatrix} T_\xi \\ T_\eta \end{bmatrix} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \cdot \begin{bmatrix} T_1 - T_3 \\ T_2 - T_3 \end{bmatrix} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\mathbf{T}^e$$

Which can be written

$$\begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{bmatrix} = \mathbf{B}_T \mathbf{T}^e \quad \leftrightarrow \quad \nabla T = \nabla\left(\mathbf{N}\mathbf{T}^e\right) = (\nabla \mathbf{N})\mathbf{T}^e = \mathbf{B}_T \mathbf{T}^e$$

$$\mathbf{B}_T = \nabla \mathbf{N} = \nabla\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} \partial N_1/\partial x & \partial N_2/\partial x & \partial N_3/\partial x \\ \partial N_1/\partial y & \partial N_2/\partial y & \partial N_3/\partial y \end{bmatrix}$$

where

$$\mathbf{B}_T = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & -y_{13} & (y_{13} - y_{23}) \\ -x_{23} & x_{13} & (x_{23} - x_{13}) \end{bmatrix} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_G (\nabla\cdot\mathbf{F})\,dV$$

Gauss' Theorem or Divergence theorem

Galerkin approach:

Consider the heat conduction problem

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q = 0$$

equivalent minimize

$$I(T) = \frac{1}{2}\iint_A\left[k\left(\frac{\partial T}{\partial x}\right)^2 + k\left(\frac{\partial T}{\partial y}\right)^2 - 2QT\right]dA + \int_{S_q} q_0 T\,ds + \int_{S_c} \tfrac{1}{2} h(T - T_\infty)^2\,ds$$

with b.c. $T=T_0$ on S_T, $q_n=q_0$ on S_q, $q_n=h(T-T_\infty)$ on S_c.

Let

$$\iint_A \phi\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]dA + \iint_A \phi Q\,dA = 0$$

for every $\phi(x,y)$ constructed from same basis function as T And satisfying $\phi=0$ on S_T.

Using

$$\phi\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\phi k\frac{\partial T}{\partial x}\right) - k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x}$$

Green's first theorem

$$\int_s \phi\frac{\partial \psi}{\partial n}\,ds = \int_V \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)dV$$

We get

$$\iint_A\left\{\left[\frac{\partial}{\partial x}\left(\phi k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi k\frac{\partial T}{\partial y}\right)\right] - \left[k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x} + k\frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y}\right]\right\}dA + \iint_A \phi Q\,dA = 0$$

Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

$$\oint_c \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R (\nabla\cdot\mathbf{F}(x,y))dA$$

Green's Theorem, curve c is +ve oriented, anticlockwise.

Using notation $q_x = -k(\partial T/\partial x),\quad q_y = -k(\partial T/\partial y)$
and divergence theorem for first term, we get

$$-\iint_A \left[\frac{\partial}{\partial x}(\phi q_x) + \frac{\partial}{\partial y}(\phi q_y)\right]dA = -\int_S \phi[q_x n_x + q_y n_y]dS = -\oint_S \phi q_n dS$$

where n_x and n_y are the direction cosines of the unit normal **n** to the boundary and $q_n = q_x n_x + q_y n_y = \mathbf{q}\cdot\mathbf{n}$ is the normal heat flow along the unit outward normal. Since $S = S_T + S_q + S_c$, $\phi = 0$ on S_T, $q_n = q_0$ on S_q, and $q_n = h(T - T_\infty)$ on S_c, we get

$$\iint_A \left\{\left[\frac{\partial}{\partial x}\left(\phi k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi k \frac{\partial T}{\partial y}\right)\right] - \left[k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x} + k\frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y}\right]\right\}dA + \iint_A \phi Q dA = 0 \;\rightarrow\; \boxed{-\int_{S_q}\phi q_0 dS - \int_{S_c}\phi h(T - T_\infty)dS - \iint_A \left(k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x} + k\frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y}\right)dA + \iint_A \phi Q dA = 0}$$

We introduce the isoparametric relations for triangular element $T = \mathbf{N}\mathbf{T}^e$. We denote global virtual temperature vector as $\boldsymbol{\psi} = [\psi_1, \psi_2, \ldots, \psi_N]^T$ where N is number of nodes. The virtual temperature is interpolated as $\phi = \mathbf{N}\boldsymbol{\psi}$ since $[\partial T/\partial x\ \partial T/\partial y]^T = \mathbf{B}_T\mathbf{T}^e$, we have

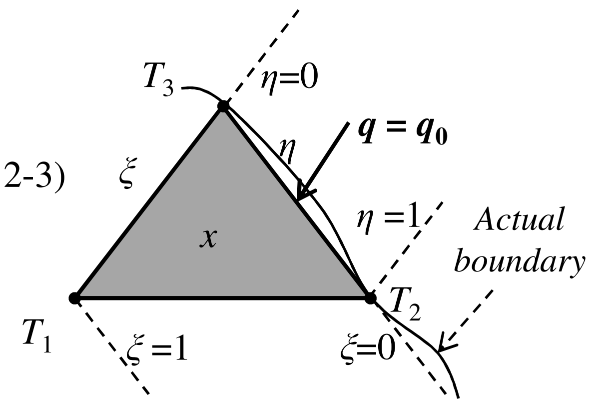
$$\nabla\phi = \begin{bmatrix}\partial\phi/\partial x \\ \partial\phi/\partial y\end{bmatrix} = (\nabla\mathbf{N})\boldsymbol{\psi} = \mathbf{B}_T\boldsymbol{\psi}$$

Now, consider the first term,

$$\int_{S_q}\phi q_0 dS = \int_{S_q}\mathbf{N}\boldsymbol{\psi} q_0 dS = \int_{S_q}\boldsymbol{\psi}^T\mathbf{N}^T q_0 dS = \sum_e \int_e \boldsymbol{\psi}^T\mathbf{N}^T q_0 dS$$

if edge 2-3 is on the boundary, so $\mathbf{N} = [0,\ \eta, 1-\eta]$, $dS = l_{2\text{-}3}d\eta$, ($l_{2\text{-}3}$=length edge 2-3)
we get

$$\boxed{\int_{S_q}\phi q_0 dS = \sum_e \int_e \boldsymbol{\psi}^T\mathbf{N}^T q_0 dS = \sum_e \boldsymbol{\psi}^T q_0 l_{2-3}\int_0^1 \mathbf{N}^T d\eta = \sum_e \boldsymbol{\psi}^T\mathbf{r}_q}$$ where $\mathbf{r}_q = \frac{q_0 l_{2-3}}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$



Specified heat flux b.c. on edge 2-3 of triangular element

Next, consider $\int_{S_c}\phi h(T - T_\infty)dS = \int_{S_c}\phi hT dS - \int_{S_c}\phi hT_\infty dS$

We get $\int_{S_c}\phi h(T - T_\infty)dS = \int_{S_c}\boldsymbol{\psi}^T\mathbf{N}^T h\mathbf{N}\mathbf{T}^e dS - \int_{S_c}\boldsymbol{\psi}^T\mathbf{N}^T hT_\infty dS$

Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

If edge 2-3 is the convection edge of the element, then

$$\int_{S_c} \phi h(T - T_\infty)dS = \sum_e \boldsymbol{\psi}^T \left[h l_{2-3} \int_0^1 \mathbf{N}^T \mathbf{N} d\eta \right] \mathbf{T}^e - \sum_e \boldsymbol{\psi}^T h T_\infty l_{2-3} \int_0^1 \mathbf{N}^T d\eta = \sum_e \boldsymbol{\psi}^T \mathbf{h}_T \mathbf{T}^e - \sum_e \boldsymbol{\psi}^T \mathbf{r}_\infty$$

Substituting for $\mathbf{N}=[0,\ \eta, 1-\eta]$, we get

$$\mathbf{h}_T = h l_{2-3} \int_0^1 \begin{bmatrix} 0 \\ \eta \\ 1-\eta \end{bmatrix} \begin{bmatrix} 0 & \eta & 1-\eta \end{bmatrix} d\eta = \frac{h l_{2-3}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{r}_\infty = h T_\infty l_{2-3} \int_0^1 \begin{bmatrix} 0 \\ \eta \\ 1-\eta \end{bmatrix} d\eta = \frac{h T_\infty l_{2-3}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Next, we get

$$\iint_A k \left(\frac{\partial \phi}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} \right) dA = \iint_A k \begin{bmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \partial T / \partial x \\ \partial T / \partial y \end{bmatrix} dA = \sum_e \boldsymbol{\psi}^T \left[k_e \int_e \mathbf{B}_T^T \mathbf{B}_T dA \right] \mathbf{T}^e = \sum_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{T}^e$$

where

$$\mathbf{k}_T = k_e \int_e \mathbf{B}_T^T \mathbf{B}_T dA = k_e \mathbf{B}_T^T \mathbf{B}_T \int_e dA = k_e A_e \mathbf{B}_T^T \mathbf{B}_T$$

Finally, if $Q=Q_e$ is constant within the element, we get $\iint_A \phi Q dA = \sum_e \boldsymbol{\psi}^T Q_e \int_e \mathbf{N}^T dA = \sum_e \boldsymbol{\psi}^T \mathbf{r}_Q$

where

$$\mathbf{r}_Q = Q_e \int_e \begin{bmatrix} \xi \\ \eta \\ 1-\xi-\eta \end{bmatrix} dA = Q_e 2A_e \int_{\eta=0}^1 \int_{\xi=0}^{1-\eta} \begin{bmatrix} \xi \\ \eta \\ 1-\xi-\eta \end{bmatrix} d\xi d\eta = \frac{Q_e A_e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Please refers to integration on curvilinear coordinates.

Finally, we get

$$-\Sigma_e \boldsymbol{\psi}^T \mathbf{r}_q - \Sigma_e \boldsymbol{\psi}^T \mathbf{h}_T \mathbf{T}^e + \Sigma_e \boldsymbol{\psi}^T \mathbf{r}_\infty - \Sigma_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{T}^e + \Sigma_e \boldsymbol{\psi}^T \mathbf{r}_Q = 0$$

or

$$\boldsymbol{\Psi}^T(\mathbf{R}_\infty - \mathbf{R}_q + \mathbf{R}_Q) - \boldsymbol{\Psi}^T(\mathbf{H}_T + \mathbf{K}_T)\mathbf{T} = 0$$

and hold for all $\boldsymbol{\Psi}$ satisfying $\boldsymbol{\Psi} = \mathbf{0}$ at nodes on S_T.

Finally, we get $\mathbf{K}^E \mathbf{T}^E = \mathbf{R}^E$ where $\mathbf{K}=\Sigma_e(\mathbf{k}_T+\mathbf{h}_T)$, $\mathbf{R}=\Sigma_e(\mathbf{r}_\infty-\mathbf{r}_q+\mathbf{r}_Q)$, superscript E represents the modifications made to $\mathbf{K}$ and $\mathbf{R}$ to handle $T=T_0$ on S_T by the elimination approach.

Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

Problem: line integral using finite element

Calculate the integration on boundary: $I = \int_s \left(x + e^y\right) ds$

For boundary s, y=0, ds=dx, so we get

$$I = \int_s \left(x + e^y\right) ds = \int_0^2 \left(x + e^0\right) dx = \int_0^2 x + 1\, dx = 4$$

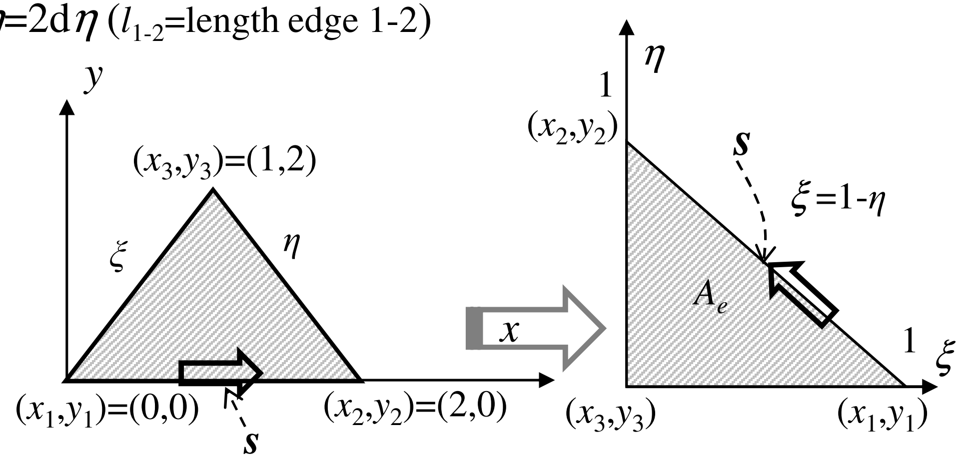
$\mathbf{N}$=[ξ, η,1 – ξ – η]

Line integral using FE.

If edge 1-2 is on the boundary, so $\mathbf{N}=[1-\eta,\eta,0]$, $dS=l_{1\text{-}2}d\eta=2d\eta$ ($l_{1\text{-}2}$=length edge 1-2)

$x=\mathbf{Nx}=2\eta$, $y=\mathbf{Ny}=0+0+0=0$,

So, we get $\int_s \left(x + e^y\right) ds = \int_0^1 \left(2\eta + e^0\right) 2d\eta = \int_0^1 4\eta + 2d\eta = 4$



No. of points, *n*	Weight, w_i	Multi-plicity	ξ_i	η_i	τ_i
One (deg.1), $O(h^2)$	1/2	1	1/3	1/3	1/3
Three (deg.2), $O(h^3)$	1/6	3	2/3	1/6	1/6
Three (deg.2), $O(h^3)$	1/6	3	1/2	1/2	0
Four (deg.3), $O(h^4)$	– 9/32	1	1/3	1/3	1/3
	25/96	3	3/5	1/5	1/5

Gauss quadrature formula for triangle $\int_0^1\int_0^{1-\xi} f(\xi,\eta)d\eta d\xi \approx \sum_{i=1}^n w_i f(\xi_i,\eta_i)$

where $\tau=1-\xi-\eta$

Double integral using FE $I = \iint_\Omega \left(x + 2y\right) dA$

$I = \int_0^2\int_{y/2}^{2-y/2} (x+2y)dxdy = 14/3$ Analytical solution

$x=\mathbf{Nx}=1-\xi+\eta$, $y=\mathbf{Ny}=2-2\xi-2\eta$

$$I = 2A\int_0^1\int_0^{1-\xi}(5-5\xi-3\eta)d\xi d\eta = 2A/6\left(f(1/2,1/2,0)+f(1/2,0,1/2)+f(0,1/2,1/2)\right)$$

$$= 4/6[(5-5/2-3/2)+(5-5/2)+(5-3/2)] = 14/3$$

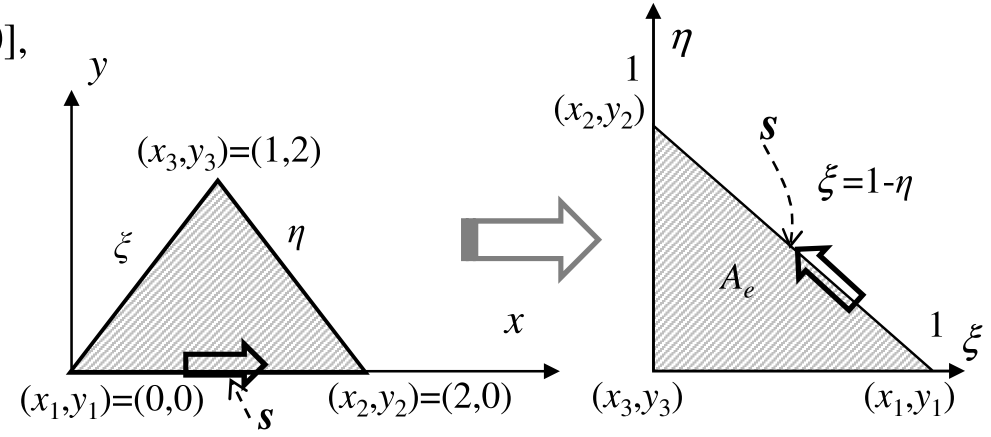
Finite Element Method

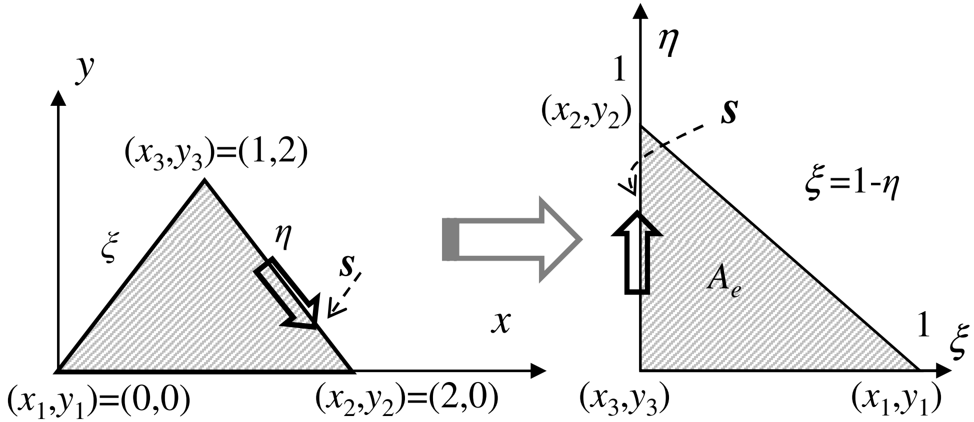
$\mathbf{N}=[\xi, \eta, 1-\xi-\eta]$

Line integral using FE.

use $\xi=1-\eta$, so $\mathbf{N}=[1-\eta,\eta,0]$,

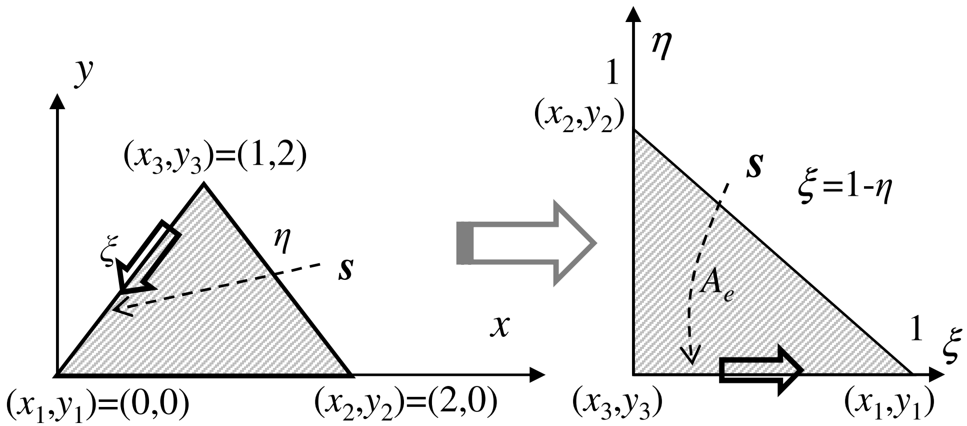
$$\int_{e,S_c} \mathbf{N}^{\mathrm{T}}\mathbf{N}dS = l_{12}\int_0^1 \mathbf{N}^{\mathrm{T}}\mathbf{N}d\eta = l_{12}\int_0^1 \begin{bmatrix}1-\eta\\ \eta\\ 0\end{bmatrix}\begin{bmatrix}1-\eta & \eta & 0\end{bmatrix}d\eta = \frac{l_{12}}{6}\begin{bmatrix}2&1&0\\1&2&0\\0&0&0\end{bmatrix}$$





$$\int_{e,S_c} \mathbf{N}^{\mathrm{T}}\mathbf{N}dS = l_{23}\int_0^1 \mathbf{N}^{\mathrm{T}}\mathbf{N}d\eta = l_{23}\int_0^1 \begin{bmatrix}0\\ \eta\\ 1-\eta\end{bmatrix}\begin{bmatrix}0 & \eta & 1-\eta\end{bmatrix}d\eta = \frac{l_{23}}{6}\begin{bmatrix}0&0&0\\0&2&1\\0&1&2\end{bmatrix}$$

$$\int_{e,S_c} \mathbf{N}^{\mathrm{T}}\mathbf{N}dS = l_{13}\int_0^1 \mathbf{N}^{\mathrm{T}}\mathbf{N}d\xi = l_{13}\int_0^1 \begin{bmatrix}\xi\\ 0\\ 1-\xi\end{bmatrix}\begin{bmatrix}\xi & 0 & 1-\xi\end{bmatrix}d\xi = \frac{l_{13}}{6}\begin{bmatrix}2&0&1\\0&0&0\\1&0&2\end{bmatrix}$$



Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

Integration on curvilinear coordinates:

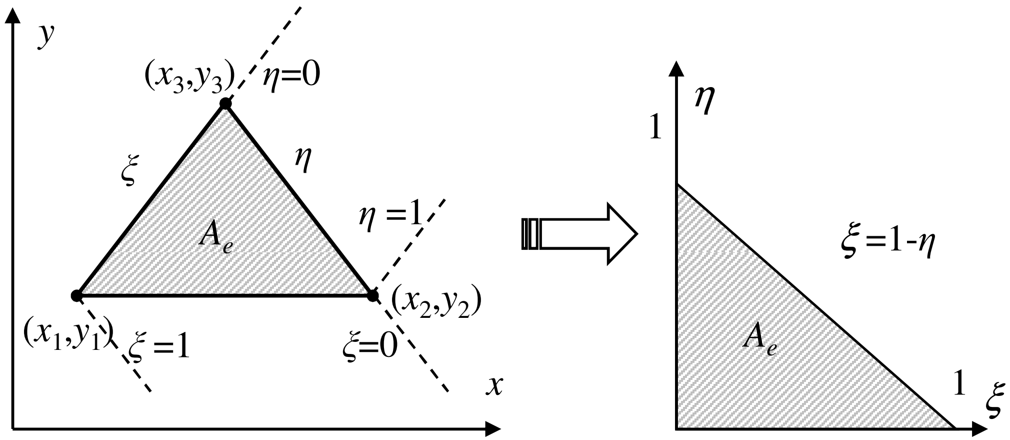
Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}$, then we get $dA=|dx\mathbf{i}\times dy\mathbf{j}|=|\mathbf{i}\times\mathbf{j}|dxdy=dxdy$.

Now, let $\mathbf{N}=[\xi, \eta, 1-\xi-\eta]$ and $\mathbf{x}=[x_1,x_2,x_3]^T$, $\mathbf{y}=[y_1,y_2,y_3]^T$, we get $\mathbf{r}=\mathbf{Nx}\mathbf{i}+\mathbf{Ny}\mathbf{j}$, so $d\mathbf{r}=\mathbf{r}_\xi d\xi+\mathbf{r}_\eta d\eta$,

where $\mathbf{r}_\xi=[1,0,-1]\mathbf{x}\mathbf{i}+[1,0,-1]\mathbf{y}\mathbf{j}$, $\mathbf{r}_\eta=[0,1,-1]\mathbf{x}\mathbf{i}+[0,1,-1]\mathbf{y}\mathbf{j}$, we get $\mathbf{r}_\xi=(x_1-x_3)\mathbf{i}+(y_1-y_3)\mathbf{j}$, $\mathbf{r}_\eta=(x_2-x_3)\mathbf{i}+(y_2-y_3)\mathbf{j}$,

$dA=|\mathbf{r}_\xi d\xi\times\mathbf{r}_\eta d\eta|=|\mathbf{r}_\xi\times\mathbf{r}_\eta|d\xi\, d\eta=2A_e d\xi\, d\eta$. Please note that

$$A_e=\int_{\eta=0}^{1}\int_{\xi=0}^{1-\eta}d\xi d\eta$$



$$\mathbf{r}_{13}\times\mathbf{r}_{23}=|\mathbf{r}_{13}|\cdot|\mathbf{r}_{23}|\sin\theta\cdot(\pm\mathbf{k})=2A_e(\pm\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_{13} & y_{13} & 0\\ x_{23} & y_{23} & 0\end{vmatrix}=\begin{vmatrix}x_{13} & y_{13}\\ x_{23} & y_{23}\end{vmatrix}\mathbf{k}=(\det\mathbf{J})\mathbf{k}$$

+ve if nodes 1, 2, 3 are Anticlockwise!

Problem: A long bar of rectangular cross section, Having thermal conductivity of 1.5 W/m·°C is subjected to the b.c. shown in figure. Two opposite sides are maintained at a uniform temperature of 180°C; one side is insulated, and the remaining side is subjected to a convection process with T_∞=25°C and h=50 W/m²·°C. Determine the temperature distribution in the bar.

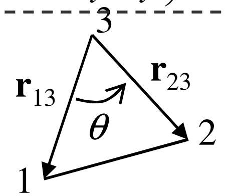
$$\iint_A\phi\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]dA+\iint_A\phi Q dA=0$$

$$-\int_{S_q}\phi q_0 dS-\int_{S_c}\phi h(T-T_\infty)dS-\iint_A\left(k\frac{\partial\phi}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial\phi}{\partial y}\frac{\partial T}{\partial y}\right)dA+\iint_A\phi Q dA=0$$

Solution:

Our model use the symmetry about the horizontal axis. Note that the line of symmetry is shown as insulated, since no heat can flow across it.

$$\det\mathbf{J}=\begin{vmatrix}x_{13} & y_{13}\\ x_{23} & y_{23}\end{vmatrix}=\pm 2A_e$$



Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

The element matrices are developed as follows. The element connectivity is defined in table:

Element	1	2	3	← local
1	1	2	3	↑
2	5	1	3	global
3	5	4	3	↓

$$\mathbf{r}_{13}\times\mathbf{r}_{23} = |\mathbf{r}_{13}|\cdot|\mathbf{r}_{23}|\sin\theta\cdot(\pm\mathbf{k}) = 2A_e(\pm\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{13} & y_{13} & 0 \\ x_{23} & y_{23} & 0 \end{vmatrix} = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}\mathbf{k} = (\det\mathbf{J})\mathbf{k}$$

We have (orient= +1, anticlockwise, – 1 clockwise)

$$\mathbf{B}_T^{(1)} = \frac{1}{0.06}\begin{bmatrix} -0.15 & 0.15 & 0 \\ 0 & -0.4 & 0.4 \end{bmatrix},\quad \mathbf{B}_T^{(2)} = \frac{1}{0.12}\begin{bmatrix} -0.15 & -0.15 & 0.3 \\ 0.4 & -0.4 & 0 \end{bmatrix},\quad \mathbf{B}_T^{(3)} = \frac{-1}{0.06}\begin{bmatrix} 0.15 & -0.15 & 0 \\ 0 & -0.4 & 0.4 \end{bmatrix}$$

For each element, we get

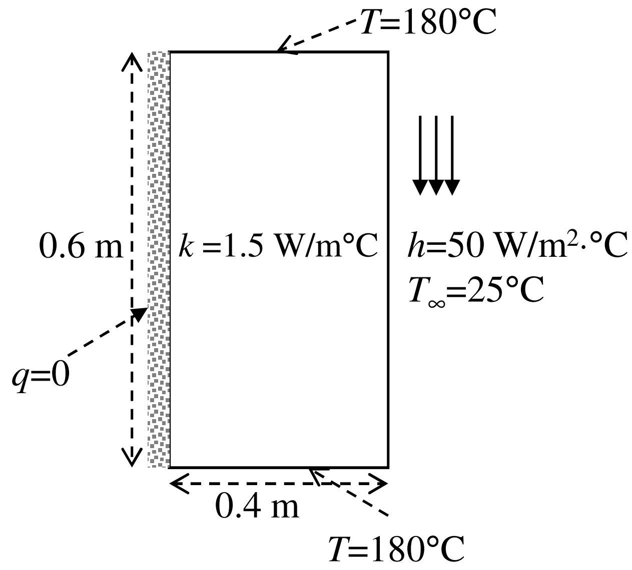
$$\mathbf{B}_T = \frac{1}{\det\mathbf{J}}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} = \frac{1}{2A_e\times orient.}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

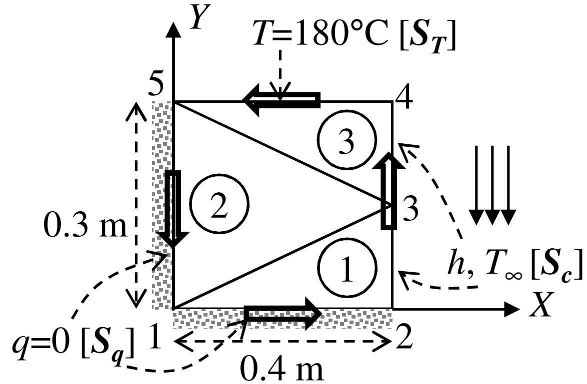
Then, $\mathbf{k}_T = kA_e\mathbf{B}_T^{\mathrm{T}}\mathbf{B}_T$ yields

$$\mathbf{k}_T^{(1)} = (1.5)(0.03)\mathbf{B}_T^{(1)\mathrm{T}}\mathbf{B}_T^{(1)} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 2.28125 & -2.0 \\ 0 & -2.0 & 2.0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

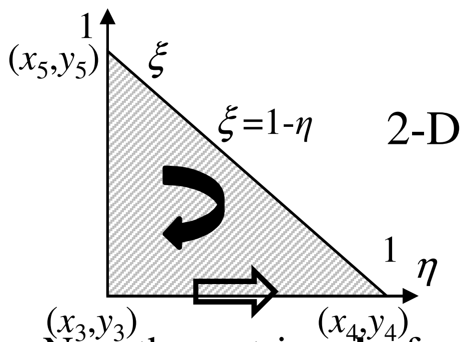
$$\Psi^T\mathbf{k}_T\mathbf{T}^e = [\psi_1 \quad \psi_2 \quad \psi_3]\mathbf{k}_T\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$\mathbf{k}_T^{(2)} = (1.5)(0.06)\mathbf{B}_T^{(2)\mathrm{T}}\mathbf{B}_T^{(2)} = \begin{array}{c} \begin{array}{ccc} 5 & 1 & 3 \end{array} \\ \begin{bmatrix} 1.14 & -0.86 & -0.28125 \\ -0.86 & 1.14 & -0.28125 \\ -0.28125 & -0.28125 & 0.5625 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 1 \\ 3 \end{array}$$





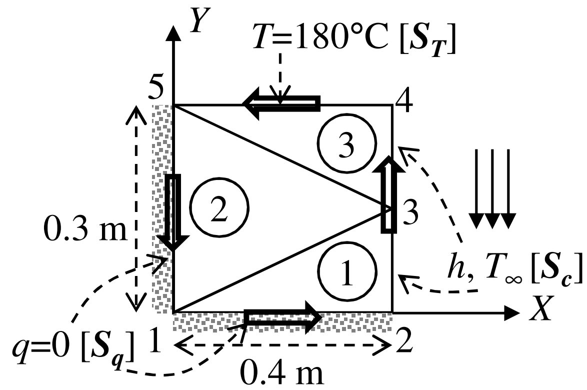
5 nodes, 3 elements for upper part with symmetry about the horizontal axis



Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

$$\mathbf{k}_T^{(3)} = (1.5)(0.03)\mathbf{B}_T^{(3)\mathrm{T}}\mathbf{B}_T^{(3)} = \begin{array}{c} \begin{matrix} 5 & \quad 4 & \quad 3 \end{matrix} \\ \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 2.28125 & -2.0 \\ 0 & -2.0 & 2.0 \end{bmatrix} \end{array} \begin{matrix} 5 \\ 4 \\ 3 \end{matrix}$$



5 nodes, 3 elements for upper part
with symmetry about the horizontal axis

Now the matrices $\mathbf{h}_T$ for elements with convection edges are developed. Since both elements 1 and 3 have edges 2-3 (in local node numbers) as convection edges, so **N**=[0, η,1 – η], dS=$l_{2\text{-}3}$dη, ($l_{2\text{-}3}$=length edge 2-3)

$$\mathbf{h}_T = h\int_{e,S_c}\mathbf{N}^{\mathrm{T}}\mathbf{N}dS = hl_{2-3}\int_0^1\mathbf{N}^{\mathrm{T}}\mathbf{N}d\eta = hl_{2-3}\int_0^1\begin{bmatrix}0\\ \eta\\ 1-\eta\end{bmatrix}\begin{bmatrix}0 & \eta & 1-\eta\end{bmatrix}d\eta = \frac{hl_{2-3}}{6}\begin{bmatrix}0&0&0\\0&2&1\\0&1&2\end{bmatrix}$$

And we get results for element 1 & 3

$$\mathbf{h}_T^{(1)} = \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.5 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}, \quad \mathbf{h}_T^{(3)} = \begin{array}{c} \begin{matrix} 5 & \quad 4 & \quad 3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.5 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix} \end{array} \begin{matrix} 5 \\ 4 \\ 3 \end{matrix}$$

Element	1	2	3	← local
1	1	2	3	↑
2	5	1	3	global
3	5	4	3	↓

The matrix **K**=Σ($\mathbf{k}_T$+$\mathbf{h}_T$) is now assembled.

$$\mathbf{K} = \begin{bmatrix} 0.28125 & -0.28125 & 0 & . & . \\ -0.28125 & 2.28125 & -2.0 & . & . \\ 0 & -2.0 & 2.0 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}^{(1)} + \begin{bmatrix} 1.14 & . & -0.28125 & . & -0.86 \\ . & . & . & . & . \\ -0.28125 & . & 0.5625 & . & -0.28125 \\ . & . & . & . & . \\ -0.86 & . & -0.28125 & . & 1.14 \end{bmatrix}^{(2)} + \begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & 2.0 & -2.0 & 0 \\ . & . & -2.0 & 2.28125 & -0.28125 \\ . & . & 0 & -0.28125 & 0.28125 \end{bmatrix}^{(3)} +$$

$$+ \begin{bmatrix} 0 & 0 & 0 & . & . \\ 0 & 2.5 & 1.25 & . & . \\ 0 & 1.25 & 2.5 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}^{(1)} + \begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & 2.5 & 1.25 & 0 \\ . & . & 1.25 & 2.5 & 0 \\ . & . & 0 & 0 & 0 \end{bmatrix}^{(3)} = \begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & . & -0.86 \\ -0.28125 & 4.78125 & -0.75 & . & . \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ . & . & -0.75 & 4.78125 & -0.28125 \\ -0.86 & . & -0.28125 & -0.28125 & 1.42125 \end{bmatrix}$$

Finite Element Method

2-D steady-state heat conduction – heat diffusion equation

Since both elements 1 and 3 have edges 2-3 (in local node numbers) as convection edges, so **N**=[0, η,1 − η], dS=l_{2-3}dη, (l_{2-3}=length edge 2-3), we get

$$\mathbf{r}_\infty = hT_\infty l_{2-3}\int_0^1 \begin{bmatrix} 0 \\ \eta \\ 1-\eta \end{bmatrix} d\eta = \frac{hT_\infty l_{2-3}}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

we get

$$\mathbf{r}_\infty^{(1)} = \frac{(50)(25)(0.15)}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad , \quad \mathbf{r}_\infty^{(3)} = \frac{(50)(25)(0.15)}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{matrix} 5 \\ 4 \\ 3 \end{matrix}$$

$$\boxed{\Psi^T \mathbf{r}_\infty = [\psi_1 \quad \psi_2 \quad \psi_3]\mathbf{r}_\infty}$$

Since no heat generation Q occurs in this problem, we get Q_e=0

$$\mathbf{r}_q = \frac{q_0 l_{2-3}}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{r}_Q = \frac{Q_e A_e}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

With **R**=Σ_e($\mathbf{r}_\infty$-$\mathbf{r}_q$+$\mathbf{r}_Q$), we get

$$\mathbf{R} = 93.75\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

Finally, we get **KT=R**

$$\mathbf{KT} = \mathbf{R} \quad \rightarrow \quad \begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & . & -0.86 \\ -0.28125 & 4.78125 & -0.75 & . & . \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ . & . & -0.75 & 4.78125 & -0.28125 \\ -0.86 & . & -0.28125 & -0.28125 & 1.42125 \end{bmatrix}\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = 93.75\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Set the boundary condition [S_T] as T_4=T_5=180°C, we get

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & . & -0.86 \\ -0.28125 & 4.78125 & -0.75 & . & . \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ . & . & 0 & 1 & 0 \\ 0 & . & 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 93.75 \\ 187.5 \\ 180 \\ 180 \end{bmatrix}$$

$$\mathbf{K}^E\mathbf{T}^E = \mathbf{R}^E \quad \rightarrow$$

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 \\ -0.28125 & 4.78125 & -0.75 \\ -0.28125 & -0.75 & 9.5625 \end{bmatrix}\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0+154.8 \\ 93.75 \\ 187.5+135+50.625 \end{bmatrix} = \begin{bmatrix} 154.8 \\ 93.75 \\ 373.125 \end{bmatrix}$$

hold for all Ψ satisfying $\Psi = \mathbf{0}$ at nodes on S_T.

T_4 and T_5 is given, so set its row/col to zero except the main diagonal. The 4 & 5 rows of **R** set to given boundary values. Note: A large temperature gradient exist along the line connecting nodes 2 & 4. We can increase the number of nodes along line 2-4.

[T_1, T_2, T_3]=[124.6, 34.1, 45.4]°C

Finite Element Method

Higher order element – six-node triangle

Six-node triangle element (quadratic triangle):

The temperature field within an element is

$$T=N_1T_1+N_2T_2+N_3T_3+N_4T_4+N_5T_5+N_6T_6$$

or $\quad T=\mathbf{N}\mathbf{T}^e$

where $[N_1=\xi(2\xi-1),\ N_2=\eta(2\eta-1),\ N_3=\tau(2\tau-1),\ N_4=4\xi\eta,\ N_5=4\tau\eta,\ N_6=4\xi\tau]$ are element shape function

$\mathbf{T}^e=[T_1,T_2,T_3,T_4,T_5,T_6]^T$ where $\tau=1-\xi-\eta$.

We have

$$x=\Sigma N_i x_i\ ,\ y=\Sigma N_i y_i.$$

Symmetry: for multiplicity of three, we get (⅔,1/6,1/6), (1/6,⅔,1/6) and (1/6,1/6, ⅔)

Using chain rule, we get

$$\frac{\partial T}{\partial \xi}=\frac{\partial T}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial \xi},\quad \frac{\partial T}{\partial \eta}=\frac{\partial T}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{bmatrix}T_\xi\\T_\eta\end{bmatrix}=\begin{bmatrix}x_\xi & y_\xi\\ x_\eta & y_\eta\end{bmatrix}\cdot\begin{bmatrix}T_x\\T_y\end{bmatrix}=\mathbf{J}\begin{bmatrix}T_x\\T_y\end{bmatrix}$$

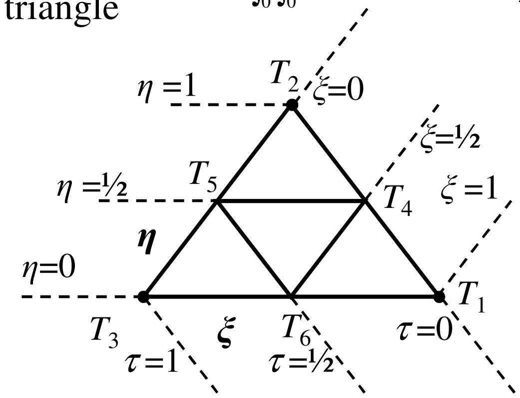
$$=\begin{bmatrix}\begin{pmatrix}(4\xi-1)x_1+(-4\tau+1)x_3+4\eta x_4\\-4\eta x_5+(4-8\xi-4\eta)x_6\end{pmatrix} & \begin{pmatrix}(4\xi-1)y_1+(-4\tau+1)y_3+4\eta y_4\\-4\eta y_5+(4-8\xi-4\eta)y_6\end{pmatrix}\\ \begin{pmatrix}(4\eta-1)x_2+(-4\tau+1)x_3+4\xi x_4\\+(4-4\xi-8\eta)x_5-4\xi x_6\end{pmatrix} & \begin{pmatrix}(4\eta-1)y_2+(-4\tau+1)y_3+4\xi y_4\\+(4-4\xi-8\eta)y_5-4\xi y_6\end{pmatrix}\end{bmatrix}\cdot\begin{bmatrix}T_x\\T_y\end{bmatrix}$$

$$\begin{bmatrix}T_x\\T_y\end{bmatrix}=\mathbf{J}^{-1}\begin{bmatrix}T_\xi\\T_\eta\end{bmatrix}=\frac{1}{\det\mathbf{J}}\begin{bmatrix}y_\eta & -y_\xi\\-x_\eta & x_\xi\end{bmatrix}\begin{bmatrix}\begin{pmatrix}(4\xi-1)T_1+(-4\tau+1)T_3+4\eta T_4\\-4\eta T_5+(4-8\xi-4\eta)T_6\end{pmatrix}\\ \begin{pmatrix}(4\eta-1)T_2+(-4\tau+1)T_3+4\xi T_4\\+(4-4\xi-8\eta)T_5-4\xi T_6\end{pmatrix}\end{bmatrix}$$

$$=\frac{1}{\det\mathbf{J}}\begin{bmatrix}y_\eta & -y_\xi\\-x_\eta & x_\xi\end{bmatrix}\begin{bmatrix}(4\xi-1) & 0 & (-4\tau+1) & 4\eta & -4\eta & (4-8\xi-4\eta)\\ 0 & (4\eta-1) & (-4\tau+1) & 4\xi & (4-4\xi-8\eta) & -4\xi\end{bmatrix}\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e$$

No. of points, n	Weight, w_i	Multi-plicity	ξ_i	η_i	τ_i
One (deg.1), $O(h^2)$	1/2	1	1/3	1/3	1/3
Three (deg.2),$O(h^3)$	1/6	3	2/3	1/6	1/6
Three (deg.2),$O(h^3)$	1/6	3	1/2	1/2	0
Four (deg.3),$O(h^4)$	– 9/32	1	1/3	1/3	1/3
	25/96	3	3/5	1/5	1/5

Gauss quadrature formula for triangle $\int_0^1\int_0^{1-\xi} f(\xi,\eta)\,d\eta\, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i,\eta_i)$



ξη space

Finite Element Method

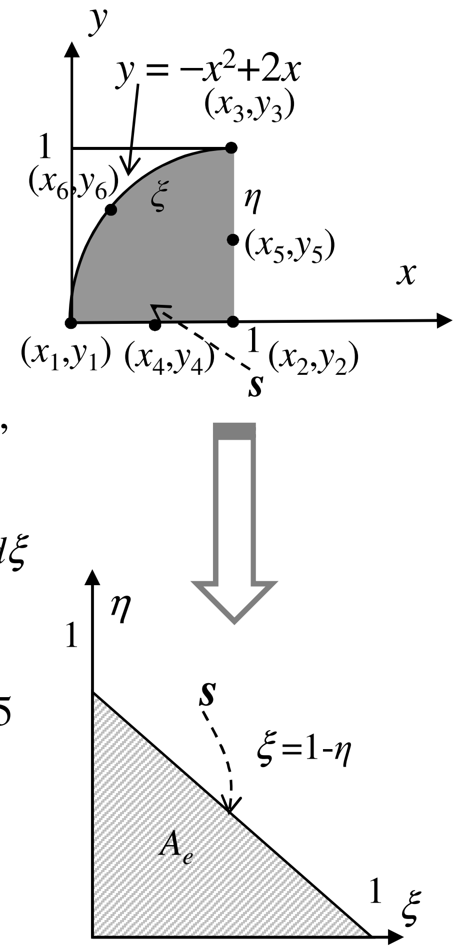
Higher order element – six-node triangle

$$\mathbf{r}_Q = Q_e\int_e \mathbf{N}^T dA = Q_e\int_{\eta=0}^{1}\int_{\xi=0}^{1-\eta}\mathbf{N}^T \det\mathbf{J}\, d\xi d\eta = Q_e\int_{\eta=0}^{1}\int_{\xi=0}^{1-\eta} f(\xi,\eta)\, d\xi d\eta$$

$$= Q_e\left[\tfrac{1}{6}\mathbf{N}^T \det\mathbf{J}\Big|_{(\frac12,\frac12,0)} + \tfrac{1}{6}\mathbf{N}^T \det\mathbf{J}\Big|_{(\frac12,0,\frac12)} + \tfrac{1}{6}\mathbf{N}^T \det\mathbf{J}\Big|_{(0,\frac12,\frac12)}\right]$$

$$\mathbf{N}^T(\tfrac12,\tfrac12,0) = \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^T,\quad \det\mathbf{J}(\tfrac12,\tfrac12,0) = \begin{vmatrix} x_1+x_3+2x_4-2x_5-2x_6 & y_1+y_3+2y_4-2y_5-2y_6 \\ x_2+x_3+2x_4-2x_5-2x_6 & y_2+y_3+2y_4-2y_5-2y_6 \end{vmatrix}$$

Numerical integration – quadratic triangle element



Problem: Calculate the integration $\int_S (x-y+1)\,dA$

$$\int_s (x-y+1)dA = \int_{x=0}^{1}\int_{y=0}^{-x^2+2x}(x-y+1)dydx = \int_{x=0}^{1}\left(-\tfrac12 x^4 + x^3 - x^2 + 2x\right)dx = \tfrac{49}{60} = \mathbf{0.8166}$$

Solution: we get $(x_1,y_1)=(0,0)$, $(x_2,y_2)=(1,0)$, $(x_3,y_3)=(1,1)$, $(x_4,y_4)=(½,0)$, $(x_5,y_5)=(1,½)$, $(x_6,y_6)=(½[3-\sqrt{5}],½[-1+\sqrt{5}])=(0.382,0.618)$. So, we get

$$I = \int_s (x-y+1)dA = \int_0^1\int_0^{1-\xi}\left(\sum_{i=1}^{6}N_i x_i - \sum_{i=1}^{6}N_i y_i + 1\right)\det(\mathbf{J})\,d\eta d\xi = \int_0^1\int_0^{1-\xi} f(\xi,\eta,\tau)\det(\mathbf{J}(\xi,\eta,\tau))\,d\eta d\xi$$

$$= \tfrac16 f(\tfrac12,\tfrac12,0)\det(\mathbf{J}(\tfrac12,\tfrac12,0)) + \tfrac16 f(\tfrac12,0,\tfrac12)\det(\mathbf{J}(\tfrac12,0,\tfrac12)) + \tfrac16 f(0,\tfrac12,\tfrac12)\det(\mathbf{J}(0,\tfrac12,\tfrac12)).$$

Finally, we get $f(½,½,0)=x_4-y_4+1=1.5$, $f(½,0,½)=x_6-y_6+1=0.764$, $f(0,½,½)=x_5-y_5+1=1.5$

$$|\mathbf{J}| = \begin{vmatrix} \begin{pmatrix}(4\xi-1)x_1+(-4\tau+1)x_3+4\eta x_4 \\ -4\eta x_5+(4-8\xi-4\eta)x_6\end{pmatrix} & \begin{pmatrix}(4\xi-1)y_1+(-4\tau+1)y_3+4\eta y_4 \\ -4\eta y_5+(4-8\xi-4\eta)y_6\end{pmatrix} \\ \begin{pmatrix}(4\eta-1)x_2+(-4\tau+1)x_3+4\xi x_4 \\ +(4-4\xi-8\eta)x_5-4\xi x_6\end{pmatrix} & \begin{pmatrix}(4\eta-1)y_2+(-4\tau+1)y_3+4\xi y_4 \\ +(4-4\xi-8\eta)y_5-4\xi y_6\end{pmatrix} \end{vmatrix}$$

Finite Element Method

Higher order element – six-node triangle

Numerical integration – quadratic triangle element

$$|\mathbf{J}(\tfrac{1}{2},\tfrac{1}{2},0)| = \begin{vmatrix} (x_1+x_3+2x_4-2x_5-2x_6) & (y_1+y_3+2y_4-2y_5-2y_6) \\ (x_2+x_3+2x_4-2x_5-2x_6) & (y_2+y_3+2y_4-2y_5-2y_6) \end{vmatrix} =$$

$$= \begin{vmatrix} (x_1+x_3+2x_4-2x_5-2x_6) & (y_1+y_3+2y_4-2y_5-2y_6) \\ x_2-x_1 & y_2-y_1 \end{vmatrix} = \begin{vmatrix} -0.764 & -1.236 \\ 1 & 0 \end{vmatrix} = 1.236$$

$$|\mathbf{J}(\tfrac{1}{2},0,\tfrac{1}{2})| = \begin{vmatrix} (x_1-x_3) & (y_1-y_3) \\ (-x_2-x_3+2x_4+2x_5-2x_6) & (-y_2-y_3+2y_4+2y_5-2y_6) \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 0.236 & -1.236 \end{vmatrix} = 1.472$$

$$|\mathbf{J}(0,\tfrac{1}{2},\tfrac{1}{2})| = \begin{vmatrix} (-x_1-x_3+2x_4-2x_5+2x_6) & (-y_1-y_3+2y_4-2y_5+2y_6) \\ (x_2-x_3) & (y_2-y_3) \end{vmatrix} = \begin{vmatrix} -1.236 & -0.764 \\ 0 & -1 \end{vmatrix} = 1.236$$

Properties of determinant

$$\det\left(A_{r_i \leftrightarrow r_j}\right) = -\det(A),\ i \neq j.$$

$$\det\left(A_{kr_i \to r_i}\right) = k \cdot \det(A),\ i \neq j.$$

$$\det\left(A_{kr_i + r_j \to r_j}\right) = \det(A),\ i \neq j.$$

Finally, we get $f(½,½,0)=x_4-y_4+1=1.5$, $f(½,0,½)=x_6-y_6+1=0.764$, $f(0,½,½)=x_5-y_5+1=1.5$

$$I = \int_s (x-y+1)dA = \tfrac{1}{6} f(\tfrac{1}{2},\tfrac{1}{2},0)\det(\mathbf{J}(\tfrac{1}{2},\tfrac{1}{2},0)) + \tfrac{1}{6} f(\tfrac{1}{2},0,\tfrac{1}{2})\det(\mathbf{J}(\tfrac{1}{2},0,\tfrac{1}{2})) + \tfrac{1}{6} f(0,\tfrac{1}{2},\tfrac{1}{2})\det(\mathbf{J}(0,\tfrac{1}{2},\tfrac{1}{2}))$$

$$= \tfrac{1}{6}(1.5)(1.236) + \tfrac{1}{6}(0.764)(1.472) + \tfrac{1}{6}(1.5)(1.236) = \mathbf{0.8055}$$

If we use 4 points Gauss quadrature (deg.3), we get I=-0.4539+0.3901+0.4833+0.3901=**0.8096**

We also can approximate the integration using averaging of function!

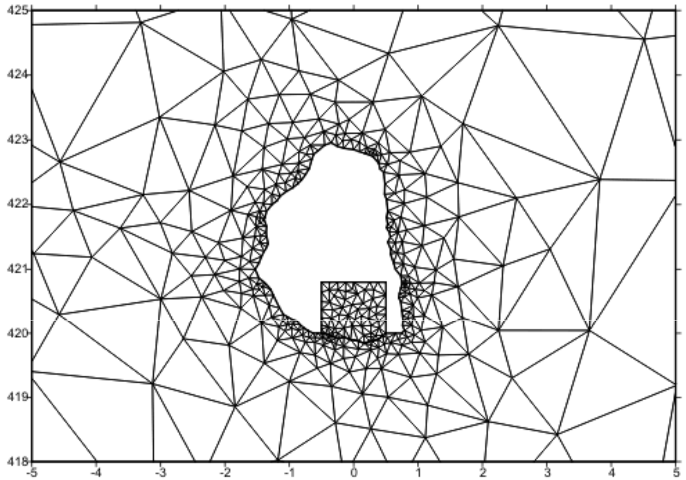
$$I = \int_s (x-y+1)dA \approx average(x-y+1)\int_{x=0}^{1}\int_{y=0}^{-x^2+2x} dydx = \frac{(0+1)+(1-0+1)+(1-1+1)}{3}\int_{x=0}^{1} -x^2+2x\,dx = \tfrac{4}{3}\left[-\tfrac{x^3}{3}+x^2\right]_0^1 = \tfrac{4}{3}\cdot\tfrac{2}{3} = \tfrac{8}{9} = \mathbf{0.889}$$

For all the six points, we get the approximation answer, $I \approx$ **0.863**

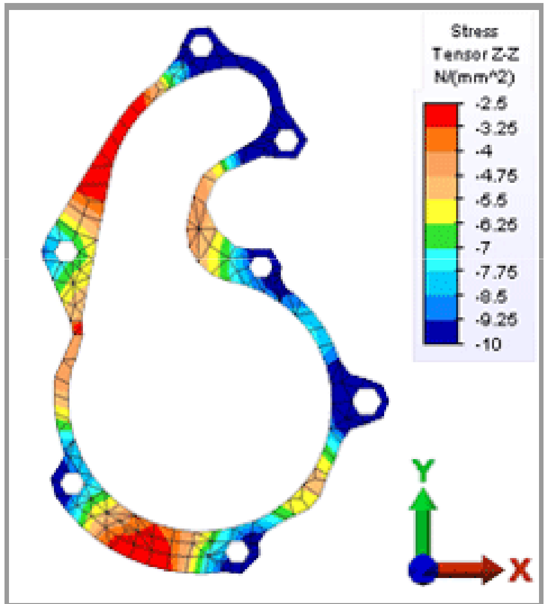
Finite Element Method

Higher order element – six-node triangle

Some examples of 3D element in FEM



Adaptive triangular elements



Finite Element Analysis modeling provides ISI application engineers with powerful technologies for analyzing joints and modeling sealing solutions.

mix triangular/rectangular elements – colour indicates stress

Finite Element Method

Isoparametric representation

The approximation of geometry similar with dependent variable (displacement) as

$$x = \sum_{i=1}^{m} x_i^e \hat{\psi}_i^e(\xi) \qquad (a)$$

where x_i^e is the global coordinate of i^{th} node of element and $\hat{\psi}_i^e$ shape functions of degree m-1.
At the same time, the approximation of dependent variable u as

$$u(x) = \sum_{j=1}^{n} u_i^e \psi_i^e(x) \qquad (b)$$

In general, the degree of approximation for coordinate transformation (a) is not equal to the degree of approximation (b) used to represent a dependable variable, $\hat{\psi}_i^e \neq \psi_i^e$. FE formulations are classified into

- Subparametric formulations: $m < n$
  The geometry is represented by lower order elements than the approximation of dependent variable. E.g. Euler-Bernoulli beam element.
- Isoparametric formulations: $m = n$
  the same element is used to approximate the geometry and dependent variable, $\hat{\psi}_i(\xi) = \psi_i(x)$.
- Superparametric formulations: $m > n$.
  The geometry is represented with higher order elements. It is seldom used in practice.

Finite Element Method

Acceptable element geometries

Numerical evaluation of integrals over actual elements involves a coordinate transformation from the actual element to a master element. The transformation is acceptable if and only if every point in the actual element is mapped uniquely into a point in the master element, and vice versa (one-to-one map).

Let, $x = \sum_i x_i^e \psi_i^e(\xi,\eta), \quad y = \sum_i y_i^e \psi_i^e(\xi,\eta)$

We get

$$\begin{bmatrix} \partial\psi_i^e/\partial\xi \\ \partial\psi_i^e/\partial\eta \end{bmatrix} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix} \begin{bmatrix} \partial\psi_i^e/\partial x \\ \partial\psi_i^e/\partial y \end{bmatrix} = \mathbf{J}\cdot \begin{bmatrix} \partial\psi_i^e/\partial x \\ \partial\psi_i^e/\partial y \end{bmatrix} \rightarrow \begin{bmatrix} \partial\psi_i^e/\partial x \\ \partial\psi_i^e/\partial y \end{bmatrix} = \mathbf{J}^{-1}\cdot \begin{bmatrix} \partial\psi_i^e/\partial\xi \\ \partial\psi_i^e/\partial\eta \end{bmatrix}$$

e.g. dilation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{J} \begin{bmatrix} x \\ y \end{bmatrix}$$

In order to compute the global derivatives of ψ_i^e, it requires inversion of Jacobian matrix. A necessary and sufficient condition for $\mathbf{J}^{-1}$ to exist is that

$$\det \mathbf{J} = \left(\partial x/\partial\xi\right)\left(\partial y/\partial\eta\right) - \left(\partial x/\partial\eta\right)\left(\partial y/\partial\xi\right) > 0$$

Geometrically, the Jabocian, det **J**, represents the ratio of an area element in the real element to the corresponding area element in the master element for integration on element.

Integration on curvilinear coordinates:

Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, $\mathrm{d}\mathbf{r}=\mathrm{d}x\mathbf{i}+\mathrm{d}y\mathbf{j}$, then we get $\mathrm{d}A=|\mathrm{d}x\mathbf{i}\times\mathrm{d}y\mathbf{j}|=|\mathbf{i}\times\mathbf{j}|\mathrm{d}x\mathrm{d}y=\mathrm{d}x\mathrm{d}y$.

Now, let $\mathbf{N}=[N_1(\xi,\eta), N_2(\xi,\eta), N_3(\xi,\eta)]$ and $\mathbf{x}=[x_1,x_2,x_3]^T$, $\mathbf{y}=[y_1,y_2,y_3]^T$, we get $\mathbf{r}=\mathbf{Nx}\mathbf{i}+\mathbf{Ny}\mathbf{j}=x\mathbf{i}+y\mathbf{j}$,

so $\mathrm{d}\mathbf{r}=\mathbf{r}_\xi\mathrm{d}\xi+\mathbf{r}_\eta\mathrm{d}\eta$, we get $\mathrm{d}A=|\mathbf{r}_\xi\mathrm{d}\xi\times\mathbf{r}_\eta\mathrm{d}\eta|=|\mathbf{r}_\xi\times\mathbf{r}_\eta|\mathrm{d}\xi\,\mathrm{d}\eta=\det\mathbf{J}\,\mathrm{d}\xi\,\mathrm{d}\eta$.

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\sin\theta\,\mathbf{n}$$

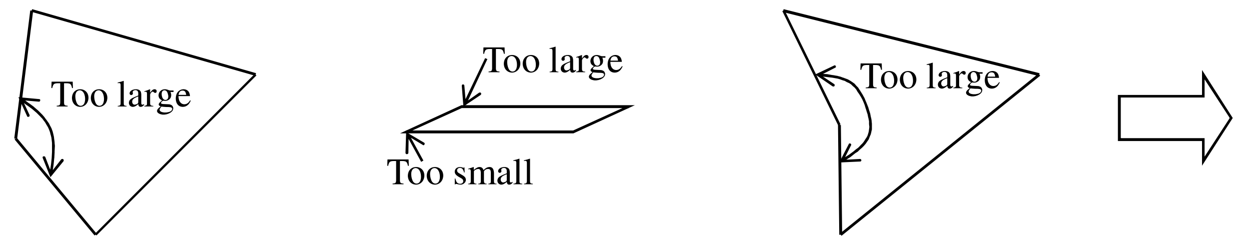
$$\mathbf{r}_\xi\times\mathbf{r}_\eta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ N_\xi\mathbf{x} & N_\xi\mathbf{y} & 0 \\ N_\eta\mathbf{x} & N_\eta\mathbf{y} & 0 \end{vmatrix} = \mathbf{k}\begin{vmatrix} N_\xi\mathbf{x} & N_\xi\mathbf{y} \\ N_\eta\mathbf{x} & N_\eta\mathbf{y} \end{vmatrix} \leftrightarrow \det\mathbf{J} = \begin{vmatrix} N_\xi\mathbf{x} & N_\xi\mathbf{y} \\ N_\eta\mathbf{x} & N_\eta\mathbf{y} \end{vmatrix} = \begin{vmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{vmatrix}$$

Finite Element Method

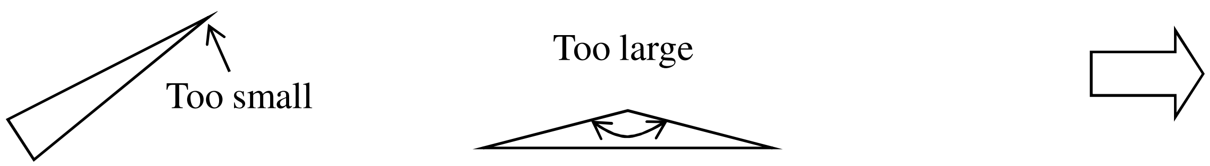
Acceptable element geometries

- If det **J** is zero, then a nonzero area element is mapped into zero area in master element (unacceptable).
- If det **J** <0, a right-handed coordinate system is mapped into a left-handed coordinate system (unacceptable).
- If det **J** ≈ 0, excessive distortion of elements is not good because a nonzero area element can be mapped into a zero or nearly zero area.

To ensure det **J** >0 and keep within the limits of acceptable distortion, certain geometric shapes of real elements must be avoided. For example, the interior angle at each vertex of a triangular element should not be equal to either 0° or 180° (angle between 0° and 180°) to avoid numerical ill-conditioning of element matrices.



Unacceptable vertex angles for linear quadrilateral elements



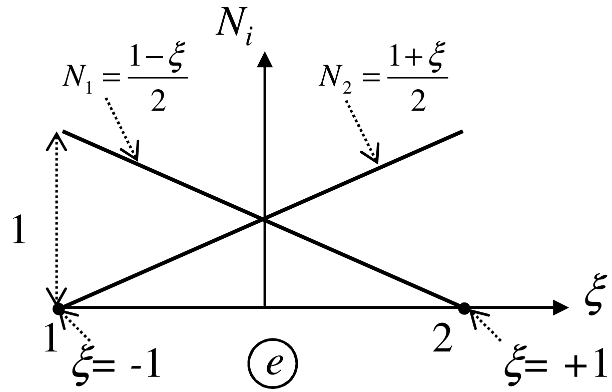
Unacceptable vertex angles for linear triangular elements

Finite Element Method

Dynamic considerations – solid body with distributed mass

Let displacement, $u(x)$, the stress-strain and strain-displacement relations are $\sigma = E\varepsilon$, $\varepsilon = du/dx$, $\sigma = F/A$ (N/m^2) where E is Young's modulus (or modulus of elasticity).

$$u(\xi)=N_1q_1+N_2q_2 = \mathbf{Nq}$$



where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{q}=[q_1, q_2]^T$. (q_i is nodal displacement)

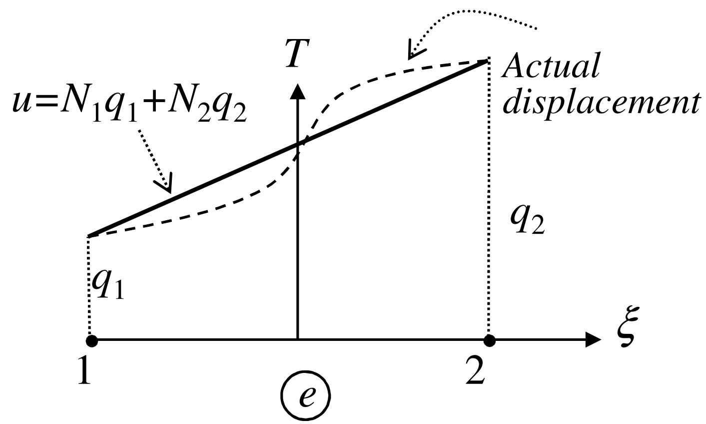
Please note $x=N_1x_1+N_2x_2$, and $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1, \; d\xi = \frac{2}{x_2 - x_1}dx = \frac{2}{l_e}dx.$

Use chain rule, $$\frac{du}{dx} = \frac{du}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{q} = \frac{1}{l_e}[-1,1]\mathbf{q} = \mathbf{Bq} = \varepsilon.$$

where matrix **B** is called element strain-displacement matrix.

The stress, from Hooke's law, is given $\sigma = E\mathbf{Bq}$

The potential-energy approach



The total potential energy, Π of an elastic body is

Π= strain energy (U) + work potential (WP)

where $U = \frac{1}{2}\int_V \sigma^T \varepsilon dV$, $WP = -\int_V u^T f dV - \int_S u^T T dS - \sum_i u_i^T P_i$

and we get

$$\Pi = \frac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T f dV - \int_S u^T T dS - \sum_i u_i^T P_i$$

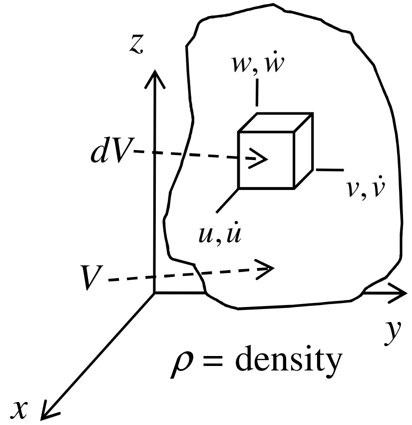
where T (distributed force per unit area, traction), f (distributed force per unit volume, e.g. weight per unit volume), P (force, a load acting at point i), u_i (displacement at point i)

Finite Element Method

Dynamic considerations – solid body with distributed mass

The kinetic energy is given by $T = \frac{1}{2}\int_V \dot{u}^T \dot{u} \rho dV$

where the velocity vector is given $\dot{\mathbf{u}} = [\dot{u}, \dot{v}, \dot{w}]^T$



1-D problem

Let **q** is the nodal displacements, we get u=**Nq**

In dynamic analysis, the elements of **q** are dependent on time, **N** represents (spatial) shape function defined on a master element. The velocity vector is $\dot{u} = \mathbf{N}\dot{\mathbf{q}}$

So, we get

$$T_e = \tfrac{1}{2}\dot{\mathbf{q}}^T \left[\int_e \rho \mathbf{N}^T \mathbf{N} dV\right] \dot{\mathbf{q}} = \tfrac{1}{2}\dot{\mathbf{q}}^T \mathbf{m}^e \dot{\mathbf{q}}$$

where $\mathbf{m}^e$ is the **element mass matrix**. This mass matrix is consistent with the shape functions chosen and is called the **consistent mass matrix**. On summing over all the elements, the kinetic energy is given

$$T = \sum_e T_e = \sum_e \tfrac{1}{2}\dot{\mathbf{q}}^T \mathbf{m}^e \dot{\mathbf{q}} = \tfrac{1}{2}\dot{\mathbf{Q}}^T \mathbf{M} \dot{\mathbf{Q}}$$

The potential energy is given by $\Pi = \tfrac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T f dV - \int_S u^T T dS - \sum_i u_i^T P_i$

using relation $\sigma = E\mathbf{Bq}$, $\varepsilon = \mathbf{Bq}$, we get

$$\Pi_e = \tfrac{1}{2}\int_e \mathbf{q}^T \mathbf{B}^T E \mathbf{Bq} A dx - \mathbf{q}^T A_e f \int_e \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx - \mathbf{q}^T T \int_e \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$\Pi = \sum_e \Pi_e - \sum_i q_i P_i = \sum_e \left(\tfrac{1}{2}\mathbf{q}^T \mathbf{k}^e \mathbf{q} - \mathbf{q}^T \mathbf{f}^e - \mathbf{q}^T \mathbf{T}^e\right) - \mathbf{Q}^T \mathbf{P} = \tfrac{1}{2}\mathbf{Q}^T \mathbf{K} \mathbf{Q} - \mathbf{Q}^T \mathbf{F}$$

K (global stiffness matrix),
F (global load vector),
Q (global displacement vector).

where element stiffness matrix, element body force vector, element traction-force vector are:

$$\mathbf{k}^e = \frac{E_e A_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{f}^e = \frac{A_e l_e f}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{T}^e = \frac{T l_e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finite Element Method

Dynamic considerations – solid body with distributed mass

Lagrangean

We define Lagrangean by $L = \mathrm{T} - \Pi$

where T (kinetic energy), Π (potential energy).

Hamilton's principle: for an arbitrary time interval from t_1 to t_2, the state of motion of a body extremizes the functional $I = \int_{t_1}^{t_2} L(t, q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$ (Euler eq.→) the equation of motion are: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \cdots, n.$

Now, we get $L = T - \Pi = \frac{1}{2}\dot{\mathbf{Q}}^T \mathbf{M} \dot{\mathbf{Q}} - \frac{1}{2}\mathbf{Q}^T \mathbf{K} \mathbf{Q} + \mathbf{Q}^T \mathbf{F}$

Matrix and vector derivatives

The derivative of the row vector **y** w.r.t scalar x is $\frac{\partial \mathbf{y}}{\partial x} = \left[\frac{\partial y_1}{\partial x} \ \cdots \ \frac{\partial y_m}{\partial x}\right]$

The derivative of a scalar y w.r.t. vector **x** is $\frac{\partial y}{\partial \mathbf{x}} = \left[\frac{\partial y}{\partial x_1} \ \cdots \ \frac{\partial y}{\partial x_n}\right]^T$

Let **x** be n×1 vector and **y** be m×1 vector, the derivative of **y** w.r.t. **x** is a matrix
$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_m/\partial x_1 \\ \vdots & \ddots & \vdots \\ \partial y_1/\partial x_n & \cdots & \partial y_m/\partial x_n \end{bmatrix}_{n\times m}$$

So, we get $\frac{\partial L}{\partial \dot{\mathbf{Q}}} = \frac{1}{2}\left(\mathbf{M}\dot{\mathbf{Q}} + \mathbf{M}^T\dot{\mathbf{Q}}\right)$ $\quad \frac{\partial L}{\partial \mathbf{Q}} = -\frac{1}{2}\left(\mathbf{K}\mathbf{Q} + \mathbf{K}^T\mathbf{Q}\right) + \mathbf{F}$

M and **K** : symmetrical matrix→ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{Q}}}\right) - \frac{\partial L}{\partial \mathbf{Q}} = \frac{d}{dt}\mathbf{M}\dot{\mathbf{Q}} - (-\mathbf{K}\mathbf{Q} + \mathbf{F}) = 0 \quad \leftrightarrow \quad \mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{F}$

y (scalar or a vector)	$\partial y/\partial \mathbf{x}$
A**x**	A^{T}
$\mathbf{x}^{\mathrm{T}}$A	A
$\mathbf{x}^{\mathrm{T}}\mathbf{x}$	2**x**
$\mathbf{x}^{\mathrm{T}}\mathrm{A}\mathbf{x}$	$\mathrm{A}\mathbf{x} + \mathrm{A}^{\mathrm{T}}\mathbf{x}$

For free vibrations, **F**=0. For steady-state condition, starting from the equilibrium state, we set **Q**=**U**sinωt, where **U** is the vector of nodal amplitudes of vibration and ω (rad/s) is the circular frequency (=2πf, f=cycles/s or Hz), we get $\mathbf{KU} = \omega^2 \mathbf{MU}$

This is generalized eigenvalue problem $\mathbf{KU} = \lambda \mathbf{MU}$

where **U** is eigenvector, representing the vibrating mode, corresponding to eigenvalue λ.

Finite Element Method

Dynamic considerations – solid body with distributed mass

Element mass matrices – 1D bar element

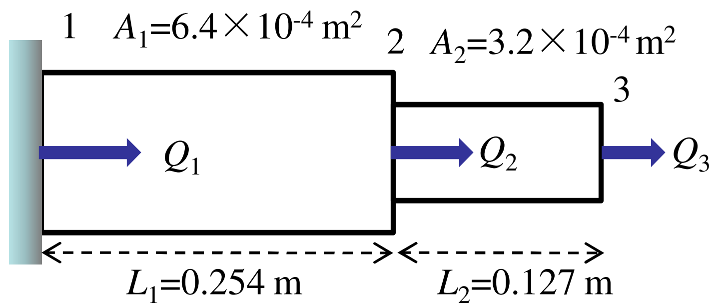
We get, $\mathbf{q}^T=[q_1\ q_2]$, $\mathbf{N}=[N_1\ N_2]$, where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$. Note: $x=N_1x_1+N_2x_2$, $\xi=\frac{2}{x_2-x_1}(x-x_1)-1,\ d\xi=\frac{2}{l_e}dx.$

We get

$$\mathbf{m}^e=\rho\int_e \mathbf{N}^T\mathbf{N}A\,dx=\frac{\rho A_e l_e}{2}\int_{-1}^{+1}\mathbf{N}^T\mathbf{N}\,d\xi=\frac{\rho A_e l_e}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

Problem: Determine the eigenvalues and eigenvectors for the stepped bar below:

Solution: generalized eigenvalue problem is given **$\mathbf{KU}=\lambda\mathbf{MU}$**



$E=206.8\times10^9$ Pa, specific weight $f=7.68\times10^4$ N/m^3.

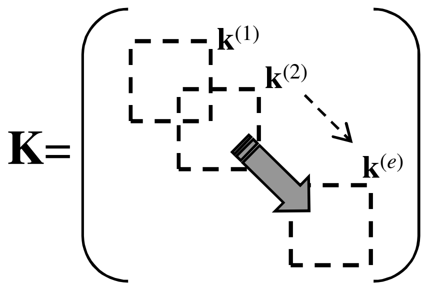
$$L=T-\Pi=\sum_e \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{m}^e\dot{\mathbf{q}}-\sum_e\left(\tfrac{1}{2}\mathbf{q}^T\mathbf{k}^e\mathbf{q}-\mathbf{q}^T\mathbf{f}^e-\mathbf{q}^T\mathbf{T}^e\right)+\mathbf{Q}^T\mathbf{P}$$

$$=\tfrac{1}{2}\dot{\mathbf{Q}}^T\mathbf{M}\dot{\mathbf{Q}}-\tfrac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q}+\mathbf{Q}^T\mathbf{F}$$

So, we get $\mathbf{k}^1=\frac{EA_1}{L_1}\begin{bmatrix}1&-1\\-1&1\end{bmatrix},\ \mathbf{k}^2=\frac{EA_2}{L_2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$

$$\mathbf{m}^1=\frac{\rho A_1L_1}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix},\quad \mathbf{m}^2=\frac{\rho A_2L_2}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

$\mathbf{Q}=\mathbf{U}\sin\omega t,\ \lambda=\omega^2$



So, we get

$$\mathbf{K}=\begin{bmatrix}\frac{EA_1}{L_1}&-\frac{EA_1}{L_1}&0\\-\frac{EA_1}{L_1}&\frac{EA_1}{L_1}&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&\frac{EA_2}{L_2}&-\frac{EA_2}{L_2}\\0&-\frac{EA_2}{L_2}&\frac{EA_2}{L_2}\end{bmatrix}=E\begin{bmatrix}\frac{A_1}{L_1}&-\frac{A_1}{L_1}&0\\-\frac{A_1}{L_1}&\left(\frac{A_1}{L_1}+\frac{A_2}{L_2}\right)&-\frac{A_2}{L_2}\\0&-\frac{A_2}{L_2}&\frac{A_2}{L_2}\end{bmatrix}$$

Pa=N/m^2, N=kg·m/s^2

Finite Element Method

Dynamic considerations – solid body with distributed mass

KU=λMU →

$$E\begin{bmatrix} A_1/L_1 & -A_1/L_1 & 0 \\ -A_1/L_1 & \left(A_1/L_1 + A_2/L_2\right) & -A_2/L_2 \\ 0 & -A_2/L_2 & A_2/L_2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \lambda \rho/6 \begin{bmatrix} 2A_1L_1 & A_1L_1 & 0 \\ A_1L_1 & 2(A_1L_1 + A_2L_2) & A_2L_2 \\ 0 & A_2L_2 & 2A_2L_2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

where u_i and U_i are local/global nodal amplitudes of vibration. Q_i and q_i are global/local displacement nodes.

Gathering the stiffness and mass values corresponding to the deg. of freedom Q_2 and Q_3, (delete 1st row, substitute U_1=0 for remaining rows), we get

$$E\begin{bmatrix} \left(A_1/L_1 + A_2/L_2\right) & -A_2/L_2 \\ -A_2/L_2 & A_2/L_2 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \lambda \rho/6 \begin{bmatrix} 2(A_1L_1 + A_2L_2) & A_2L_2 \\ A_2L_2 & 2A_2L_2 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

The density is given $\rho = \dfrac{f}{g} = \dfrac{7.68\times10^4}{9.81} = 7828.75\ kg/m^3$

Substitute this values, we get $206.8\times10^9\begin{bmatrix} 5\times10^{-3} & -2.5\times10^{-3} \\ -2.5\times10^{-3} & 2.5\times10^{-3} \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \lambda\cdot 7828.75\begin{bmatrix} 2.4\times10^{-4} & 4\times10^{-5} \\ 4\times10^{-5} & 8\times10^{-5} \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$

KU=λMU → (K – λM)U = 0 → det(**K** – λ**M**)=0 Get eigenvalue λ_1, λ_2. (**K** – λ_i**M**)**U**$_i$ **= 0** get eigenvector U_1, U_2.

We get eigenvalue λ_1, λ_2= 0.1650×10^{10}, 0.1501×10^{9}; Eigenvectors **U**$_1$, **U**$_2$=[0.4996, –1]^T, [–0.7484, –1]^T.

For normalization, we use **U**$_i^T$**MU**$_i$=1

Finally, we get

Eigenvectors **U**$_1$, **U**$_2$=[0.5650, –1.131]^T, [–0.5109, –0.6826]^T.

Finite Element Method

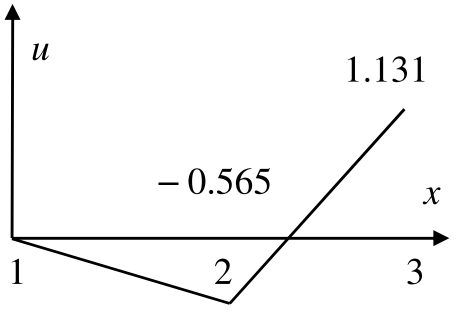
Dynamic considerations – solid body with distributed mass

Note that $\lambda=\omega^2$, where ω is the circular frequency ($\omega=2\pi f$), f=frequency in hertz, Hz (cycles/s). Period, $T=2\pi/\omega$

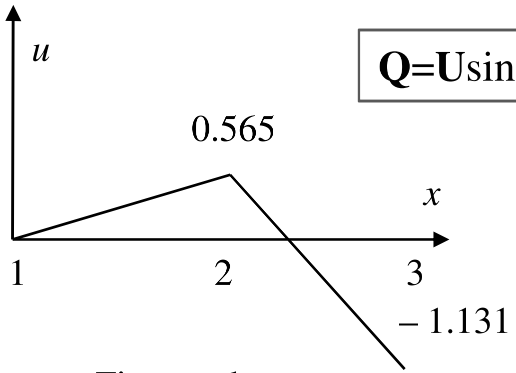
The mode shapes are given

Free vibration at L_1=0.254m, L_2=0.127m! Magnitude depend on elongation.

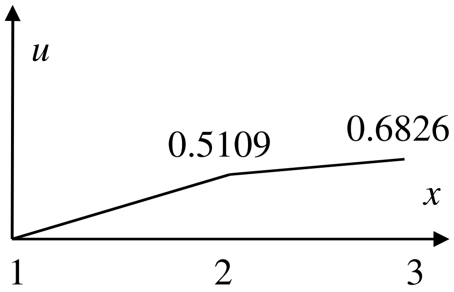
$\mathbf{Q=U}\sin\omega t$



vibration



First mode



Second mode

Generalized Eigenvalue problem

The generalized eigenvalue problem is defined by $\mathbf{KU}=\lambda\mathbf{MU}$

There are n (dim. of system) eigenpairs. The i^{th} eigenpair is denoted by $(\lambda_i,\mathbf{U}_i)$, where the eigenvalues are ordered according to their magnitude: $0\le\lambda_1\le\lambda_2\le\ldots\le\lambda_{n-1}\le\lambda_n$.

And the eigenvectors are mass-normalized as:

$$\mathbf{U}_i^T\mathbf{M}\mathbf{U}_j=\begin{cases}1, & i=j\\ 0, & i\ne j\end{cases}\quad,\quad \mathbf{U}_i^T\mathbf{K}\mathbf{U}_j=\begin{cases}\lambda_i, & i=j\\ 0, & i\ne j\end{cases}$$

or we get $\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}_{n\times n}$, $\mathbf{U}^T\mathbf{K}\mathbf{U}=\Lambda$

The above properties are hold if **K** is symmetric, **M** is symmetric and positive definite. Using Cholesky factorization, the generalized eigenproblem can be written $\mathbf{KU}=\lambda\mathbf{LL}^T\mathbf{U}\rightarrow\mathbf{L}^{-1}\mathbf{KU}=\lambda\mathbf{L}^T\mathbf{U}\rightarrow\mathbf{Cy}=\lambda\mathbf{y}$ (with $\mathbf{y}=\mathbf{L}^T\mathbf{U}$, $\mathbf{C}=\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$, sym.)

If **A** is symmetric, we get (eigenvalues of sym. matrix are real)

$$\begin{bmatrix}2&4&6\\4&7&8\\6&8&5\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\3&4&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&4\\0&0&1\end{bmatrix}$$

If **A** is symmetric and positive definite, then we don't have diagonal matrix, and diagonal entries are not necessary 1 in matrix **L**.

Finite Element Method

Generalized Eigenvalue problem

Let **A** be the matrix representation of any inner product on V. Then **A** is a positive definite matrix.

Let u_1=(1,1,0), u_2=(1,2,3), u_3=(1,3,5) form a basis *S* for Euclidean space $\mathbf{R}^3$. We get $\langle u_1,u_1\rangle$=1+1+0=2, $\langle u_1,u_2\rangle$=1+2+0=3,…

$$\mathbf{A}=\begin{bmatrix} 2 & 3 & 4 \\ 3 & 14 & 22 \\ 4 & 22 & 35 \end{bmatrix}$$

A=positive definite matrix
- $\mathbf{A}^{-1}$ exist
- $a_{ii}>0$
- $(a_{ij})^2<a_{ii}a_{jj}$

Let **A** be a real symmetric matrix, then it is positive definite if, for every nonzero vector **u**, $\langle \mathbf{u},\mathbf{Au}\rangle=\mathbf{u}^T\mathbf{Au}>0$

Inner Product spaces: Let *V* real vector space. Let $u,v\in V$, there is assigned a real number, denoted by $\langle u,v\rangle$. This function is called a **inner product** on *V* if it satisfied the following axioms:
- (linear property): $\langle au_1+bu_2,v\rangle= a\langle u_1,v\rangle+ b\langle u_2,v\rangle$
- (symmetric property): $\langle u,v\rangle= \langle v,u\rangle$
- (positive definite property): $\langle u,u\rangle\geq 0$; and $\langle u,u\rangle=0$ if and only if $u=0$.

Examples of inner product spaces:
- Euclidean Space, $\mathbf{R}^n$: $\langle u,v\rangle=u\cdot v=a_1b_1+\ldots a_nb_n$.
- Function Space, $C[a,b]$: $\langle f,g\rangle=\int_a^b f(t)g(t)dt$
- Matrix Space, $\mathbf{M}_{m\times n}$: $\langle \mathbf{A},\mathbf{B}\rangle=\mathrm{tr}(\mathbf{B}^T\mathbf{A})$
- Hilbert Space, l_2-space: $\langle u,v\rangle=u\cdot v=a_1b_1+a_2b_2+\ldots$.

Cholesky factorization
$\mathbf{A=LL^T}$
The algorithm is given

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$l_{kk}=\sqrt{a_{kk}-\sum_{s=1}^{k-1}l_{ks}^2}$$
$$l_{ik}=\left(a_{ik}-\sum_{s=1}^{k-1}l_{is}l_{ks}\right)\Big/l_{kk}, \quad i=k,k+1,\cdots,n.$$

Finite Element Method

Generalized Eigenvalue problem

Inverse Iteration Method

Let $u_1=\Sigma_{i=1}^n a_i\phi_i$ (ϕ_i are normalized eigenvector)
use $\mathbf{K}u_{k+1}=\mathbf{M}u_k$, we get
$u_2=\mathbf{K}^{-1}\mathbf{M}u_1= \Sigma_{i=1}^n a_i\mathbf{K}^{-1}\mathbf{M}\phi_i$
Since $\mathbf{K}\phi_i=\lambda_i\mathbf{M}\phi_i$, we get
$u_2=\Sigma_{k=1}^n a_k\mathbf{K}^{-1}\mathbf{M}\phi_k=\Sigma_{k=1}^n a_k\mathbf{K}^{-1}(\lambda_k^{-1}\mathbf{K}\phi_k)=\Sigma_{k=1}^n a_k\lambda_k^{-1}\phi_k$.
Similarly,
$u_3=\mathbf{K}^{-1}\mathbf{M}u_2=\Sigma_{k=1}^n a_k\lambda_k^{-1}\mathbf{K}^{-1}\mathbf{M}\phi_k=\Sigma_{k=1}^n a_k\lambda_k^{-2}\phi_k$.
And generally,
$u_{k+1}=\Sigma_{i=1}^n a_i\lambda_i^{-k}\phi_i= \lambda_1^{-k}\left(a_i\phi_1+\sum_{i=2}^n a_i\left[\lambda_1/\lambda_i\right]^k\phi_i\right)$
When k becomes large, $(\lambda_1/\lambda_i)^k\rightarrow 0$ (since $\lambda_1<\lambda_i$) and we get
$u_{k+1}= \lambda_1^{-k} a_1\phi_1 \propto \phi_1$ (proportional to), which after normalization, becomes the 1st eigenvector (λ_1 is eigenvalue).
Smallest eigenvalue/eigenvector are: $(\boldsymbol{\lambda_1}, \boldsymbol{u_{k+1}})$.

Find the lowest eigenvalue (with eigenvector) for $\mathbf{KU}=\lambda\mathbf{MU}$

The procedure are:

1.Choose a guess starting vector u_1,
2.Solve w_{k+1} from: $\mathbf{K}w_{k+1}=\mathbf{M}u_k$,
3.Normalize w_{k+1} as: $u_{k+1}=w_{k+1}\Big/\sqrt{w_{k+1}^T\mathbf{M}w_{k+1}}$
4.Compute λ_{k+1} using: $\lambda_{k+1}=u_{k+1}^T\mathbf{K}u_{k+1}$,
5.Check the convergence: $|\lambda_{k+1}-\lambda_k|/\lambda_{k+1}\le\varepsilon$. ($\varepsilon$ is error tolerance and stop iteration when satisfied)

Finite Element Method

Generalized Eigenvalue problem

Problem: Find the lowest eigenvalue and eigenvector of the system with following mass and stiffness matrices.

$$\mathbf{K} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution: Start with our first guess eigenvector, u_1=[1 1 1 1]^T.

$\mathbf{M}u_1$=[3 3 3 2]^T.

$$\mathbf{K}w_2 = \mathbf{M}u_1 \rightarrow \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} w_2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

Using Thomas algorithm, we get w_2=[11 15 20 21]^T. $w_2^T\mathbf{M}w_2$=3120.

So, $u_2=w_2/\sqrt{(3120)}$=[0.1969 0.2685 0.3580 0.3760]^T.

$\lambda_2=u_2^T\mathbf{K}u_2$=0.05768

$\mathbf{M}u_2$=[0.5907 0.8055 1.0740 0.7520]^T

Solve $\mathbf{K}w_3=\mathbf{M}u_2$, we get w_3=[3.2224 4.5381 6.3641 6.7401]^T

$w_3^T\mathbf{M}w_3$=305.30, and we get $u_3=w_3/\sqrt{(305.3)}$=[0.1844 0.2597 0.3642 0.3857]^T.

$\lambda_3=u_3^T\mathbf{K}u_3$=0.05719

So, after 2 iterations, we get

λ_3=0.05719, u_3=[0.1844 0.2597 0.3642 0.3857]^T.

The exact eigensolution is:

$\boldsymbol{\lambda_1}$**=0.05719096**, λ_2=0.42264973

λ_3=1.57735027, λ_4=1.94280904

$$[\phi] = \begin{bmatrix} \mathbf{0.182574} & 0.365148 & 0.365148 & -0.182574 \\ \mathbf{0.258198} & 0.316227 & -0.316227 & 0.258198 \\ \mathbf{0.365148} & -0.182574 & -0.182574 & -0.365148 \\ \mathbf{0.387298} & -0.316227 & 0.316227 & 0.387298 \end{bmatrix}$$

The convergence rate is **very fast** since $\boldsymbol{\lambda_1}$ is **far smaller** than λ_2 (and other eigenvalues).

Finite Element Method

Generalized Eigenvalue problem

Forward Iteration Method

Find the **largest** eigenvalue (with eigenvector) for **KU=λMU**

The procedure are:

1.Choose a guess starting vector u_1,

2.Solve w_{k+1} from: $\mathbf{M}w_{k+1}=\mathbf{K}u_k$,

3.Normalize w_{k+1} as: $u_{k+1} = w_{k+1} \Big/ \sqrt{w_{k+1}^T \mathbf{M} w_{k+1}}$

4.Compute λ_{k+1} using: $\lambda_{k+1}=u_{k+1}^T\mathbf{K}u_{k+1}$,

5.Check the convergence: $|\lambda_{k+1} - \lambda_k| / \lambda_{k+1} \leq \varepsilon$. ($\varepsilon$ is error tolerance and stop iteration when satisfied)

Problem: Find the largest eigenvalue and eigenvector of the system with following mass and stiffness matrices.

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Start with our first guess eigenvector, u_1=[1 1 1 1]^T.

Ku_1=[2 -1 -1 2]^T.

$$\mathbf{M}w_2 = \mathbf{K}u_1 \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

So, we get w_2=[1 -0.5 -1 2]^T. $w_2^T\mathbf{M}w_2$=7.5

So, $u_2=w_2/\sqrt{(7.5)}$=[0.3651 -0.1826 -0.3651 0.7303]^T.

$\lambda_2=u_2^T\mathbf{K}u_2$=5.9333

Finite Element Method

Generalized Eigenvalue problem

$\mathbf{K}u_2$=[2.1909 -0.3652 -4.0166 4.9295]^T

Solve $\mathbf{M}w_3=\mathbf{K}u_2$, we get w_3=[1.0955 -0.1826 -4.0166 4.9295]^T

$w_3^T\mathbf{M}w_3$=42.900, and we get $u_3=w_3/\sqrt{(42.9)}$=[0.1673 -0.0279 -0.0557 0.1115]^T.

$\lambda_3=u_3^T\mathbf{K}u_3$=8.5788

u_4=[0.0184 0.1306 -0.7068 0.6823]^T

$\lambda_4=u_4^T\mathbf{K}u_4$=10.1597

$\Downarrow$

u_{11}=**[-0.1073 0.2554 -0.7283 0.5623]**^{T}

$\lambda_{11}=u_{11}^T\mathbf{K}u_{11}$=**10.6384**

The exact eigensolution is:

λ_1=0.09653732, λ_2=1.39146545

λ_3=4.37354955, $\boldsymbol{\lambda_4}$=**10.63844766**

$$[\phi]=\begin{bmatrix} 0.31262952 & 0.44526615 & -0.43866985 & -\mathbf{0.1075620} \\ 0.49547585 & 0.12443600 & 0.41674029 & \mathbf{0.25563036} \\ 0.47911662 & -0.4894418 & 0.02322175 & -\mathbf{0.72825457} \\ 0.28979330 & -0.5770218 & -0.51696549 & \mathbf{0.56197181} \end{bmatrix}$$

The convergence rate is **not fast** since $\boldsymbol{\lambda_4}$ is **not far bigger** than λ_3 (and other eigenvalues).

Problem: Find the largest/smallest eigenpairs of the system with following mass and stiffness matrices (in 3 D.P.).

$$\mathbf{K}=\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{M}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

The exact eigensolution is:

$\lambda_1=-0.1679$, λ_2=1.1675

λ_3=10.20

$$\phi=\begin{bmatrix} -0.8706 & 1.000 & 0.0022 \\ 0.5084 & 0.0838 & 0.0100 \\ -1.000 & -0.1897 & 1.000 \end{bmatrix}$$

Some notes

How to create Generalized Eigenvalue problem?

$$\mathbf{KU}=\lambda\mathbf{MU}$$

The above properties are hold if **K** is symmetric, **M** is symmetric and positive definite. Using Cholesky factorization, the generalized eigenproblem can be written $\mathbf{KU}=\lambda\mathbf{LL}^T\mathbf{U} \rightarrow \mathbf{L}^{-1}\mathbf{KU}=\lambda\mathbf{L}^T\mathbf{U} \rightarrow \mathbf{Cy}=\lambda\mathbf{y}$ (with $\mathbf{y}=\mathbf{L}^T\mathbf{U}$, $\mathbf{C}=\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$, sym.) So, with given example of **C** (give simple eigenpairs), we can get **K** with $\mathbf{K}=\mathbf{LCL}^T$. We also get **M** as $\mathbf{M}=\mathbf{LL}^T$.

Problem: Solve the generalized eigenproblem using Cholesky factorization.

$$\mathbf{KU}=\lambda\mathbf{MU} \quad \rightarrow \quad \begin{bmatrix} 1 & 5 \\ 5 & 25 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Answer: eigenvalues are $(\lambda_1,\lambda_2)=(0,5)$, $[u_1,u_2]=[(1,-0.2)^T, (0,-1)^T]$.

$$\mathbf{L}=\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad , \quad \mathbf{C}=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

ξ=xi, ϕ=phi, η=eta, δ=delta, ε=epsilon, ψ=psi.