Finite Element Method
MSM 1333

Chapter 1 & 2

Preliminary concepts, Variational methods & Raleigh-Ritz method,
Time-dependent problems

Prepared by:

Dr. Yeak Su Hoe
C22-432, U.T.M. Skudai
s.h.yeak@utm.my
012-7116604

Feb 2015

Finite Element Method (FEM)

Comparison with the finite difference method (FDM)

The finite difference method (FDM) is an alternative way of approximating solutions of PDEs.
The differences between FEM and FDM are:

- The finite difference method is an approximation to the differential equation;
  the finite element method is an approximation to its solution.
- The most attractive feature of the FEM is its ability to handle complex geometries (and boundaries) with relative ease.
  While FDM in its basic form is restricted to handle rectangular shapes and simple alterations.
- The most attractive feature of finite differences is that it can be very easy to implement.
- The quality of the approximation between grid points is poor in FDM comparing to FEM.
- The quality of a FEM approximation is often higher than in the corresponding FDM approach,
  but this is extremely problem dependent and several examples to the contrary can be provided.

Generally, FEM is the method of choice in all types of analysis in **structural mechanics** while computational fluid dynamics (CFD) tends to use FDM or other methods (e.g., finite volume method). CFD problems usually require discretization of the problem into a **large number** of cells/gridpoints (millions and more), therefore cost of the solution favors **simpler**, **lower order approximation** within each cell.

Comparison with the finite difference method (FDM), finite element (FEM), finite volume (FVM) and boundary element method (BEM)

FDM	approximation to the differential equation very easy to implement Less accuracy
FEM	approximation to its solution Complicated geometries High-order approximations Use weak formulation, Strong mathematical foundation
FVM	Rather than pointwise approximations on a grid, FVM approximates the average integral value on a reference volume. Applies to integral form of conservation law. Handles discontinuities in solutions volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem.
BEM	Use boundary integral equation, use Green's function Accuracy – due to the semi-analytical nature and use of integral Solution time is long and memory size if large Limited to solve small-scale models. Good for complicated geometry and stress concentration problems.

Finite Element Method – brief history

- In 1943, Richard Courant proposed breaking a continuous system into triangular segments. (The unveiling of ENIAC at the University of Pennsylvania.)

- In the 1950s, a team form Boeing demonstrated that complex surfaces could be analyzed with a matrix of triangular shapes.

- Dr. Ray Clough coined the term "finite element" in 1960. The 1960s saw the true beginning of commercial FEA as digital computers replaced analog ones with the capability of thousands of operations per second.

- In the early 1960s, the MacNeal-Schwendler Corporation (MSC) develop a general purpose FEA code. This original code had a limit of 68,000 degrees of freedom. When the NASA contract was complete, MSC continued development of its own version called MSC/NASTRAN, while the original NASTRAN become available to the public and formed the basis of dozens of the FEA packages available today. Around the time MSC/NASTRAN was released, ANSYS, MARC, and SAP were introduced.

Finite Element Method – brief history

- By the 1970s, Computer-aided design, or CAD, was introduced later in the decade
- In the 1980s, the use of FEA and CAD on the same workstation with developing geometry standards such as IGES and DXF. Permitted limited geometry transfer between the systems.
- In the 1980s,CAD progressed from a 2D drafting tool to a 3D surfacing tool, and then to a 3D solid modeling system. Design engineers began to seriously consider incorporating FEA into the general product design process.
- As the 1990s draw to a place, the PC platform has become a major force in high end analysis. The technology has become to accessible that it is actually being “hidden” inside CAD packages.

Mathematical background

Gradient $\quad \mathbf{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Directional derivative in direction **u** $\quad D_{\mathbf{u}}\phi = \nabla\phi\cdot\hat{\mathbf{u}}$

Isotimic/equipotential/isothermal of surface/line $\quad \phi(x, y, z) = k$

$\nabla\phi$ is perpendicular to the surface/curve.

Gauss' Theorem or divergence Theorem $\quad \iint_\sigma \mathbf{F}\cdot\mathbf{n}dS = \iiint_G (\nabla\cdot\mathbf{F})dV$

Stokes' Theorem $\quad \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_\sigma (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$

Gradient Theorem $\quad \int_\Omega \nabla\phi\,dxdy = \oint_\Gamma \phi\mathbf{n}\,ds \qquad$ Try with $\phi = x+2y$

divergence Theorem $\quad \int_\Omega \nabla\cdot\mathbf{F}dxdy = \oint_\Gamma \mathbf{F}\cdot\mathbf{n}ds \qquad$ Try with $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j}$

Integration by parts – higher dimension (Green-Gauss Theorem)
Ω=open bounded subset of R^n, Γ=piecewise smooth boundary, **n** outward unit vector normal to Γ

$$\int_\Omega \nabla u\cdot\mathbf{v}d\Omega = \int_\Gamma u\mathbf{v}\cdot\mathbf{n}d\Gamma - \int_\Omega u\nabla\cdot\mathbf{v}d\Omega$$

$$\int_\Omega \nabla u\cdot\nabla v d\Omega = \int_\Gamma u\nabla v\cdot\mathbf{n}d\Gamma - \int_\Omega u\nabla\cdot\nabla v d\Omega = \int_\Gamma u\nabla v\cdot\mathbf{n}d\Gamma - \int_\Omega u\nabla^2 v d\Omega$$

Mathematical background

Green's first theorem

$$\int_s \phi \frac{\partial \psi}{\partial n} ds = \int_V \left(\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi\right) dV$$

$$\oint_c \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \left(\nabla \cdot \mathbf{F}(x, y)\right) dA$$

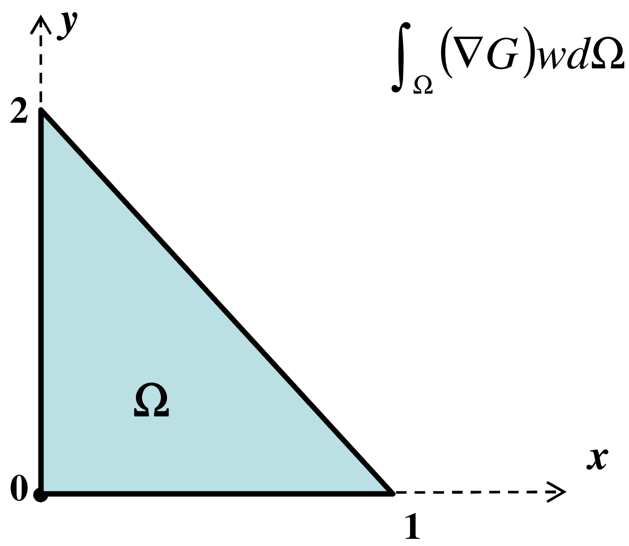
Green's Theorem, curve c is +ve oriented, anticlockwise.

$$\int_\Omega (\nabla G) w d\Omega = \int_\Gamma wG\, \mathbf{n} d\Gamma - \oint_\Omega G(\nabla w) d\Omega$$

$$-\int_\Omega \left(\nabla^2 G\right) w d\Omega = -\oint_\Gamma \frac{\partial G}{\partial n} w d\Gamma + \int_\Omega \nabla w \cdot \nabla G d\Omega$$

Integration by parts – higher dimension (Green-Gauss Theorem)

e.g.



$$\int_\Omega (\nabla G) w d\Omega = \oint_\Gamma wG\, \mathbf{n} d\Gamma - \int_\Omega G(\nabla w) d\Omega$$

where $G=x+2y$, $w=x^2$

Mathematical background

Numerical analysis: initial value problems

$$f(x+h)=f(x)+h\frac{f'(x)}{1!}+h^2\frac{f''(x)}{2!}+\cdots+h^n\frac{f^{(n)}(x)}{n!}+R_n^*.$$

Euler method

Taylor's series at *x*=*a*

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n.$$

$$R_n=\frac{f^{(n+1)}(\theta x)}{(n+1)!}(x-a)^{n+1},\quad a<\theta x<x.$$

Let $y=f$, $x_i=a$, $x_{i+1}=x$, $h=x-a$, and truncated the series after the second term, for $h\to 0$, we get

$$\boxed{y_{i+1}=y_i+hf(x_i,y_i)+O(h^2),\quad y_{i+1}\approx y_i+hf(x_i,y_i).}$$

Basic Euler formula, First order

Truncation error

"Big O" notation→$O(g(x))$. **Some finite value × $g(x)$**

$O(h^2)$=Some finite value × h^2
$|O(h^2)|\le M|h^2|$

Definition: $f(x)$ has order $O(g(x))$ as $x\to a$, if and only if $|f(x)|\le M|g(x)|$ for $|x-a|<\delta$, where $0<M<\infty$, $\delta>0$.

E.g. initial value problem:
$y'=y-t^2+1$, $0\le t\le 1$, $y(0)=0.5$

Let h =0.2, y_0=0.5, we get (here use 7 decimal places, D.P.)

$$\boxed{y_{i+1}=y_i+hf(t_i,y_i)=y_i+h(y_i-t_i^2+1),\quad i=0,\cdots,4.}$$

The exact solution is $y(t)=(t+1)^2-0.5e^t$.

Time, t_i	Approx, y_i	Exact, $y(t_i)$	Absolute error, $\lvert y(t_i)-y_i\rvert$
0.0	0.5	0.5	0
0.2	0.8	0.8292986	0.0292986
0.4	1.152	1.2140877	0.0620877
0.6	1.5504	1.6489406	0.0985406
0.8	1.98848	2.1272295	0.1387495
1.0	2.458176	2.6408591	0.1826831

Mathematical background

Numerical analysis: initial value problems

e.g. convert the problem

$(x'')^2+te^y+y'=x'-x$, $y'y''-\cos(xy)+\sin(tx'y)=x$.

into a system of first order ODEs.

Solution: introduce new variables as: $x_1=x$, $x_2=x'$, $x_3=y$ and $x_4=y'$. The system of ODEs for $X=[x_1, x_2, x_3, x_4]^T$ is

$$
\begin{aligned}
x_1' &= x_2 \\
x_2' &= \left(x_2 - x_1 - x_4 - te^{x_3}\right)^{1/2} \\
x_3' &= x_4 \\
x_4' &= \left[x_1 - \sin\left(tx_2x_3\right) + \cos\left(x_1x_3\right)\right]/x_4
\end{aligned}
$$

$$\boxed{\begin{array}{l} X''(t) = {}^{d}\!/_{dt}\, F\left(t, X\right) \\ x_1(t),\ x_2(t),\cdots \end{array}}$$

Taylor series for column vector of X can be written as:

$$X(t+h) = X(t) + hX'(t) + \frac{h^2}{2!}X''(t) + \frac{h^3}{3!}X^{(3)}(t) + \cdots$$

$X(t+h) \approx X(t) + hX'(t) + \frac{h^2}{2!}X''(t) + O(h^3)$ ←---- **Second-order Taylor series method**

$X(t+h) \approx X(t) + hX'(t) + O(h^2)$ ←------------- **First-order Taylor series method**

Mathematical background

Systems of first-order initial value problems

Example of system of first-order ODEs is given:

$$x' = x + 4y - e^t \quad , \quad y' = x + y + 2e^t.$$

with initial conditions, $x(0)=4$, $y(0)=5/4$. Calculate $x(0.2)$ and $y(0.2)$ with Euler method.

The particular solution is given $x=4e^{3t}+2e^{-t}-2e^t$, $y=2e^{3t}-e^{-t}+1/4\ e^t$. $x(0.2)=6.483131$, $y(0.2)=3.130858$.

Here, h=0.2.

$$X(0+h) = X(0.2) = X(0) + hF(t=0)$$

$$\begin{bmatrix} x_{0.2} \\ y_{0.2} \end{bmatrix} \approx \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} + 0.2 \begin{bmatrix} 4+4(5/4)-e^0 \\ 4+5/4+2e^0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5/4 \end{bmatrix} + 0.2 \begin{bmatrix} 8 \\ 7.25 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 2.7 \end{bmatrix}.$$

$$F' = \tfrac{d}{dt} F = \frac{d}{dt} \begin{bmatrix} x+4y-e^t \\ x+y+2e^t \end{bmatrix} = \begin{bmatrix} x'+4y'-e^t \\ x'+y'+2e^t \end{bmatrix} = \begin{bmatrix} 5x+8y+6e^t \\ 2x+5y+3e^t \end{bmatrix}$$

The error vector, E is given as:

E=trues value-approximate values=[6.483131, 3.130858]-[5.6, 2.7]=[0.883131, 0.430858]

The size of error vector can be measured using different norms as below:

Euclidean norm: ----------→ $\|E\|_e = \sqrt{\sum_{i=1}^{n} e_i^2} = \sqrt{0.883131^2 + 0.430858^2} = 0.9826$

***p*-norm: --------------------→** $\|E\|_p = \left(\sum_{i=1}^{n} |e_i|^p\right)^{1/p} = \left(|0.883131|^p + |0.430858|^p\right)^{1/p}$

1-norm: --------------------→ $\|E\|_1 = \sum_{i=1}^{n} |e_i| = |0.883131| + |0.430858| = 1.313989$

***Maximum-magnitude* -----→** $\|E\|_\infty = \max_{1 \le i \le n} |e_i| = \max\left(|0.883131|, |0.430858|\right) = 0.883131$
or *uniform-vector* norm:

X_0	$\rightarrow$ $X_{0.1}$	$\rightarrow$ $X_{0.2}$		
Error: $O(0.1^2)$	$O(0.1^2)$		**Σerror=const×0.02**	

$X_0 \rightarrow X_{0.2}$

Error: $O(0.2^2)$ **Σerror=const×0.04**

Mathematical background

Finite difference method for linear second-order boundary value problem

Solve the linear boundary value problem

$$y''+(1/x)y'-(1/x^2)y=3, \quad y(1)=2, \ y(2)=3$$

for x=1(0.2)2 using finite difference method. Analytical solution: $y(x)=x(x-1)+2/x$.

Let h=0.2, $x_0=a=1$, $x_1=1.2$, $x_2=1.4$, $x_3=1.6$, $x_4=1.8$ and $x_5=b=2$. Find $y_i \approx y(x_i)$, i=1,2,3,4.

At x_i, we get

$$y_i''+\left(\frac{1}{x_i}\right)y_i'-\left(\frac{1}{x_i^2}\right)y_i=3 \quad \rightarrow \quad \left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)+\left(\frac{1}{x_i}\right)\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)-\left(\frac{1}{x_i^2}\right)y_i=3$$

Multiply with h^2, here we use 4 decimal place.

$$(y_{i+1}-2y_i+y_{i-1})+\frac{h}{2}\left(\frac{1}{x_i}\right)(y_{i+1}-y_{i-1})-h^2\left(\frac{1}{x_i^2}\right)y_i=3h^2 \quad \rightarrow \quad \left(1-\frac{0.1}{x_i}\right)y_{i-1}-\left[2+\left(\frac{0.2}{x_i}\right)^2\right]y_i+\left(1+\frac{0.1}{x_i}\right)y_{i+1}=0.12$$

For i=1,
$$\left(1-\frac{0.1}{x_1}\right)y_0-\left[2+\left(\frac{0.2}{x_1}\right)^2\right]y_1+\left(1+\frac{0.1}{x_1}\right)y_2=0.12 \quad \rightarrow \quad \left(1-\frac{0.1}{1.2}\right)2-\left[2+\left(\frac{0.2}{1.2}\right)^2\right]y_1+\left(1+\frac{0.1}{1.2}\right)y_2=0.12$$

$$\rightarrow \quad -2.0278y_1+1.0833y_2=-1.7133$$

For i=2,
$$\left(1-\frac{0.1}{x_2}\right)y_1-\left[2+\left(\frac{0.2}{x_2}\right)^2\right]y_2+\left(1+\frac{0.1}{x_2}\right)y_3=0.12 \quad \rightarrow \quad \left(1-\frac{0.1}{1.4}\right)y_1-\left[2+\left(\frac{0.2}{1.4}\right)^2\right]y_2+\left(1+\frac{0.1}{1.4}\right)y_3=0.12$$

$$\rightarrow \quad 0.9286y_1-2.0204y_2+1.0714y_3=0.12$$

For i=3,
$$\left(1-\frac{0.1}{x_3}\right)y_2-\left[2+\left(\frac{0.2}{x_3}\right)^2\right]y_3+\left(1+\frac{0.1}{x_3}\right)y_4=0.12 \quad \rightarrow \quad \left(1-\frac{0.1}{1.6}\right)y_2-\left[2+\left(\frac{0.2}{1.6}\right)^2\right]y_3+\left(1+\frac{0.1}{1.6}\right)y_4=0.12$$

$$\rightarrow \quad 0.9375y_2-2.0156y_3+1.0625y_4=0.12$$

For i=4,
$$\left(1-\frac{0.1}{x_4}\right)y_3-\left[2+\left(\frac{0.2}{x_4}\right)^2\right]y_4+\left(1+\frac{0.1}{x_4}\right)y_5=0.12 \quad \rightarrow \quad \left(1-\frac{0.1}{1.8}\right)y_3-\left[2+\left(\frac{0.2}{1.8}\right)^2\right]y_4+\left(1+\frac{0.1}{1.8}\right)(3)=0.12$$

$$\rightarrow \quad 0.9444y_3-2.0123y_4=-3.0468$$

Mathematical background

Finite difference method for linear second-order boundary value problem

Finally, we get the tridiagonal system as below:

$$\mathbf{Ay} = \mathbf{b} \quad \rightarrow \quad \begin{pmatrix} -2.0278 & 1.0833 & 0 & 0 \\ 0.9286 & -2.0204 & 1.0714 & 0 \\ 0 & 0.9375 & -2.0156 & 1.0625 \\ 0 & 0 & 0.9444 & -2.0123 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1.7133 \\ 0.1200 \\ 0.1200 \\ -3.0468 \end{pmatrix}$$

Using Thomas algorithm, we get

i	1	2	3	4
d_i e_i c_i b_i	-2.0278 - 1.0833 -1.7133	-2.0204 0.9286 1.0714 0.1200	-2.0156 0.9375 1.0625 0.1200	-2.0123 0.9444 - -3.0468
$(\alpha_1=d_1)$ $\alpha_i=d_i-c_i\beta_{i-1}$,	-2.0278	-1.5243	-1.3566	-1.2726
$\beta_i=e_i/\alpha_i$,	-0.5342	-0.7029	-0.7832	-
$(w_1=b_1/\alpha_1)$ $w_i=(b_i-c_iw_{i-1})/\alpha_i$,	0.8449	0.4360	0.2128	2.5521
$(y_n=w_n)$ $y_i=w_i-\beta_iy_{i+1}$,	1.9082	1.9905	2.2116	2.5521

Finally, we get $y(1.2)\approx y_1$=1.9082, y_2=1.9905, y_3=2.2116 and $y(1.8)\approx y_4$=2.5521.
The exact solution is given as $y(1.2)$=1.9067, $y(1.4)$=1.9886, $y(1.6)$=2.2100, $y(1.8)$=2.5511.
So, finite difference method produce results accurate up to 2 decimal places.

Mathematical background

Extra notes

Finite difference method for nonlinear second-order boundary value problem

For the general nonlinear boundary value problem

$$y''=f(x,y,y'), \qquad a\le x\le b,\ y(a)=\alpha,\ y(b)=\beta, \qquad (a)$$

Let divide the interval $[a,b]$ into N equal subintervals where $x_0=a$, $x_i=x_0+ih$, $\{i=1,2,\ldots,N\}$, $x_N=b$ and $h=(b\text{-}a)/N$.

At point $x=x_i$, equation (a) becomes

$$-\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+f\left(x_i,y_i,\frac{y_{i+1}-y_{i-1}}{2h}\right)=0 \;\rightarrow\; \boxed{-y_{i-1}+2y_i-y_{i+1}+h^2 f\left(x_i,y_i,\frac{y_{i+1}-y_{i-1}}{2h}\right)=0=f_i\left(y_1,\cdots,y_{N-1}\right)}$$

For $i=1,2,\ldots,N\text{-}1$, the above equation will produce nonlinear system $(N\text{-}1)$ equations with unknowns $y_0,y_1,\ldots,y_N$. The above nonlinear system has a unique solution if $h<2/L$, $L=\max|f_{y'}(x,y,y')|$. With the given boundary condition, $y_0=\alpha$ and $y_N=\beta$, the system can be solved by Newton's method for nonlinear systems. A sequence of iteration will converge to solution if the guess initial approximation is sufficiently close to solution.

The Jacobian matrix, $J(y_1,\ldots,y_{N\text{-}1})$ is tridiagonal with ij-th entry:

$$J\left(y_1,\cdots,y_{N-1}\right)_{ij}=\begin{cases}-1+\dfrac{h}{2}f_{y'}\left(x_i,y_i,\dfrac{y_{i+1}-y_{i-1}}{2h}\right), & for\ i=j-1\ and\ j=2,\cdots,N-1,\\ 2+h^2 f_y\left(x_i,y_i,\dfrac{y_{i+1}-y_{i-1}}{2h}\right), & for\ i=j\ and\ j=1,\cdots,N-1,\\ -1-\dfrac{h}{2}f_{y'}\left(x_i,y_i,\dfrac{y_{i+1}-y_{i-1}}{2h}\right), & for\ i=j+1\ and\ j=1,\cdots,N-2.\end{cases}$$

Correction vector can be calculated using Thomas algorithm:

$$\boxed{J\cdot\begin{bmatrix}h_1^{(k)}\\ \vdots\\ h_{N-1}^{(k)}\end{bmatrix}=-\begin{bmatrix}f_1\left(y_1,\cdots,y_{N-1}\right)\\ \vdots\\ f_{N-1}\left(y_1,\cdots,y_{N-1}\right)\end{bmatrix}\rightarrow\begin{bmatrix}y_1^{(k+1)}\\ \vdots\\ y_{N-1}^{(k+1)}\end{bmatrix}=\begin{bmatrix}y_1^{(k)}\\ \vdots\\ y_{N-1}^{(k)}\end{bmatrix}+\begin{bmatrix}h_1^{(k)}\\ \vdots\\ h_{N-1}^{(k)}\end{bmatrix}.}$$

The Newton iteration will stop when the solutions converge to certain decimal places or some norm stopping criteria.

13

Mathematical background **Extra notes**

Finite difference method for nonlinear second-order boundary value problem

Newton's method for nonlinear systems

$$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The system of equations $g_i(y_1,y_2,\ldots,y_n)=0$ $(1\le i\le n)$
can be expressed simply as $\mathbf{G}(\mathbf{Y})=\mathbf{0}$
by letting $\mathbf{Y}=(y_1,y_2,\ldots,y_n)^T$ and $\mathbf{G}=(g_1,g_2,\ldots,g_n)^T$. Using the Taylor's series expansion, we get

$$\mathbf{0}=\mathbf{G}(\mathbf{Y}+\mathbf{H}) \approx \mathbf{G}(\mathbf{Y})+\mathbf{G}'(\mathbf{Y})\mathbf{H}\ , \quad \textit{(where } \mathrm{Y+H} \textit{ is more accurate solution)}$$

where $\mathbf{H}=(h_1,h_2,\ldots,h_n)^T$ and $\mathbf{G}'(\mathbf{Y})$ is the $n\times n$ Jacobian matrix $\mathbf{J}(\mathbf{Y})$:

$$\mathbf{J}(\mathbf{Y}) = \mathbf{G}'(\mathbf{Y}) = \begin{bmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \cdots & \partial g_1/\partial y_n \\ \partial g_2/\partial y_1 & \partial g_2/\partial y_2 & \cdots & \partial g_2/\partial y_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial g_n/\partial y_1 & \partial g_n/\partial y_2 & \cdots & \partial g_n/\partial y_n \end{bmatrix}$$

$y'' + (y')^3 y = 0$

$ans: y^3/3 - 2c_1 y = 2x + c2$

$let\ c_1 = c_2 = 0$

$y^3 = 6x$

$x = 1(0.25)2$

$$f(x+a, y+b) = f(x,y) + \frac{1}{1!}D_1[f(x,y)] + \frac{1}{2!}D_2[f(x,y)] + \cdots + \frac{1}{n!}D_n[f(x,y)] + R_n.$$

$$D_n = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)^n \quad R_n = \frac{1}{(n+1)!}D_{n+1}[f(x+\theta_1 a, y+\theta_2 b)]$$

$$0<\theta_1<1,\ 0<\theta_2<1.$$

The correction vector $\mathbf{H}$ is obtained by solving linear system

$$\mathbf{J}(\mathbf{Y})\mathbf{H}= -\mathbf{G}(\mathbf{Y})$$

If Jacobian matrix is tridiagonal matrix, then $\mathbf{H}$ can be solved using Thomas algorithm. If the matrix size is 2×2, then just use the inverse of matrix $\mathbf{J}$, $\mathbf{H}=\mathbf{J}^{-1}(-\mathbf{G})$.
Finally, Newton's iteration for n nonlinear equations in n variables is given by

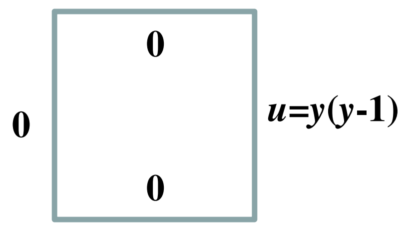
$$\mathbf{Y}^{(k+1)}=\mathbf{Y}^{(k)}+\mathbf{H}^{(k)}$$

where the Jacobian system is

$$\mathbf{J}(\mathbf{Y}^{(k)})\mathbf{H}^{(k)}= -\mathbf{G}(\mathbf{Y}^{(k)}).$$

Mathematical background

PDE : Elliptic equation : Laplace equation



E.g. Solve the Laplace equation

$$u_{xx}+u_{yy}=0,\ 0<x<1,\ 0<y<1$$

with boundary conditions

$$u(x,0) = x(x-1),\ u(x,1)=0, \qquad 0< x <1$$
$$u(0,y)=u(1,y)=0, \qquad 0\le y \le 1$$

using finite difference with SOR iteration, ω=1.25 and $h=k=1/3$, $\mathbf{u}^{(0)}=\mathbf{0}$. All calculation in 3 DP.

Solution:

x-axis: $h=⅓$, $x_0=0$, $x_1=⅓$, $x_2=⅔$, $x_3=1$. y-axis: $k=⅓$, $y_0=0$, $y_1=⅓$, $y_2=⅔$, $y_3=1$.

Given $u(x,0)=x(x-1)$, $0\le y \le 1$, we get

$u_{0,0}=u(0,0)=0$; $u_{1,0}=u(⅓,0)=⅓(-⅔)= -0.222$; $u_{2,0}=u(⅔,0)= -0.222$; $u_{3,0}=u(1,0)=0$.

Given $u(x,1)=0$, $0\le x \le 1$, we get

$u_{0,3}=u(0,1)=0$; $u_{1,3}=u(⅓,1)=0$; $u_{2,3}=u(⅔,1)=0$; $u_{3,3}=u(1,1)=0$.

Given $u(0,y)=u(1,y)=0$, $0\le y \le 1$, we get

$u_{0,0}=u(0,0)=0$; $u_{0,1}=u(0,⅓)=0$;

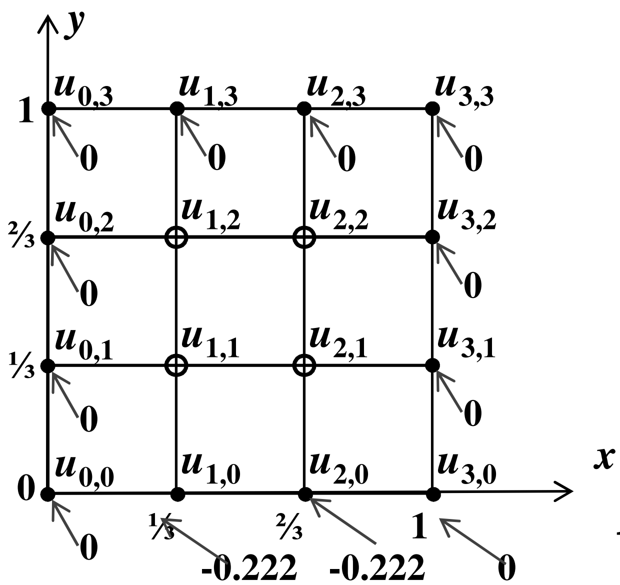
$u_{0,2}=u(0,⅔)=0$; $u_{0,3}=u(0,1)=0$;

$u_{3,0}=u(1,0)=0$; $u_{3,1}=u(1,⅓)=0$;

$u_{3,2}=u(1,⅔)=0$; $u_{3,3}=u(1,1)=0$.

((boundary conditions symmetry on x=½))

We need to calculate for $\boldsymbol{u}_{1,1}$, $\boldsymbol{u}_{2,1}$, $\boldsymbol{u}_{1,2}$ and $\boldsymbol{u}_{2,2}$ only.



Mathematical background

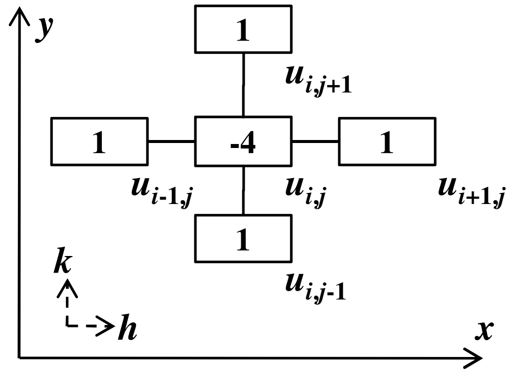
PDE : Elliptic equation : Laplace equation

Using finite difference, at point (x_i, y_i), we get

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = 0 \quad \rightarrow \quad \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

Let $h=k$, we get

$$\boxed{\boldsymbol{u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = 0}}$$



Computational grid - implicit

at $i=1$, $j=1$;

$u_{1,0}+u_{0,1} - 4u_{1,1} + u_{2,1} + u_{1,2}=0$
$-0.222+0 - 4u_{1,1} + u_{2,1} + u_{1,2}=0$
$- 4u_{1,1} + u_{2,1} + u_{1,2}=0.222$

at $i=2$, $j=1$;

$u_{2,0}+u_{1,1} - 4u_{2,1} + u_{3,1} + u_{2,2}=0$
$-0.222+u_{1,1} - 4u_{2,1} +0 + u_{2,2}=0$
$u_{1,1} - 4u_{2,1} + u_{2,2}=0.222$

at $i=1$, $j=2$;

$u_{1,1}+u_{0,2} - 4u_{1,2} + u_{2,2} + u_{1,3}=0$
$u_{1,1}+0 - 4u_{1,2} + u_{2,2} + 0 =0$
$u_{1,1} - 4u_{1,2} + u_{2,2} =0$

at $i=2$, $j=2$;

$u_{2,1}+u_{1,2} - 4u_{2,2} + u_{3,2} + u_{2,3}=0$
$u_{2,1}+u_{1,2} - 4u_{2,2} +0 + 0 =0$
$u_{2,1}+u_{1,2} - 4u_{2,2} =0$

$$\mathbf{Au} = \mathbf{b} \quad \rightarrow \quad \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 0.222 \\ 0.222 \\ 0 \\ 0 \end{bmatrix}$$

Mathematical background

Linear and bilinear functionals

Consider the integral form

$$I(u) = \int_a^b F(x, u, u')dx, \;\; u = u(x), \;\; u' = \frac{du}{dx}$$

The integrand $F(x,u,u')$ is a given function of coordinate x (independent variable), dependent variable u, and its derivative du/dx. Given a realfunction u=$u(x)$, $I(u)$ is a real number. A functional s a "function of functions".

Example of functionals:

$$I(u) = \int_a^b \left(p(x)\frac{du}{dx} + q(x)u^2 \right)dx + Pu(a)$$

$$I(u, v) = \int_\Omega \left(p(x, y)\frac{du}{dx}\frac{dv}{dx} + q(x, y)v \right)dxdy + \int_\Gamma Quds$$

Mathematical background

Linear and bilinear functionals

A functional $I(u)$ is linear in u if and only if it satisfies (any real number α, β)

$$I(\alpha u+\beta v)=\alpha I(u)+\beta I(v)$$

Example of linear functional

$$I(u)=\int_a^b f(x)u\,dx+qu(b),\quad I(u,v)=\int_\Omega \big(f(x,y)u+q(x,y)v\big)dxdy$$

Example of nonlinear functional

$$I(u)=\int_a^b u\frac{du}{dx}dx,\quad I(u)=\int_a^b f(x)u\,dx+c$$

A functional $B(u,v)$ is bilinear if it is linear in each of its arguments u and v:

$$B(\alpha u_1+\beta u_2,v)=\alpha B(u_1,v)+\beta B(u_2,v),$$ linear in first argument

$$B(u,\alpha v_1+\beta v_2)=\alpha B(u,v_1)+\beta B(u,v_2),$$ linear in second argument

Example of bilinear functional

$$B(u,v)=\int_a^b\left(p(x)uv+q(x)\frac{du}{dx}\frac{dv}{dx}\right)dx+ku(a)v(a)$$

$$B(\mathbf{u},\mathbf{v})=\int_a^b\left(p(x)\mathbf{u}\cdot\mathbf{v}+q(x)\frac{d\mathbf{u}}{dx}\cdot\mathbf{v}\right)dx$$

$$B(\mathbf{u},\mathbf{v})=\int_\Omega\big(p(\mathbf{x})\mathbf{u}\cdot\mathbf{v}+q(\mathbf{x})\nabla\mathbf{u}\cdot\nabla\mathbf{v}\big)d\mathbf{x}$$

symmetric

Example of non bilinear functional

$$I(u,v)=\int_a^b\left[p(x)u^2+q(x)\left(\frac{dv}{dx}\right)^2\right]dx$$

$$I(\mathbf{u},\mathbf{v})=\int_a^b\left(p(x)\mathbf{u}\cdot\mathbf{u}+q(x)\frac{d\mathbf{v}}{dx}\cdot\frac{d\mathbf{v}}{dx}\right)dx$$

$$I(u,v)=\int_\Omega\left[p(x,y)\left(\frac{\partial u}{\partial x}\right)^2 v+q(x,y)u\right]dxdy$$

Mathematical background

Linear and bilinear functionals

A bilinear $B(u,v)$ is symmetric if $B(u,v)=B(v,u)$

A quadratic functional $Q(u)$ is one that satisfies the relation

$$Q(\alpha u)=\alpha^2 Q(u), \qquad \alpha\text{=real number}$$

Fundamental lemma of calculus of variations: for any integrable function $G(x)$, if the statement

$$\int_a^b G(x)\eta(x)dx = 0$$

For any arbitrary continuous function $\eta(x)$

⇒ $G(x)=0$ in (a,b)

Fundamental lemma of calculus of variations: for any integrable function $G(x)$, if the statement

$$\int_a^b G(x)\eta(x)dx + B(a)\eta(a) = 0$$

For any arbitrary continuous function $\eta(x)$ and $\eta(a)$ is arbitrary

⇒ $G(x)=0$ in (a,b) and $B(a)=0$

Taylor series for arbitrary function: $\eta(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Finite Element Method

Calculus of variations

Let $F(x,u,u')$ with fixed value of independent variable x, F depends on u and u'. The change εv in u, where ε is a constant, v is a function, is called **variation** of u (denoted by δu):

$\delta u = \varepsilon v$. →(**variation** of u), operator δ is called **variational operator**.

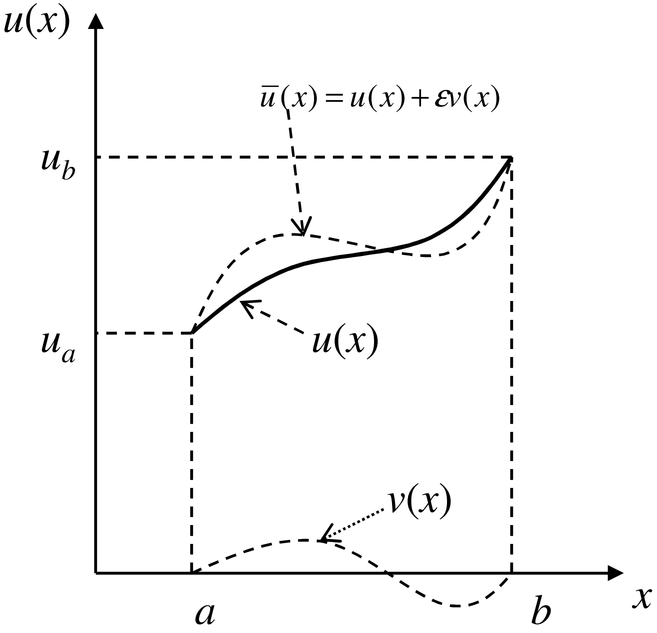
The variation δu represents an admissible change in function $u(x)$ at fixed value of x.

Expand in powers of ε gives ($[u+\varepsilon v]$, $[u'+\varepsilon v']$ are dependent functions)

$$\Delta F = F(x,u+\varepsilon v,u'+\varepsilon v') - F(x,u,u') = F(x,u,u') + \varepsilon v\frac{\partial F}{\partial u} + \varepsilon v'\frac{\partial F}{\partial u'} +$$

$$\frac{(\varepsilon v)^2}{2!}\frac{\partial^2 F}{\partial u^2} + \frac{2(\varepsilon v)(\varepsilon v')}{2!}\frac{\partial^2 F}{\partial u\partial u'} + \frac{(\varepsilon v')^2}{2!}\frac{\partial^2 F}{\partial u'^2} + \cdots - F(x,u,u')$$

$$= \varepsilon v\frac{\partial F}{\partial u} + \varepsilon v'\frac{\partial F}{\partial u'} + O(\varepsilon^2). \quad \text{where } \lim_{\varepsilon\to 0} O(\varepsilon^2) = 0.$$



The **first variation** of F is

$$\delta F = \varepsilon\left[\lim_{\varepsilon\to 0}\frac{F(x,u+\varepsilon v,u'+\varepsilon v') - F(x,u,u')}{\varepsilon}\right] = \varepsilon\left[\lim_{\varepsilon\to 0}\frac{\Delta F}{\varepsilon}\right]$$

$$= \varepsilon\left[\frac{d}{d\varepsilon}\left(F(u+\varepsilon v)\right)\right]_{\varepsilon=0} = \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'.$$

$O(h^2)$=Some finite value $\times$ h^2

$|O(h^2)| \le M|h^2|$

Analogy for total differential of F with fixed x, $dx=0$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'.$$

$f(x)$ has order $O(g(x))$ as $x \to a$, if and only if $|f(x)| \le M|g(x)|$ for $|x-a| < \delta$, where $0 < M < \infty$, $\delta > 0$.

Finite Element Method

Calculus of variations

Let $F=F(x,y,u,v,u_x,v_x,u_y,v_y)$, where $u=u(x,y)$ and $v=v(x,y)$ are dependent variables,

The first variation of F is $\delta F=\delta_u F+\delta_v F$, where $\delta_u F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y$, $\delta_v F = \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_y}\delta v_y$.

If $F_1=F_1(u)$ and $F_2=F_2(u)$, then

$$\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2,\quad \delta(F_1 \cdot F_2) = (\delta F_1)F_2 + F_1(\delta F_2)$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{(\delta F_1)F_2 - F_1(\delta F_2)}{F_2^2},\quad \delta(F_1)^n = n(F_1)^{n-1}\delta F_1$$

$$\frac{d}{dx}(\delta u) = \frac{d}{dx}(\varepsilon v) = \varepsilon\frac{dv}{dx} = \delta\left(\frac{du}{dx}\right)$$

$$\delta\int_a^b u(x)dx = \int_a^b \delta u(x)dx, \quad \textit{where } a,b \textit{ are fixed.}$$

Some examples

$$I(u) = \int_a^b F(x,u,u')dx \to \delta I(u) = \delta\int_a^b F(x,u,u')dx = \int_a^b \delta F(x,u,u')dx = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right)dx$$

$$I(u) = \int_a^b \left(p(x)\frac{du}{dx} + q(x)u^2 \right)dx + Pu(a) \to \delta I(u) = \int_a^b \left(p(x)\frac{d\delta u}{dx} + 2q(x)u\delta u \right)dx + P\delta u(a)$$

$$I(u,v) = \int_\Omega \left(p(x,y)\frac{du}{dx}\frac{dv}{dx} + q(x,y)v \right)dxdy + \int_\Gamma Quds \to \delta I = \int_\Omega \left(p(x,y)\left(\frac{d\delta u}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{d\delta v}{dx}\right) + q(x,y)\delta v \right)dxdy + \int_\Gamma Q\delta uds$$

→functions of position, p and q, do not undergo variation since not functions of dependent variables.

Finite Element Method

Calculus of variations

Euler equations

Find a function $u=u(x)$ such that $u(a)=u_a$, $u(b)=u_b$,

and $$I(u) = \int_a^b F(x, u(x), u'(x))dx$$

is extremum.

The second variation $\delta^2 I(u)$ of functional $I(u)$ is given, $\delta^2 I(u) = \frac{\varepsilon^2}{2}\left[\frac{d^2}{d\varepsilon^2} I(u+\varepsilon v)\right]_{\varepsilon=0}$

sufficient condition for $I(u)$ relative minimum (max) is $\delta^2 I(u)$ is greater (less) than zero.

We get,

$$0 = \delta I(u) = \varepsilon \left.\frac{dI(u+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \delta F dx = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right)dx$$

$$= \varepsilon \int_a^b \left(\frac{\partial F}{\partial u} v + \frac{\partial F}{\partial u'} v'\right)dx$$

Integration by part: $$\int_{x=a}^{x=b} udv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} vdu$$

$$\rightarrow 0 = \int_a^b v\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx + \left(\frac{\partial F}{\partial u'} v\right)\Big|_a^b$$

Without constraint on u', the boundary term vanished if v zero at $x=a$ & b.

Finally, we get **Euler equation or Euler-Lagrange equation**

$$\boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \ , \ a < x < b.}$$

Finite Element Method

Calculus of variations

Euler equations

$$I(u)=\int_a^b F(x,u(x),u'(x))dx \qquad \text{extremum} \Rightarrow \boxed{\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0 \ , \ a<x<b.}$$

Problem: finding the shortest curve in plane that connect two points (x_1,y_1) and (x_2,y_2)

Arc length is given by $I[f]=\int_{x_1}^{x_2}\sqrt{1+[f'(x)]^2}\,dx$ with $f'(x)=\frac{df}{dx},\ y=f(x)$

Let $f+\delta f=f(x)+\varepsilon\eta(x)$

$$0=\delta I(f)=\varepsilon\frac{dI(f+\varepsilon\eta)}{d\varepsilon}\bigg|_{\varepsilon=0}$$

$$\frac{d}{d\varepsilon}\int_{x_1}^{x_2}\sqrt{1+\left[f'(x)+\varepsilon\eta'(x)\right]^2}\,dx\bigg|_{\varepsilon=0}=\int_{x_1}^{x_2}\frac{\left(f'(x)+\varepsilon\eta'(x)\right)\eta'(x)}{\sqrt{1+\left[f'(x)+\varepsilon\eta'(x)\right]^2}}\bigg|_{\varepsilon=0}dx=\int_{x_1}^{x_2}\frac{f'(x)\eta'(x)}{\sqrt{1+[f'(x)]^2}}dx=0$$

$\int_a^b u(x)\eta'(x)dx=[u(x)\eta(x)]_a^b-\int_a^b u'(x)\eta(x)dx$ With substitution $u(x)=\frac{f'(x)}{\sqrt{1+[f'(x)]^2}}$ And let $\eta(x_1)=\eta(x_2)=0$

$$\downarrow$$

$$0-\int_{x_1}^{x_2}\eta(x)\frac{d}{dx}\left[\frac{f'(x)}{\sqrt{1+[f'(x)]^2}}\right]dx=0$$

Fundamental lemma of calculus of variation $\frac{d}{dx}\left[\frac{f'(x)}{\sqrt{1+[f'(x)]^2}}\right]=0 \Rightarrow \frac{\sqrt{1+f'^2}\cdot f''-f'f'f''\big/\sqrt{1+f'^2}}{1+f'^2}=0$

$\Rightarrow$ $\frac{d^2f}{dx^2}=0 \rightarrow f(x)=ax+b$ The extremals are straight lines

Arc length is given as $\int_C ds=\int_C\left|\frac{d\mathbf{r}}{dt}\right|dt$ Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}=t\mathbf{i}+y\mathbf{j}$ $\Rightarrow$ $\int_{t_1}^{t_2}\sqrt{1+[y'(t)]^2}\,dt$

Finite Element Method

Natural and essential boundary conditions

Find the extremum of $I(u)$ subject to no end conditions [the set of v is arbitrary even at end point, i.e., $v(a)\neq 0$ and $v(b)\neq 0$], the functional has the form

$$I(u)=\int_a^b F\big(x,u(x),u'(x)\big)dx-Q_a u(a)-Q_b u(b)$$

where Q_a and Q_b are known values.

We get,

$$0=\delta I(u)=\int_a^b \varepsilon v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left(\frac{\partial F}{\partial u'}\varepsilon v\right)\Bigg|_a^b-Q_a\varepsilon v(a)-Q_b\varepsilon v(b)$$

To eliminate boundary conditions, let $\left(-\frac{\partial F}{\partial u'}-Q_a\right)v\Big|_{x=a}=0,\ \left(\frac{\partial F}{\partial u'}-Q_b\right)v\Big|_{x=b}=0.$

And we get,

(1) $v(a)=0, v(b)=0,$

(2) $v(a)=0,\ \frac{\partial F}{\partial u'}\Big|_{x=b}-Q_b=0,$

(3) $-\frac{\partial F}{\partial u'}\Big|_a-Q_a=0,\ v(b)=0,$

(4) $-\frac{\partial F}{\partial u'}\Big|_a-Q_a=0,\ \frac{\partial F}{\partial u'}\Big|_{x=b}-Q_b=0.$

u is fixed at x=0, L

Note that v=0 at end point is equivalent to requirement that u is specified (some value) at that point.

Essential boundary conditions: v (and its derivatives) to vanish at boundary. E.g. v=0 on boundary.

Natural boundary conditions: coefficient of v (and its derivatives) is specified some value.

e.g. $\frac{\partial F}{\partial u'}=Q$ on boundary.

Finite Element Method – Common 1D problems

Linear operator: L

Sturm-Liouville operator $$L = \frac{d}{dx}\left[p(x)\frac{d}{dx} \right] + q(x)$$ $$\int LG(x,s)f(s)ds = L\left(\int G(x,s)f(s)ds\right)$$

Problems	Strong form	Constitutive law (flux)
Heat distribution	$\frac{d}{dx}\left(A(x)k(x)\frac{dT}{dx} \right) + Q(x) = 0$	Fourier's law: $q = -k\frac{dT}{dx}$
Diffusion problem	$\frac{d}{dx}\left(A(x)D(x)\frac{dc}{dx} \right) + Q(x) = 0$	Fick's law: $q = -D\frac{dc}{dx}$
Axially loaded elastic bar	$\frac{d}{dx}\left(A(x)E(x)\frac{du}{dx} \right) + Q(x) = 0$	Hooke's law: $\sigma = E\frac{du}{dx}$

Finite Element Method

Integral formulations

Strong form

Given $-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]=f(x),\quad 0<x<L$ Subject to boundary conditions $u(0)=u_0,\ a(x)\frac{du}{dx}\Big|_{x=L}=Q_L$

Assume $u(x)\approx U_N(x)=\sum_{j=1}^{N}c_j\phi_j(x)+\phi_0(x)$ ϕ_0 for BC, ϕ_j for homogeneous $\Rightarrow$ $-\frac{d}{dx}\left[a(x)\frac{dU_N}{dx}\right]=f(x)$

Residual function $R(x,c_j)=-\frac{d}{dx}\left[a(x)\frac{dU_N}{dx}\right]-f(x)\neq 0$

Collocation method: $\boldsymbol{R(x_i,c_j)=0}$, for $x=x_i$, $i=1,2,\ldots,N$

Least-squares method $\delta I\equiv\delta\int_0^L R^2dx=0\ \rightarrow\ \frac{\partial}{\partial c}\int_0^L R^2dx=0$

Weighted-residual method: determine value of c_j $\int_0^L w_i(x)R(x,c_j)dx=0,\quad i=1,2,\cdots,N$

where $w_i(x)$ is weight function

Petrov-Galerkin method: $w_i=\psi_i\neq\phi_i$

Galerkin's method: $w_i=\phi_i$

Least-squares method: $w_i=\frac{d}{dx}\left(a(x)\frac{d\phi_i}{dx}\right)$

Collocation method: $w_i=\delta(x-x_i)$ $\Rightarrow$ $\delta(x-x_i)=0,\ x\neq x_i$

$\int_{-\infty}^{\infty}f(x)\delta(x-x_i)dx=f(x_i)$

$$0=\int_0^L w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f\right]dx=\int_0^L\left[a\frac{dw}{dx}\frac{du}{dx}-wf\right]dx-\left[wa\frac{du}{dx}\right]_0^L$$ $\rightarrow$ Weak form

$$\int_{x=a}^{x=b}udv=uv\Big|_{x=a}^{x=b}-\int_{x=a}^{x=b}vdu$$

Finite Element Method

$$u(0)=u_0,\ a(x)\frac{du}{dx}\bigg|_{x=L}=Q_L$$

Integral formulations – linear and bilinear forms, quadratic functionals

$$0=\int_0^L w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f\right]dx=\int_0^L\left[a\frac{dw}{dx}\frac{du}{dx}-wf\right]dx-\left[wa\frac{du}{dx}\right]_0^L \qquad w(0)=0\ \rightarrow\ 0=\int_0^L\left[a\frac{dw}{dx}\frac{du}{dx}-wf\right]dx-w(L)Q_L \qquad \textbf{(a)}$$

Let

$$B(w,u)=\int_0^L a\frac{dw}{dx}\frac{du}{dx}dx,\quad l(w)=\int_0^L wf\,dx+w(L)Q_L$$

(a) $\rightarrow$ 0=$B(w,u)$-$l(w)$ or $\boldsymbol{B(w,u)=l(w)}$ **(b)** Variational problem

B is bilinear and symmetric, $l(w)$ is linear

Function w is a variation of actual solution, $\rightarrow w=\delta u$ $\rightarrow$ $\boldsymbol{0=B(\delta u,u)-l(\delta u)}$

$$B(\delta u,u)=\frac{1}{2}\delta[B(u,u)],\quad l(\delta u)=\delta[l(u)]$$

$$0=\frac{1}{2}\delta[B(u,u)]-\delta[l(u)]\equiv\delta I(u)\quad\rightarrow\ I(u)=\frac{1}{2}B(u,u)-l(u)$$

$$\delta^2B(u,u)=\frac{\varepsilon^2}{2}\left[\frac{d^2}{d\varepsilon^2}B(u+\varepsilon v,u+\varepsilon v)\right]_{\varepsilon=0}$$

$$=\frac{\varepsilon^2}{2}\left[\frac{d}{d\varepsilon}2B(v,u+\varepsilon v)\right]_{\varepsilon=0}=\varepsilon^2B(v,v)=B(\varepsilon v,\varepsilon v)$$

$\Rightarrow$ $\delta I=0=B(\delta u,u)-l(\delta u),\ \rightarrow\delta^2I=B(\delta u,\delta u)>0\ \ ,\delta u\neq 0$ Function u that **minimize** $I(u)$ is solution of (a) & (b)

Note: if $w(L)$=0, then $w\in H_0^1(\Omega)$ (suitable space of "once differentiable" function of Ω that zero on $\partial\Omega$.

Variational method $\rightarrow$ Raleigh-Ritz or Ritz method

$H_0^1(0,1)\rightarrow$ abs. cont.. Functions on (0,1)

$$u(x)\approx U_N(x)=\sum_{j=1}^N c_j\phi_j(x)+\phi_0(x)$$

$$u(x)\approx U_N(x)=\sum_{j=1}^N c_j\phi_j(x)$$

$$I(u)=\frac{1}{2}B(u,u)-l(u)$$

minimize $\underbrace{I(U_N)=\frac{1}{2}B(U_N,U_N)-l(U_N)}_{\left[wa\frac{du}{dx}\right]_0^L=w(L)Q_L-0}$ $\Rightarrow$ $\dfrac{\partial I}{\partial c_1}=0,\ \cdots,\ \dfrac{\partial I}{\partial c_N}=0$

Not fulfill essential boundary condition

Finite Element Method

Variational method

$$\frac{\partial F}{\partial T} = -Q,\ \frac{\partial F}{\partial T'} = kT'$$

$$I(T) = \int_a^b F(x, T(x), T'(x))dx$$

Examples , let total potential energy is

$$I(u) = \int_0^L \left[\frac{A}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{h}{2}[u(L)]^2$$

Use principle of minimum total potential energy, we need to find the minimum with $\delta I(u)=0$,

$$\delta I(u) = \int_0^L \left(A\frac{du}{dx}\frac{d\delta u}{dx} - f\delta u \right) dx + hu(L)\delta u(L) = \int_0^L \left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right]\delta u\, dx + \left[A\frac{du}{dx}\delta u \right]_0^L + hu(L)\delta u(L)$$

$$0 \ = \int_0^L \delta u\left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[A\frac{du}{dx} + hu(L)\right]_{x=L} - \delta u(0)\left[A\frac{du}{dx}\right]_{x=0}$$

Let δu is arbitrary in $0<x\le L$ but with <u>$\boldsymbol{\delta u(0)=0}$</u>, the above boundary term vanishes and we get

Euler equation $\quad -\frac{d}{dx}\left(A\frac{du}{dx}\right) - f = 0,\quad 0 < x < L \quad$ **Natural boundary condition** $\quad A\frac{du}{dx} + hu(L)\Big|_{x=L} = 0$

$$\boxed{\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0 = -\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q \right]}$$

Based on the above example, the problem of

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0 \qquad T|_{x=0} = T_0,\quad q|_{x=L} = h(T_L\text{-}T_\infty).$$

Is equivalent to the minimizing the functional with $\boldsymbol{\delta T(0)=0}$ or T is fixed at x=0.

$$\boxed{I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\, dx + \frac{h}{2}(T_L - T_\infty)^2}$$

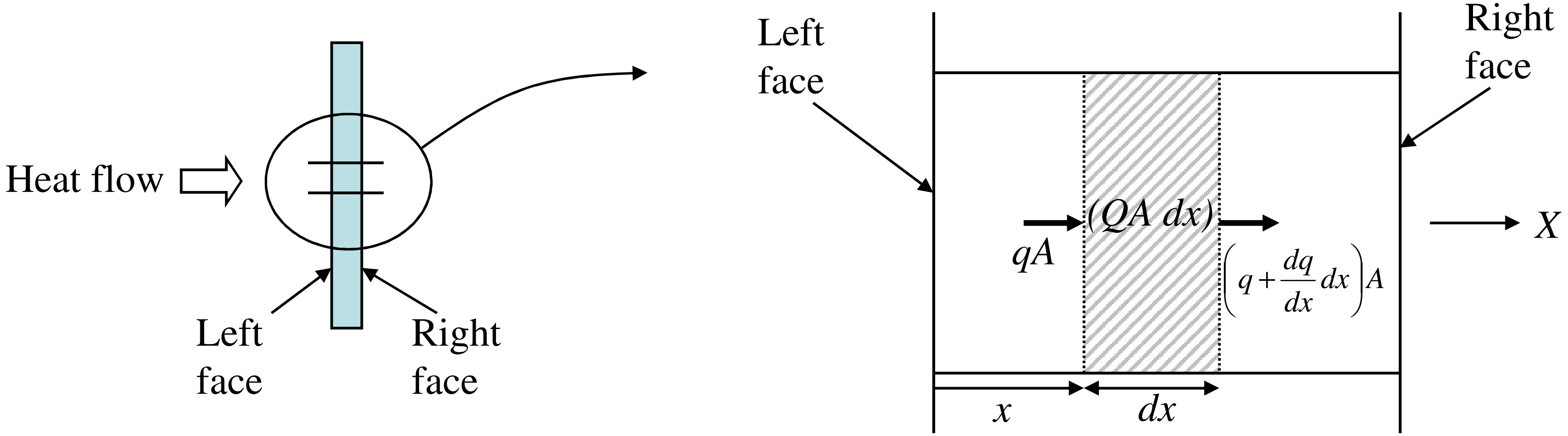
$$I(u) = \int_0^L \left[\frac{A}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{h}{2}[u(L)]^2 + \frac{h}{2}[u(0)]^2 \Longrightarrow 0 = \delta I(u) = \int_0^L \delta u\left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[A\frac{du}{dx} + hu(L)\right]_{x=L} - \delta u(0)\left[A\frac{du}{dx} - hu(0)\right]_{x=0}$$

with $\boldsymbol{\delta u(0)\neq 0}$, $\boldsymbol{\delta u(L)\neq 0}$,and $\quad A\frac{du}{dx} + hu(L)\Big|_{x=L} = 0 \quad A\frac{du}{dx} - hu(0)\Big|_{x=0} = 0$

Finite Element Method

Steady-state 1-D heat conduction

Governing equation (heat conduction in plane wall with uniform heat generation)



Let A =area normal to direction of heat flow,

Q(W/m^3) =internal heat generated per unit volume.

Heat rate (heat flux × area) enter the control volume **+** heat rate generated **=** Heat rate leaving control volume.

$$qA + QAdx = \left(q + \frac{dq}{dx}dx\right)A \xrightarrow{\text{simplify}} Q = \frac{dq}{dx}$$

$$q = -k\frac{small - big}{dx} = +ve$$

+ve=heat flux same direction with x-axis

Substitute Fourier's law $q = -k\dfrac{dT}{dx}$ $\Longrightarrow$ $\dfrac{d}{dx}\left(k\dfrac{dT}{dx}\right) + Q = 0$

Q is called source when +ve (heat is generated) and is called sink when –ve (heat is consumed)

Here, Q is referred as source.

Finite Element Method

Steady-state 2-D heat conduction

The heat flow through the wall of a heated room on a winter day is an example of conduction. In a thermally isotropic medium, Fourier's law for 2-D heat flow is:

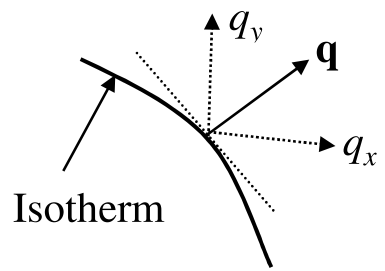
Index, not partial derivative!

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}$$

T=$T(x,y)$=temperature, q_x and q_y are components of heat flux (W/m^2), k is thermal conductivity (W/m. °C). (1W=1J/s=1Nm/s). Minus sign: heat is transferred in direction of decreasing temperature. k is material property.

$\mathbf{q}=q_x\mathbf{i}+q_y\mathbf{j}$, resultant heat flux (at right angles to an isotherm or a line of constant temperature.

$\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ are temperature gradients along x and y.



Constitutive relation- contains a material property.

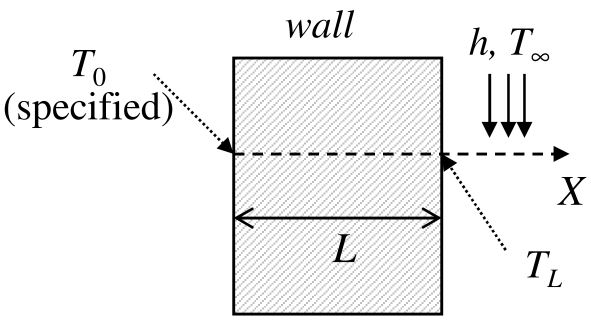
Convection-the flow of heat through a gas or a liquid

$q = h(T_s - T_\infty)$, q is convective heat flux (W/m^2), h is convection heat transfer coefficient or film coef (W/m^2·°C), T_s and T_∞ are surface and fluid temperature.

Finite Element Method

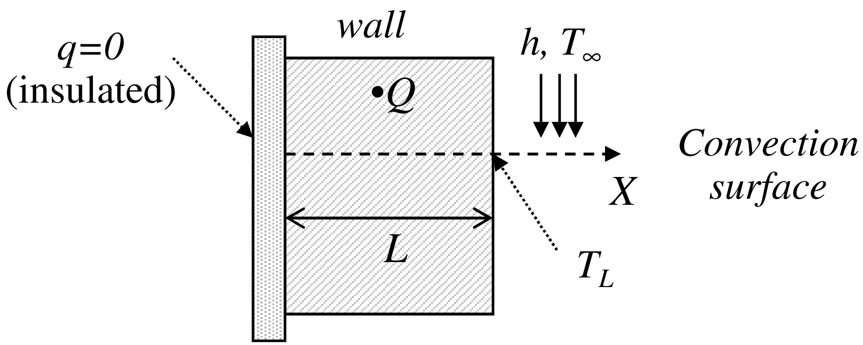
Steady-state 1-D heat conduction, Boundary conditions

Specified temperature



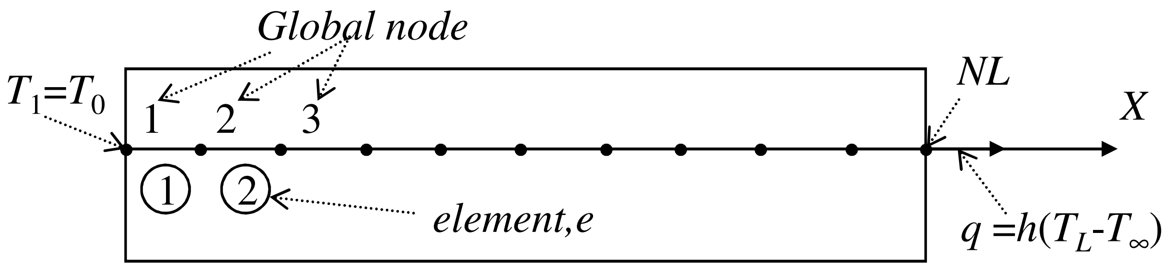
Wall of tank contain hot liquid at T_0,
airstream of T_∞ passed on outside,
maintain T_L at boundary.
$T|_{x=0} = T_0$, $q|_{x=L} = h(T_L - T_\infty)$. [note: $T_L > T_\infty$]

Specified heat flux



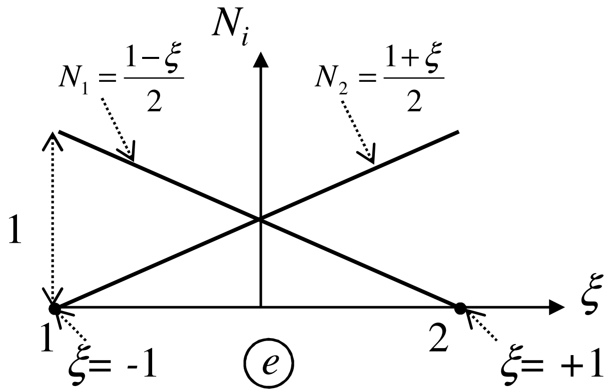
A wall where the inside surface is insulated
And outside is convection surface.
$q|_{x=0} = 0$, $q|_{x=L} = h(T_L - T_\infty)$.

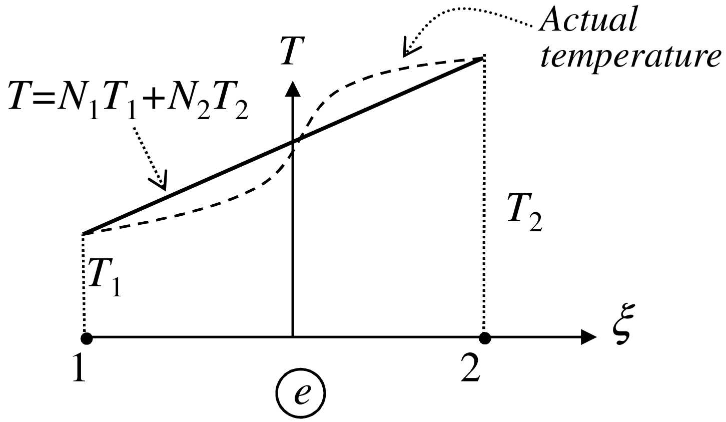
1-D element : two-node element with linear shape functions



Finite Element Method

1-D element





$$\boxed{T(\xi) = N_1T_1 + N_2T_2 = \mathbf{N}\mathbf{T}^e}$$

where N_1=(1-ξ)/2, N_2=(1+ ξ)/2, ξ varies from -1 to +1, $\mathbf{N}$=[N_1, N_2], $\mathbf{T}^e$=[T_1, T_2]$^\mathrm{T}$.

Please note $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1,\ d\xi = \frac{2}{x_2 - x_1}dx = \frac{2}{l_e}dx.$ $\boxed{x = N_1x_1 + N_2x_2}$ $x = \frac{(1-\xi)}{2}x_1 + \frac{(1+\xi)}{2}x_2$

Use chain rule, $\frac{dT}{dx} = \frac{dT}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e = \frac{1}{x_2 - x_1}[-1,1]\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e.$

$$\int_e f dx = \int_{-1}^{1} f\ J d\xi\ ,\quad J = \frac{l_e}{2} = Jacobian$$

where $\mathbf{B}_T = \frac{d}{dx}\mathbf{N} = \frac{1}{x_2 - x_1}[-1,\ 1] = \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}$

$$N_1 = \frac{x - x_2}{x_1 - x_2},\ \rightarrow N_1 = \frac{\xi - 1}{-1-1}$$

$$N_2 = \frac{x - x_1}{x_2 - x_1},\ \rightarrow N_2 = \frac{\xi - (-1)}{1-(-1)}$$

Lagrange interpolation $P(x) = \sum_{1}^{N} L_i(x) f_i,\ L_i(x) = \Pi_{\substack{j=0\\ j\neq i}}^{N}\frac{(x - x_j)}{(x_i - x_j)}$

Finite Element Method

Raleigh-Ritz variational method

$$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0}\qquad \boxed{T|_{x=0}=T_0,\quad q|_{x=L}=h(T_L\text{-}T_\infty).} \xrightarrow{\text{equivalent}} \boxed{I(T)=\int_0^L\frac{k}{2}\left(\frac{dT}{dx}\right)^2dx-\int_0^L QT\,dx+\frac{h}{2}(T_L-T_\infty)^2}$$

Let

$$I(T)=\frac{1}{2}B(T,T)-l(T)$$

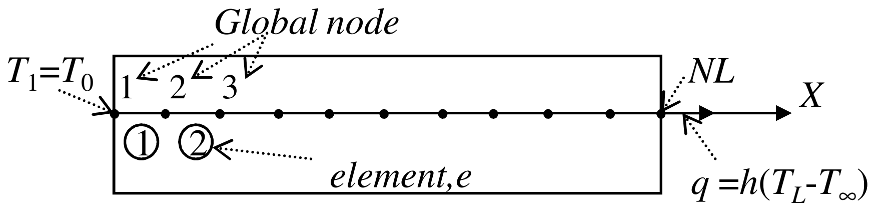
Use global node: $T=N_1^gT_1+\cdots+N_{NL}^gT_{NL}=\mathbf{NT}$

$N_0(x)$ satisfy specified essential boundary conditions, lowest order ($a+bx$).

$$\boxed{I(T)=\int_0^L\frac{k}{2}\left(\sum_i T_i\frac{dN_i^g}{dx}\right)^2dx-\int_0^L Q\left(\sum_i N_i^gT_i\right)dx+\frac{h}{2}(T_{NL}-T_\infty)^2}$$

The minimization of energy is equivalent to $\frac{\partial I(T)}{\partial T_i}=0\ ,\quad i=1,2,\cdots,NL.$

For i=1, involve e=1 only



$$\boxed{\text{Local node}\rightarrow N_i}$$

$$\frac{\partial I(T)}{\partial T_1}=\int_0^L k\left(\sum_i T_i\frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_1^g\,dx\bigg|_{e=1}$$

$$0=\int_{e=1}k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1}k_{e=1}\left(\frac{dN_2}{dx}T_2\right)\frac{dN_1}{dx}dx-r_1^{e=1}$$

$$=k_{11}^{e=1}T_1+k_{12}^{e=1}T_2-r_1^{e=1}+\gamma(T_1-T_0)=\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\end{bmatrix}-r_1^{e=1}.$$

$$\boxed{\mathbf{k}_T=k_e\int\mathbf{B}_T^T\mathbf{B}_Tdx=k_e\int\tfrac{d}{dx}\left(\mathbf{N}^T\right)\tfrac{d}{dx}(\mathbf{N})dx}$$

Finite Element Method

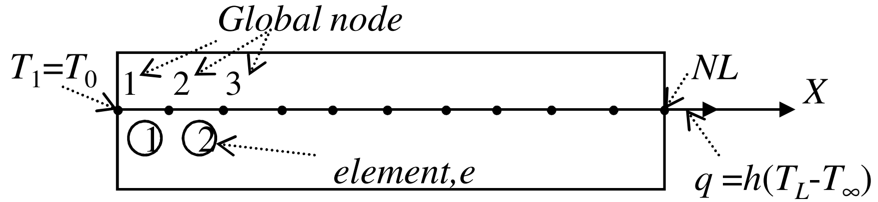
Raleigh-Ritz variational method

For i=2, involve e= 1& 2.

$$\frac{\partial I(T)}{\partial T_2} = \int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\bigg|_{e=1,2} - \int_0^L QN_2^g\, dx\bigg|_{e=1,2} + 0$$

$$0 = \int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1 + \frac{dN_2}{dx}T_2\right)\frac{dN_2}{dx}dx + \int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_2 + \frac{dN_2}{dx}T_3\right)\frac{dN_1}{dx}dx - \int_{e=1} QN_2 dx - \int_{e=2} QN_1 dx$$

$$0 = \begin{bmatrix} k_{21} & k_{22} \end{bmatrix}^{e=1}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \end{bmatrix}^{e=2}\begin{bmatrix} T_2 \\ T_3 \end{bmatrix} - r_2^{e=1} - r_1^{e=2}$$



For i=NL, involve e= NL-1.

$$\frac{\partial I(T)}{\partial T_{NL}} = \int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_{NL}^g}{dx}dx\bigg|_{e=NL-1} - \int_0^L QN_{NL}^g\, dx\bigg|_{e=NL-1} + h(T_{NL} - T_\infty)$$

$$0 = \int_{e=NL-1} k_{e=NL-1}\left(\frac{dN_1}{dx}T_{NL-1} + \frac{dN_2}{dx}T_{NL}\right)\frac{dN_2}{dx}dx - \int_{e=NL-1} QN_2 dx + h(T_{NL} - T_\infty)$$

$$0 = \begin{bmatrix} k_{21} & k_{22} \end{bmatrix}^{e=NL-1}\begin{bmatrix} T_{NL-1} \\ T_{NL} \end{bmatrix} - r_2^{e=NL-1} + h(T_{NL} - T_\infty).$$

Finite Element Method

Raleigh-Ritz variational method

Combining all *NL* equations, we finally get

$$\begin{bmatrix} (K_{11}) & K_{12} & \cdots & K_{1,NL} \\ K_{21} & K_{22} & \cdots & K_{2,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,1} & K_{NL,2} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NL} \end{bmatrix} = \begin{bmatrix} (R_1) \\ R_2 \\ \vdots \\ (R_{NL}+hT_\infty) \end{bmatrix}$$

Global stiffness matrix   Global load matrix

Note: The minimization of the functional $\quad I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_L - T_\infty)^2$

$$\delta I(T) = 0 = \int_0^L k\left(\frac{dT}{dx}\right)\delta\left(\frac{dT}{dx}\right)dx - \int_0^L Q\,\delta T\,dx + h(T_L - T_\infty)\delta(T_L)$$

$$0 = \int_0^L k\,\frac{dT}{dx}\frac{d\delta T}{dx}dx - \int_0^L Q\,\delta T\,dx + h(T_L - T_\infty)\delta(T_L)$$

Set the arbitrary function $\boldsymbol{\delta T \to \phi}$, where $\boldsymbol{\phi(0)=0}$ or $\delta(T_0=0)=0$.

$$0 = \int_0^L k\,\frac{dT}{dx}\frac{d\phi}{dx}dx - \int_0^L Q\phi\,dx + h(T_L - T_\infty)\phi(L).$$

Finite Element Method

Raleigh-Ritz variational method

Problem: A composite wall consists of 3 materials. The outer temperature is T_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with T_∞=800ºC and h=25 W/m²·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k\left.\frac{\partial T}{\partial x}\right|_{x_1} = -k\frac{T_{1.1}-T_1}{\Delta x} \approx -k\frac{400-500}{+0.1} = +ve = -h(T_1 - T_\infty)$$

Solution: we use 3 elements of linear element.

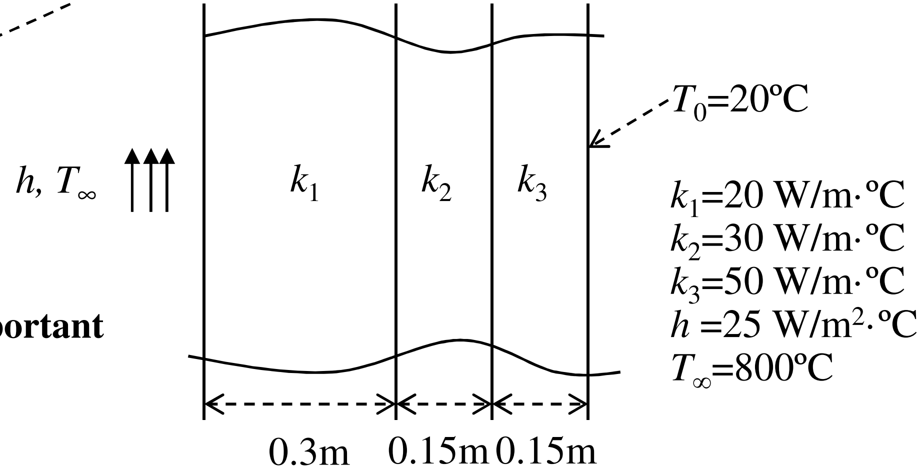
B.C.: $T_4 = T_0$=20, $q|_{x=0} = -h(T_1\text{-}T_\infty)$. We get

$$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0} \qquad \boxed{T|_{x=L} = T_0=20,\quad q|_{x=0} = -h(T_1\text{-}T_\infty).}$$

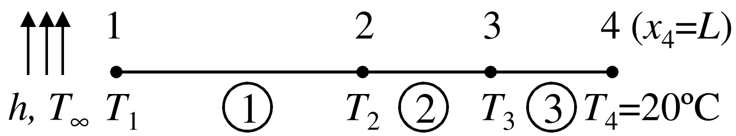
$\Downarrow$ equivalent

Sign not important

$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2}$$



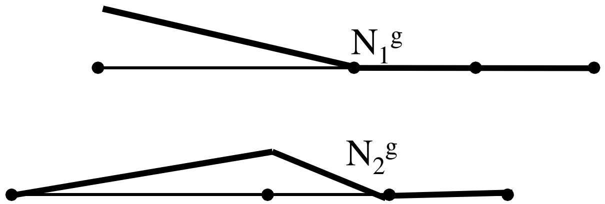
Now, let
$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2}$$



3 elements of linear FE

Use global node: $T = N_1^g T_1 + \cdots + N_4^g T_4$

$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g T_i\right)dx + \frac{h}{2}(T_\infty - T_1)^2}$$

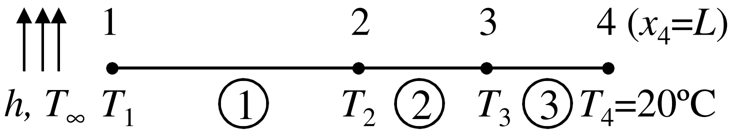


The minimization of energy is equivalent to $\dfrac{\partial I(T)}{\partial T_i} = 0\,,\quad i = 1,2,3,4.$

Finite Element Method

Raleigh-Ritz variational method

$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}\left(T_\infty - T_1\right)^2}$$



3 elements of linear FE

For i=1, involve e=1 only

$$\frac{\partial I(T)}{\partial T_1}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\Bigg|_{e=1}-\int_0^L QN_1^g dx\Bigg|_{e=1}+h\left(T_\infty - T_1\right)(-1)$$

Same final form

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_2}{dx}T_2\right)\frac{dN_1}{dx}dx-r_1^{e=1}+h\left(T_1-T_\infty\right)$$

$$=k_{11}^{e=1}T_1+k_{12}^{e=1}T_2-r_1^{e=1}+h\left(T_1-T_\infty\right)=\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\end{bmatrix}-r_1^{e=1}+h\left(T_1-T_\infty\right).$$

$$\mathbf{k}_T=\frac{k_e}{l_e}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix},\quad \mathbf{r}_Q=\frac{Q_e l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

For i=2, involve e= 1& 2.

$$\frac{\partial I(T)}{\partial T_2}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\Bigg|_{e=1,2}-\int_0^L QN_2^g\,dx\Bigg|_{e=1,2}+0$$

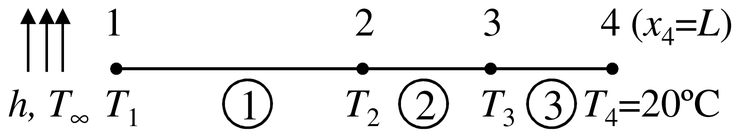
$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1+\frac{dN_2}{dx}T_2\right)\frac{dN_2}{dx}dx+\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_2+\frac{dN_2}{dx}T_3\right)\frac{dN_1}{dx}dx-\int_{e=1}QN_2dx-\int_{e=2}QN_1dx$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=2}\begin{bmatrix}T_2\\T_3\end{bmatrix}-r_2^{e=1}-r_1^{e=2}$$

Finite Element Method

Raleigh-Ritz variational method

$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}\left(T_\infty - T_1\right)^2}$$



3 elements of linear FE

For i=3, involve e= 2 & 3.

$$\frac{\partial I(T)}{\partial T_3}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_3^g}{dx}dx\bigg|_{e=2,3}-\int_0^L QN_3^g\, dx\bigg|_{e=2,3}+0$$

$$0=\int_{e=2}k_{e=2}\left(\frac{dN_1}{dx}T_2+\frac{dN_2}{dx}T_3\right)\frac{dN_2}{dx}dx+\int_{e=3}k_{e=3}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4\right)\frac{dN_1}{dx}dx-\int_{e=2}QN_2dx-\int_{e=3}QN_1dx$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=2}\begin{bmatrix}T_2\\ T_3\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=3}\begin{bmatrix}T_3\\ T_4\end{bmatrix}-r_2^{e=2}-r_1^{e=3}$$

For i=4, involve e= 3.

$$\frac{\partial I(T)}{\partial T_4}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_4^g}{dx}dx\bigg|_{e=3}-\int_0^L QN_4^g\, dx\bigg|_{e=3}$$

$$0=\int_{e=3}k_{e=3}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4\right)\frac{dN_2}{dx}dx-\int_{e=3}QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=3}\begin{bmatrix}T_3\\ T_4\end{bmatrix}-r_2^{e=3}.$$

$$\boxed{\mathbf{k}_T=k_e\int \mathbf{B}_T^T\mathbf{B}_T dx=k_e\int \tfrac{d}{dx}\left(\mathbf{N}^T\right)\tfrac{d}{dx}\left(\mathbf{N}\right)dx}$$

Finite Element Method

What is the element assembly matrix?

Some matrix concept: **AMC+ANC=(AM+AN)C=A(M+N)C.**

Let: $\mathbf{k}_T^{e=i} = \begin{bmatrix} k_{i,i} & k_{i,i+1} \\ k_{i+1,i} & k_{i+1,i+1} \end{bmatrix}, \quad \mathbf{r}_Q^{e=i} = \begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}.$

Please note: k_T are symmetry! $\rightarrow k_{i,j}=k_{j,i}$

2 elements example:

$$\boldsymbol{\psi}^T\mathbf{k}_T^{e=1}\mathbf{T}^{e=1} + \boldsymbol{\psi}^T\mathbf{k}_T^{e=2}\mathbf{T}^{e=2} = [\psi_1 \quad \psi_2]\begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} k_{2,2} & k_{2,3} \\ k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_2 \\ T_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & 2k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \boldsymbol{\Psi}^T\mathbf{K}_T\mathbf{T}.$$

We also get:

$$\boldsymbol{\psi}^T\mathbf{r}_Q^{e=1} + \boldsymbol{\psi}^T\mathbf{r}_Q^{e=2} = [\psi_1 \quad \psi_2]\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} r_2 \\ r_3 \end{bmatrix} = [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} + [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} 0 \\ r_2 \\ r_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} r_1 \\ 2r_2 \\ r_3 \end{bmatrix} = \boldsymbol{\Psi}^T\mathbf{R} = [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}.$$

Finite Element Method

Raleigh-Ritz variational method

Try insulation at x=L, $\phi(L)$=0
Try Q=2

Combining all 4 equations, we finally get

$$\begin{bmatrix} (K_{11}+h) & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & (K_{44}) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} (R_1+hT_\infty) \\ R_2 \\ R_3 \\ (R_4) \end{bmatrix}$$

We get

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 25\times 800 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global $\mathbf{K}_T$=Σ $\mathbf{k}_T$ is obtained

$$\mathbf{K}_T = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, we get $\mathbf{r}_Q$=[0 0]^T, $\mathbf{R}$=[0 0 0]^T.

Given T_0=20ºC, T_∞=800ºC and h=25 W/m^2·ºC, a tridiagonal linear system can be solved using Thomas algorithm and we get

[T_1, T_2, T_3, T_4]=**[304.6, 119.0, 57.1, 20.0]** **ºC**

Row and column adjustment method

Set boundary condition, T_4=20, we get

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 25\times 800 \\ 0 \\ 0 \\ 66.7\times 20 \end{bmatrix}$$

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 25\times 800 \\ 0 \\ 66.7\times 5\times 20 \\ 66.7\times 20 \end{bmatrix} \rightarrow 66.7\begin{bmatrix} 1.375 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 8 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 25\times 800 \\ 0 \\ 66.7\times 5\times 20 \end{bmatrix}$$

Other method : penalty method

$$\boxed{I(T) = \int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_L - T_\infty)^2 + \frac{\gamma}{2}(T_1 - T_0)^2}$$

Finite Element Method

Galerkin's approach for heat conduction

Problem: $$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0 \qquad T|_{x=0}=T_0,\quad q|_{x=L}=h(T_L\text{-}T_\infty).$$

Assume: $$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q\right]dx=0$$

$\phi(x)$ constructed from same basis function of T, with $\phi(0)=0$. ϕ as a virtual temperature change that is consistent with boundary conditions.

Weighted-Residual Method

First term use integration by part: $$\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du \quad \Longrightarrow \quad \boxed{\phi k\frac{dT}{dx}\Bigg|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q\,dx = 0}$$

Now, $$\phi k\frac{dT}{dx}\Bigg|_0^L = \phi(L)k(L)\frac{dT}{dx}(L) - \phi(0)k(0)\frac{dT}{dx}(0)$$

Since, $q=-k\dfrac{dT}{dx}$ So, $\phi(0)=0$, $q(L)=-k(L)[dT(L)/dx]=h(T_L\text{-}T_\infty)$, we get $\phi k\dfrac{dT}{dx}\Bigg|_0^L = -\phi(L)h(T_L-T_\infty)$.

Finally, we get $$\boxed{-\phi(L)h(T_L-T_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q\,dx = 0} \Longleftarrow$$

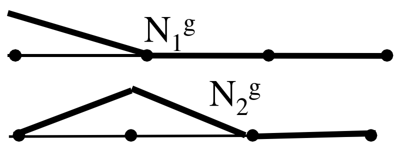
Weak form – reduced (weakened) continuity of T

Use global node: $T = N_1^g T_1 + \cdots + N_4^g T_4$

A global virtual-temperature vector is denoted: $\boldsymbol{\psi}=[\psi_1, \psi_2, \ldots, \psi_{NL}]^T$, or element-wise: $\boldsymbol{\psi}^e=[\psi_i, \psi_{i+1}]^T$.

The test function within each element is interpolated as: (global nodes) $\phi=\mathbf{N}\boldsymbol{\psi}$, or element-wise $\phi^e=\mathbf{N}^e\boldsymbol{\psi}^e$.

$$\frac{d\phi^e}{dx} = \frac{d}{dx}\phi^e = \frac{d}{dx}\left(\mathbf{N}^e\boldsymbol{\psi}^e\right) = \left(\frac{d\mathbf{N}^e}{d\xi}\frac{d\xi}{dx}\right)\cdot\boldsymbol{\psi}^e = \mathbf{B}_T\boldsymbol{\psi}^e$$



Finite Element Method

Galerkin's approach for heat conduction

<u>Some matrix concept</u>: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.
Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$=row vector, $\mathbf{AB}^T$=scalar $\rightarrow$ $\mathbf{AB}^T = (\mathbf{AB}^T)^T$.
$\mathbf{AB}^T\mathbf{CD}^T = (\mathbf{AB}^T)^T\mathbf{CD}^T = \mathbf{B}^{TT}\mathbf{A}^T\mathbf{CD}^T = \mathbf{B}(\mathbf{A}^T\mathbf{C})\mathbf{D}^T$=scalar

$N_i(x_j)=\delta_{ij}$ (Kronecker delta function,global)

We get,

$$-\phi(L)h(T_L - T_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q\,dx = 0 = -(\mathbf{N}(L)\boldsymbol{\psi})h(T_L - T_\infty) - \sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}} k\frac{d\phi}{dx}\frac{dT}{dx}dx + \sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}} \phi Q\,dx$$

$$= -\psi_{NL}h(T_L - T_\infty) - \sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dT^e}{dx}d\xi + \sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi = 0$$

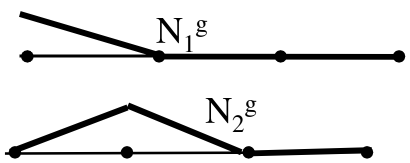
$$\boxed{d\xi = \tfrac{2}{l_e}\,dx}$$

Note that, $\frac{d\phi^e}{dx}\frac{dT^e}{dx} = (\mathbf{B}_T\boldsymbol{\psi}^e)(\mathbf{B}_T\mathbf{T}^e) = (\mathbf{B}_T\boldsymbol{\psi}^e)^T(\mathbf{B}_T\mathbf{T}^e) = \boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{T}^e$ and, $\mathbf{N}^e\boldsymbol{\psi}^e$=scalar=$(\mathbf{N}^e\boldsymbol{\psi}^e)^T = \boldsymbol{\psi}^T\mathbf{N}^T$.

$$0 = -\psi_{NL}h(T_L - T_\infty) - \sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{T}^e d\xi + \sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T\mathbf{N}^T d\xi$$

$$\boxed{\mathbf{k}_T = k_e\int \mathbf{B}_T^T\mathbf{B}_T dx = k_e\int \tfrac{d}{dx}(\mathbf{N}^T)\tfrac{d}{dx}(\mathbf{N})dx}$$

$$0 = -\psi_{NL}h(T_L - T_\infty) - \sum_e \boldsymbol{\psi}^T\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi\,\mathbf{T}^e + \sum_e \boldsymbol{\psi}^T\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi$$



Note that: $\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi = \frac{1}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int_{-1}^{1}d\xi = \frac{2}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\int_{-1}^{1}\mathbf{N}^T d\xi = \int_{-1}^{1}\begin{Bmatrix}(1-\xi)/2\\ (1+\xi)/2\end{Bmatrix}d\xi = \begin{bmatrix}1\\ 1\end{bmatrix}$.

Finally, $\boxed{0 = -\psi_{NL}h(T_L - T_\infty) - \sum_e \boldsymbol{\psi}^T\mathbf{k}_T\mathbf{T}^e + \sum_e \boldsymbol{\psi}^T\mathbf{r}_Q}$ where, $\mathbf{k}_T = \frac{k_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\mathbf{r}_Q = \frac{Q_e l_e}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$.

Finite Element Method

Galerkin's approach for heat conduction

Finally, we get:

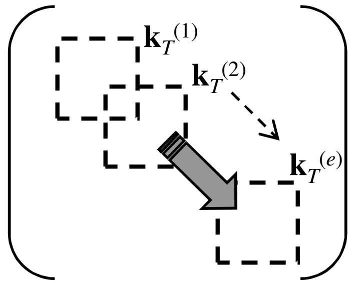
$$0 = -\psi_{NL} h\left(T_L - T_\infty\right) - \sum_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{T}^e + \sum_e \boldsymbol{\psi}^T \mathbf{r}_Q$$

$$0 = -\psi_{NL} h T_L + \psi_{NL} h T_\infty - \left(\boldsymbol{\psi}_{e=1}^T \mathbf{k}_T^{e=1} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \cdots + \boldsymbol{\psi}_{e=NL-1}^T \mathbf{k}_T^{e=NL-1} \begin{bmatrix} T_{NL-1} \\ T_{NL} \end{bmatrix} \right)$$

$$+ \left(\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \cdots + \begin{bmatrix} \psi_{NL-1} & \psi_{NL} \end{bmatrix} \begin{bmatrix} r_{NL-1} \\ r_{NL} \end{bmatrix} \right)$$

$$0 = -\psi_{NL} h T_L + \psi_{NL} h T_\infty - \boldsymbol{\Psi}^T \mathbf{K}_T \mathbf{T} + \boldsymbol{\Psi}^T \mathbf{R}.$$

$\mathbf{K}_T =$



The global matrices $\mathbf{K}_T$ and $\mathbf{R}$ are assembled from element matrices $\mathbf{k}_T$ and $\mathbf{r}_Q$.

Now, let $\boldsymbol{\Psi}^{\mathrm{T}}=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[0,1,0,\ldots,0]$, and $T_1=T_0$, we get

$$-0+0-\begin{bmatrix} K_{21} & K_{22} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NL} \end{bmatrix} + R_2 = 0 \rightarrow \begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{NL} \end{bmatrix} = R_2 - K_{21} T_0.$$

Continue the process, finally let $\boldsymbol{\Psi}^{\mathrm{T}}=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[0,0,\ldots,1]$, we get ($T_L=T_{NL}$)

$$-1\cdot hT_L + 1\cdot hT_\infty - \begin{bmatrix} K_{NL,1} & K_{NL,2} & \cdots & K_{NL,NL} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NL} \end{bmatrix} + R_{NL} = 0 \rightarrow \begin{bmatrix} K_{NL,2} & K_{NL,3} & \cdots & \left(K_{NL,NL}+h\right) \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{NL} \end{bmatrix} = \left(R_{NL} + hT_\infty\right) - K_{NL,1} T_0.$$

Finite Element Method

Galerkin's approach for heat conduction

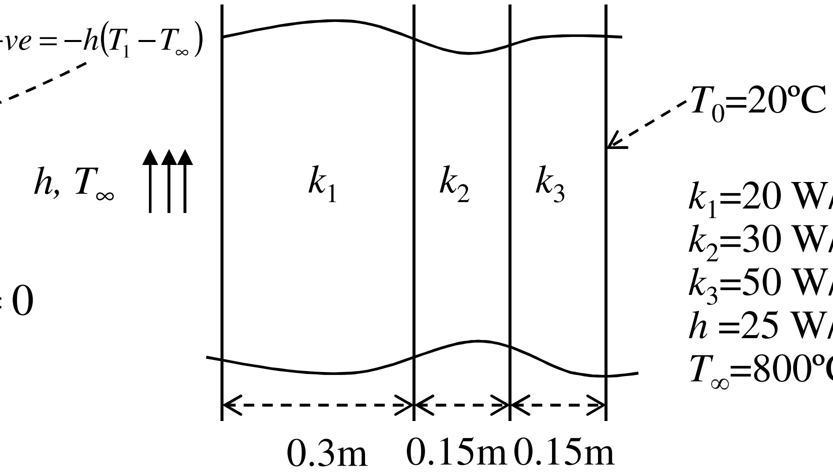
Finally, the compact form is given:

$$\begin{bmatrix} K_{2,2} & K_{2,3} & \cdots & K_{2,NL} \\ K_{3,2} & K_{3,3} & \cdots & K_{3,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,2} & K_{NL,3} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{NL} \end{Bmatrix} = \begin{Bmatrix} R_2 \\ R_3 \\ \vdots \\ R_{NL}+hT_\infty \end{Bmatrix} - \begin{Bmatrix} K_{2,1}T_0 \\ K_{3,1}T_0 \\ \vdots \\ K_{NL,1}T_0 \end{Bmatrix}.$$

Try insulation at $x=L$, $\phi(L)=0$
Try $Q=2$

Problem: A composite wall consists of 3 materials. The outer temperature is T_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with T_∞=800ºC and h=25 W/m^2·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k\left.\frac{\partial T}{\partial x}\right|_{x_1} = -k\frac{T_{1.5}-T_1}{\Delta x} \approx -k\frac{400-500}{+0.1} = +ve = -h(T_1 - T_\infty)$$



k_1=20 W/m·ºC
k_2=30 W/m·ºC
k_3=50 W/m·ºC
h =25 W/m^2·ºC
T_∞=800ºC

Solution: we use 3 elements of linear element.

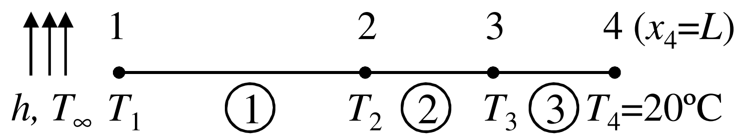
B.C.: $T_4 = T_0$=20, $q|_{x=0} = -h(T_1 - T_\infty)$. [$T_\infty > T_1$] We get

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q\right]dx = 0 \rightarrow \phi k \left.\frac{dT}{dx}\right|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q dx = 0$$

$$\phi k\left.\frac{dT}{dx}\right|_0^L = \phi(L)k(L)\frac{dT}{dx}(L) - \phi(0)k(0)\frac{dT}{dx}(0)$$

So, let $\boldsymbol{\phi(L)=0}$, $q(0) = -k(0)[dT(0)/dx] = -h(T_1 - T_\infty)$, we get

$$\phi k\left.\frac{dT}{dx}\right|_0^L = -\phi(0)h(T_1 - T_\infty).$$



3 elements of linear FE

Finite Element Method

Galerkin's approach for heat conduction

Let ϕ=**NΨ**, we get

Use global node: $T = N_1^g T_1 + \cdots + N_4^g T_4$

$$0 = -\psi_1 h(T_1 - T_\infty) - \sum_e \frac{k_e l_e}{2} \int_{-1}^{1} \frac{d\phi^e}{dx} \frac{dT^e}{dx} d\xi + \sum_e \frac{Q_e l_e}{2} \int_{-1}^{1} \mathbf{N}^e \boldsymbol{\psi}^e d\xi$$

Finally,

$$\boxed{0 = -\psi_1 h(T_1 - T_\infty) - \sum_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{T}^e + \sum_e \boldsymbol{\psi}^T \mathbf{r}_Q} \Longrightarrow \boxed{0 = -\psi_1 h T_1 + \psi_1 h T_\infty - \boldsymbol{\Psi}^T \mathbf{K}_T \mathbf{T} + \boldsymbol{\Psi}^T \mathbf{R}.}$$

Now, let $\boldsymbol{\Psi}^T$=[ψ_1, ψ_2, …, ψ_{NL}]=[1,0,0,0], and T_4=T_0, we get

$$\phi = N_1^g \rightarrow \quad -h(T_1 - T_\infty) - [K_{11} \quad K_{12} \quad K_{13} \quad K_{14}] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_1 = 0 \rightarrow [(K_{11} + h) \quad K_{12} \quad K_{13}] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = R_1 + hT_\infty - K_{14} T_0$$

let $\boldsymbol{\Psi}^T$=[ψ_1, ψ_2, …, ψ_{NL}]=[0,1,0,0], we get

$$\phi = N_2^g \rightarrow \quad -0 + 0 - [K_{2,1} \quad K_{2,2} \quad K_{2,3} \quad K_{2,4}] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_2 = 0 \rightarrow [K_{2,1} \quad K_{2,2} \quad K_{2,3}] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = R_2 - K_{2,4} T_0.$$

Finally, let $\boldsymbol{\Psi}^T$=[ψ_1, ψ_2, …, ψ_{NL}]=[0,0,1,0], we get

$$\phi = N_3^g \rightarrow \quad -0 + 0 - [K_{3,1} \quad K_{3,2} \quad K_{3,3} \quad K_{3,4}] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_3 = 0 \rightarrow [K_{3,1} \quad K_{3,2} \quad K_{3,3}] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = R_3 - K_{3,4} T_0.$$

Finite Element Method

Galerkin's approach for heat conduction

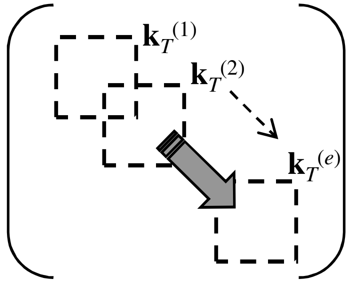
Finally, we get

$$\boxed{\begin{bmatrix}(K_{11}+h) & K_{1,2} & K_{1,3}\\ K_{2,1} & K_{2,2} & K_{2,3}\\ K_{3,1} & K_{3,2} & K_{3,3}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}=\begin{bmatrix}R_1+hT_\infty\\ R_2\\ R_3\end{bmatrix}-\begin{bmatrix}K_{1,4}T_0\\K_{2,4}T_0\\K_{3,4}T_0\end{bmatrix}} \quad \ldots\ldots\ldots\ldots\ldots \ (a)$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)}=\frac{20}{0.3}\begin{array}{c}\begin{matrix}\mathbf{1} & \mathbf{2}\end{matrix}\\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array},\quad \mathbf{k}_T^{(2)}=\frac{30}{0.15}\begin{array}{c}\begin{matrix}\mathbf{2} & \mathbf{3}\end{matrix}\\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array},\quad \mathbf{k}_T^{(3)}=\frac{50}{0.15}\begin{array}{c}\begin{matrix}\mathbf{3} & \mathbf{4}\end{matrix}\\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}$$

$\mathbf{K}_T =$



The global $\mathbf{K}_T=\Sigma\ \mathbf{k}_T$ is obtained

$$\mathbf{K}_T=66.7\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 4 & -3 & 0\\ 0 & -3 & 8 & -5\\ 0 & 0 & -5 & 5\end{bmatrix}$$

Since no heat generation Q occurs in this problem, we get $\mathbf{r}_Q=[0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0]^T$.

Given $T_0=20$ºC, $T_\infty=800$ºC and $h=25$ W/m$^2\cdot$ºC, eq. (a) becomes

$$\boxed{66.7\begin{bmatrix}1.375 & -1 & 0\\ -1 & 4 & -3\\ 0 & -3 & 8\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}=\begin{bmatrix}0+25(800)\\0\\0\end{bmatrix}-\begin{bmatrix}0\\0\\-5(66.7)(20)\end{bmatrix}=\begin{bmatrix}20{,}000\\0\\6670\end{bmatrix}}$$

This linear system can be solved using Thomas algorithm and we get $[T_1, T_2, T_3]$=**[304.6, 119.0, 57.1] ºC**

$$\mathbf{Ax}=\mathbf{b},\ \ \mathbf{A}=\mathbf{LU}\rightarrow\begin{pmatrix}d_1 & e_1 & 0 & \cdots & 0\\ c_2 & d_2 & e_2 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1}\\ 0 & \cdots & 0 & c_n & d_n\end{pmatrix}=\begin{pmatrix}\alpha_1 & 0 & 0 & \cdots & 0\\ c_2 & \alpha_2 & 0 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0\\ 0 & \cdots & 0 & c_n & \alpha_n\end{pmatrix}\cdot\begin{pmatrix}1 & \beta_1 & 0 & \cdots & 0\\ 0 & 1 & \beta_2 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & 0 & 1 & \beta_{n-1}\\ 0 & \cdots & 0 & 0 & 1\end{pmatrix}$$

The whole Thomas algorithm can be summarized :

1. $\alpha_1=d_1$
2. $\alpha_i=d_i-c_i\beta_{i-1}$, $i=2,3,\ldots,n$
3. $\beta_i=e_i/\alpha_i$, $i=1,2,\ldots,n-1$.
4. $w_1=b_1/\alpha_1$
5. $w_i=(b_i-c_iw_{i-1})/\alpha_i$, $i=2,3,\ldots,n$.
6. $x_n=w_n$
7. $x_i=w_i-\beta_ix_{i+1}$, $i=n-1, n-2,\ldots,1$.

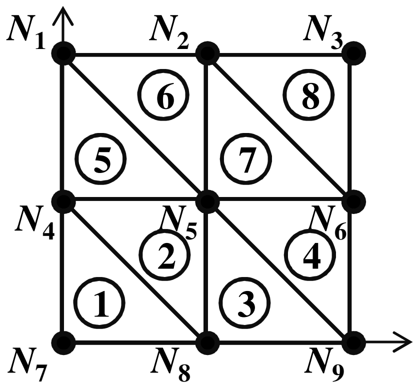
Finite Element Method

Galerkin's approach for heat conduction

Preprocessing

Preprocessing of the problem includes one or more of the following tasks:

- Read geometry and material data (E), and boundary and initial conditions of the problem.
- Mesh generation.
- Generation of node numbers.
- Generation of coordinates and connectivity.



element	1	2	3	← local
1	7	8	4	
2	8	5	4	
3	8	9	5	↑
4	9	6	5	Global
5	4	5	1	
6	5	2	1	↓
7	5	6	2	
8	6	3	2	

Linear triangular element

Processing of FEM

Processing of the FEM includes one or more of the following tasks:

- Calculate element matrices.
- Assemble element equations.
- Solve the system of equations.

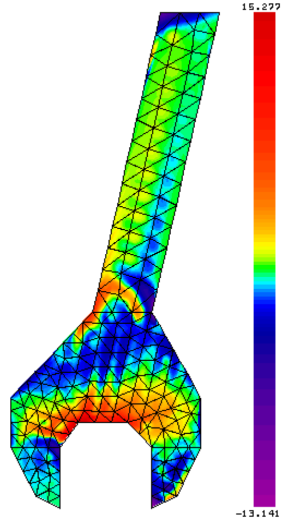
Finite Element Method

Galerkin's approach for heat conduction

Postprocessing

Postprocessing of the FEM includes one or more of the following tasks:

- Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
- Interpretation of the results to check whether the solution makes sense (based on physical Process and experience when other solutions are not available.
- Tabular and/or graphical presentation of the results. Contour plotting uses $\xi = \dfrac{2}{x_2 - x_1}(x - x_1) - 1$



Contour plot for stress

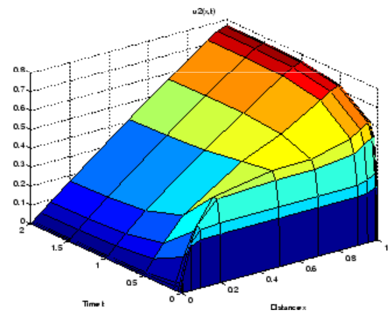
Interpolation of temperature within each element is given

$$T(\xi) = N_1 T_1 + N_2 T_2 = \mathbf{N}\mathbf{T}^e$$

where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{T}^e=[T_1, T_2]^{\mathrm{T}}$.

The derivative of the solution is obtained by differentiation

Use chain rule, $$\frac{dT}{dx} = \frac{dT}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e = \frac{1}{l_e}[-1,1]\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e.$$



Contour plot for $u_2(x,t)$

For element 1, we get $$\frac{dT^{e=1}}{dx} = \mathbf{B}_T\mathbf{T}^{e=1} = \frac{1}{l_e}[-1,1]\mathbf{T}^{e=1} = \frac{1}{0.3}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}304.6\\119.0\end{bmatrix} = -618.67$$

For element 2, we get $$\frac{dT^{e=2}}{dx} = \mathbf{B}_T\mathbf{T}^{e=2} = \frac{1}{l_e}[-1,1]\mathbf{T}^{e=2} = \frac{1}{0.15}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}119.0\\57.1\end{bmatrix} = -412.67$$

Note that the derivative above is discontinuous, for any order element, at the nodes connecting the different elements because the continuity of the derivative of FE solution at the connecting nodes is not imposed.

Finite Element Method

Galerkin's method with penalty approach

Problem: **minimize** the quadratic function

$$f(x,y)=4x^2-3y^2+2xy+6x-3y+5$$

subject to the **constraint** $G(x,y)=2x+3y=0$

Lagrange Multiplier Method. The modified functional is

$$F(x,y)=f(x,y)+\lambda G(x,y)$$

$$\frac{\partial F}{\partial x}=8x+2y+6+2\lambda=0$$

$$\frac{\partial F}{\partial y}=-6y+2x-3+3\lambda=0$$

$$\frac{\partial F}{\partial \lambda}=2x+3y=0$$

Solve 3 algebraic equations, we get $x = -3$, $y = 2$, $\lambda = 7$.

Penalty Function Method. The modified functional is

$$F(x,y)=f(x,y)+\frac{\gamma}{2}G^2(x,y)$$

$$\frac{\partial F}{\partial x}=8x+2y+6+2\gamma(2x+3y)=0$$

$$\frac{\partial F}{\partial y}=-6y+2x-3+3\gamma(2x+3y)=0$$

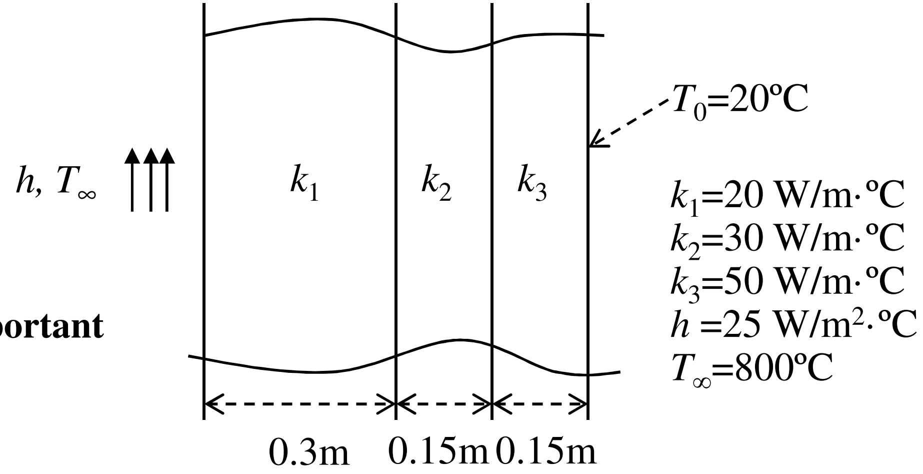
In the limit $\gamma \rightarrow \infty$, the results approaches the exact solution

γ	x	y	$G(x,y)$
0	-0.5769	-0.6923	-3.2308
1	1.5	-3	-6
10	-3.6702	2.7447	0.8936
100	-3.0537	2.0596	0.0716
1000	-3.0053	2.0058	0.0068
10000	-3.0005	2.0006	0.0008
∞	-3	2	0

Galerkin's method with penalty approach– quadratic shape functions

Penalty method is not in syllabus

Problem: A composite wall consists of 3 materials. The outer temperature is T_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with T_∞=800ºC and h=25 W/m^2·ºC. Determine the temperature distribution in the wall using quadratic shape functions.



Solution: we use 3 elements of quadratic shape functions.

B.C.: $T_7 = T_0$=20, $q|_{x=0} = -h(T_1-T_\infty)$. We get

$$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0} \quad \boxed{T|_{x=L} = T_0=20, \quad q|_{x=0} = -h(T_1-T_\infty).}$$

⇓ equivalent

Sign not important

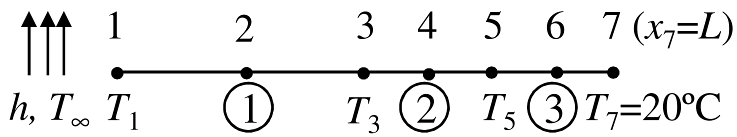
$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2}$$

Now, let $$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx + \frac{h}{2}(T_\infty - T_1)^2 + \frac{\gamma}{2}(T_7 - T_0)^2}$$

Use global node: $T = N_1^g T_1 + \cdots + N_7^g T_7$

$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g T_i\right)dx + \frac{h}{2}(T_\infty - T_1)^2 + \frac{\gamma}{2}(T_7 - T_0)^2}$$

The minimization of energy is equivalent to $\dfrac{\partial I(T)}{\partial T_i}=0$, $i = 1,\cdots,7.$



3 elements of quadratic FE

Galerkin's method with penalty approach– quadratic shape functions

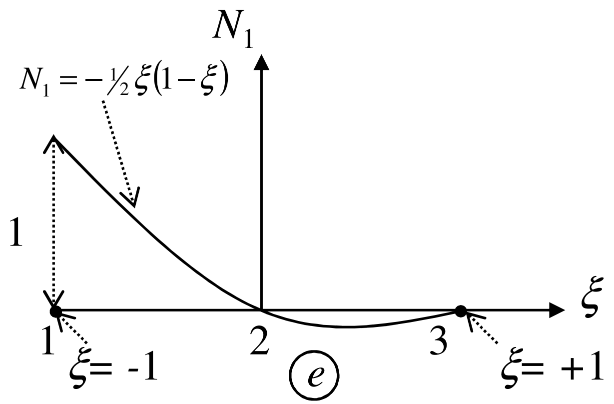
The temperature field within the element is written in terms of the nodal temperature as

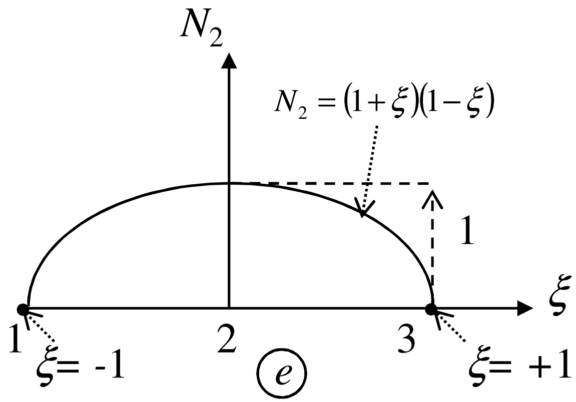
$$P(x)=\sum_{1}^{N}L_i(x)f_i,\ L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{N}\frac{(x-x_j)}{(x_i-x_j)}$$

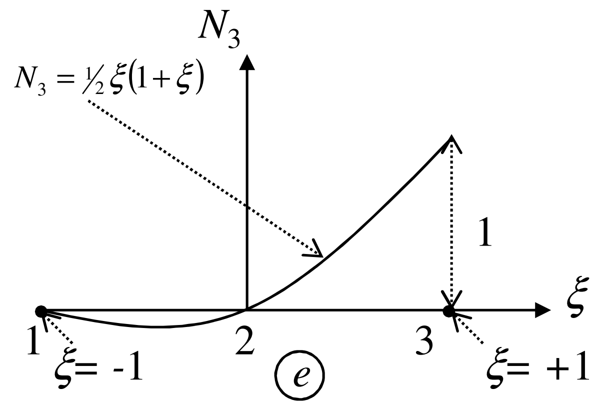
$$\boxed{T(\xi)=N_1T_1+N_2T_2+N_3T_3=\mathbf{N}\mathbf{T}^e}$$ Lagrange's interpolation

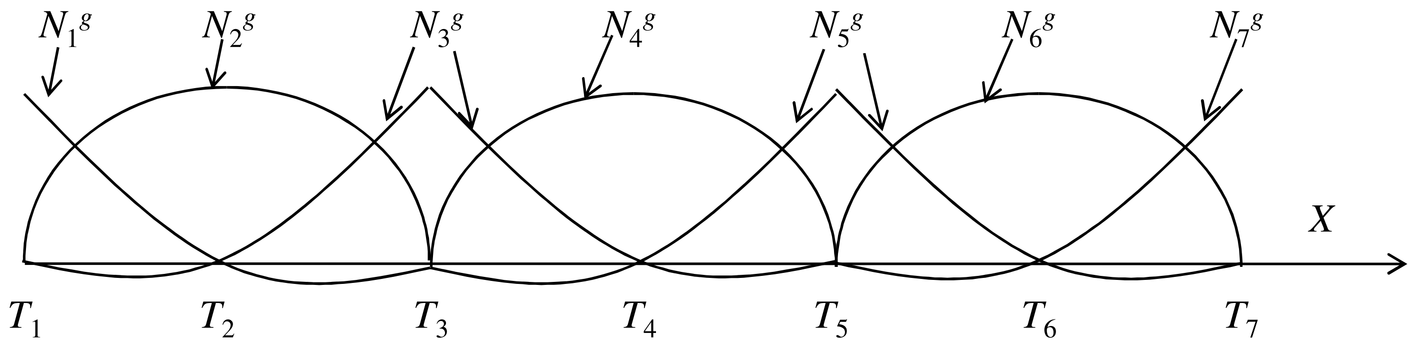
$$N_1(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

Where $N_1(\xi)= -\tfrac{1}{2}\xi(1-\xi)$, $N_2(\xi)= (1+\xi)(1-\xi)$, $N_3(\xi)= \tfrac{1}{2}\xi(1+\xi)$, ξ varies from -1 to $+1$, $\mathbf{N}=[N_1, N_2, N_3]$, $\mathbf{T}^e=[T_1, T_2, T_3]^T$.









Please note

$$\xi=\frac{2(x-x_2)}{x_3-x_1},\quad d\xi=\frac{2}{x_3-x_1}dx=\frac{2}{l_e}dx.$$

$$\int_{-1}^{1}\mathbf{N}^T d\xi=\int_{-1}^{1}\begin{bmatrix}-\tfrac{1}{2}\xi(1-\xi)\\(1+\xi)(1-\xi)\\\tfrac{1}{2}\xi(1+\xi)\end{bmatrix}d\xi=\begin{bmatrix}1/3\\4/3\\1/3\end{bmatrix}$$

Use chain rule, $$\frac{dT}{dx}=\frac{dT}{d\xi}\cdot\frac{d\xi}{dx}=\frac{2}{x_3-x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e=\frac{2}{x_3-x_1}[-{}^{1-2\xi}\!/_2,-2\xi,{}^{1+2\xi}\!/_2]\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e.$$

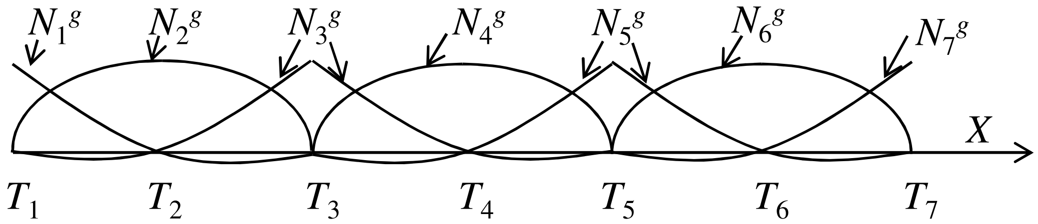
$$\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{2}{3l_e^2}\begin{bmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{bmatrix}$$

$$\mathbf{B}_T=\frac{d}{dx}\mathbf{N}=\frac{2}{x_3-x_1}[-{}^{1-2\xi}\!/_2,-2\xi,{}^{1+2\xi}\!/_2]=\frac{2}{l_e}[-{}^{1-2\xi}\!/_2,-2\xi,{}^{1+2\xi}\!/_2]$$

Galerkin's method with penalty approach– quadratic shape functions

$$\boxed{I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}(T_\infty-T_1)^2+\frac{\gamma}{2}(T_7-T_0)^2}$$

Use global node: $T=N_1^g T_1+\cdots+N_7^g T_7$



For i=1, involve e=1 only

$$\frac{\partial I(T)}{\partial T_1}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_1^g dx\bigg|_{e=1}+h(T_\infty-T_1)(-1)$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_2}{dx}T_2\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_3}{dx}T_3\right)\frac{dN_1}{dx}dx-r_1^{e=1}+h(T_1-T_\infty)$$

$$=k_{11}^{e=1}T_1+k_{12}^{e=1}T_2+k_{13}^{e=1}T_3-r_1^{e=1}+h(T_1-T_\infty)=\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}-r_1^{e=1}+h(T_1-T_\infty).$$

For i=2, involve e= 1.

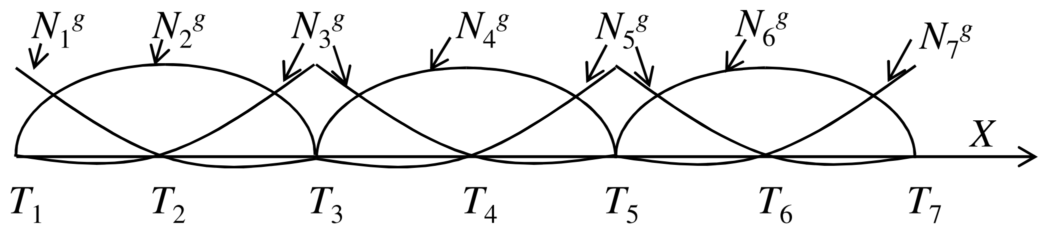
$$\frac{\partial I(T)}{\partial T_2}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_2^g\, dx\bigg|_{e=1}+0$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1+\frac{dN_2}{dx}T_2+\frac{dN_3}{dx}T_3\right)\frac{dN_2}{dx}dx-\int_{e=1} QN_2 dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}-r_2^{e=1}$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}(T_\infty-T_1)^2+\frac{\gamma}{2}(T_7-T_0)^2$$



Use global node: $T=N_1^g T_1+\cdots+N_7^g T_7$

For i=3, involve e= 1& 2.

$$\frac{\partial I(T)}{\partial T_3}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_3^g}{dx}dx\bigg|_{e=1,2}-\int_0^L QN_3^g\,dx\bigg|_{e=1,2}+0$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1+\frac{dN_2}{dx}T_2+\frac{dN_3}{dx}T_3\right)\frac{dN_3}{dx}dx+\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4+\frac{dN_3}{dx}T_5\right)\frac{dN_1}{dx}dx-\int_{e=1} QN_3dx-\int_{e=2} QN_1dx$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=1}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=2}\begin{bmatrix}T_3\\T_4\\T_5\end{bmatrix}-r_3^{e=1}-r_1^{e=2}$$

For i=4, involve e= 2.

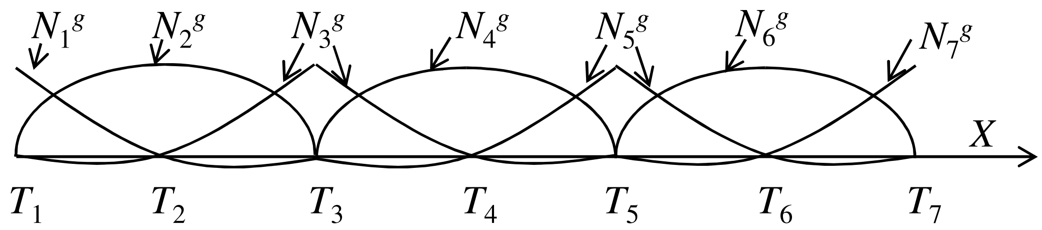
$$\frac{\partial I(T)}{\partial T_4}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_4^g}{dx}dx\bigg|_{e=2}-\int_0^L QN_4^g\,dx\bigg|_{e=2}+0$$

$$0=\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4+\frac{dN_3}{dx}T_5\right)\frac{dN_2}{dx}dx-\int_{e=2} QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=2}\begin{bmatrix}T_3\\T_4\\T_5\end{bmatrix}-r_2^{e=2}$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}(T_\infty-T_1)^2+\frac{\gamma}{2}(T_7-T_0)^2$$



Use global node: $T=N_1^g T_1+\cdots+N_7^g T_7$

For i=5, involve e=2 & 3.

$$\frac{\partial I(T)}{\partial T_5}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_5^g}{dx}dx\Bigg|_{e=2,3}-\int_0^L QN_5^g\,dx\Bigg|_{e=2,3}+0$$

$$0=\int_{e=2}k_{e=2}\left(\frac{dN_1}{dx}T_3+\frac{dN_2}{dx}T_4+\frac{dN_3}{dx}T_5\right)\frac{dN_3}{dx}dx+\int_{e=3}k_{e=3}\left(\frac{dN_1}{dx}T_5+\frac{dN_2}{dx}T_6+\frac{dN_3}{dx}T_7\right)\frac{dN_1}{dx}dx-\int_{e=2}QN_3dx-\int_{e=3}QN_1dx$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=2}\begin{bmatrix}T_3\\T_4\\T_5\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=3}\begin{bmatrix}T_5\\T_6\\T_7\end{bmatrix}-r_3^{e=2}-r_1^{e=3}$$

For i=6, involve e= 3.

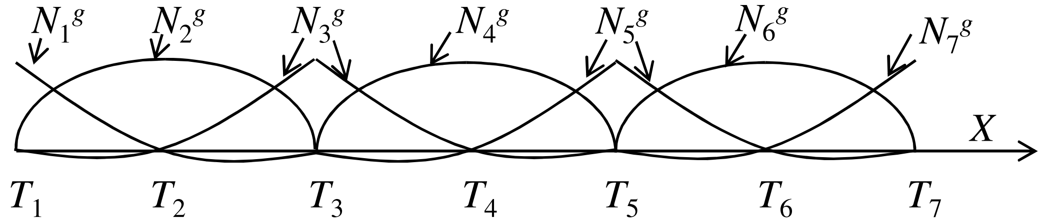
$$\frac{\partial I(T)}{\partial T_6}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_6^g}{dx}dx\Bigg|_{e=3}-\int_0^L QN_6^g\,dx\Bigg|_{e=3}+0$$

$$0=\int_{e=3}k_{e=3}\left(\frac{dN_1}{dx}T_5+\frac{dN_2}{dx}T_6+\frac{dN_3}{dx}T_7\right)\frac{dN_2}{dx}dx-\int_{e=3}QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=3}\begin{bmatrix}T_5\\T_6\\T_7\end{bmatrix}-r_2^{e=3}$$

Galerkin's method with penalty approach– quadratic shape functions

$$I(T)=\int_0^L \frac{k}{2}\left(\sum_i T_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g T_i\right)dx+\frac{h}{2}\left(T_\infty - T_1\right)^2+\frac{\gamma}{2}\left(T_7-T_0\right)^2$$



Use global node: $T = N_1^g T_1 + \cdots + N_7^g T_7$

For i=7, involve e= 3.

$$\frac{\partial I(T)}{\partial T_7}=\int_0^L k\left(\sum_i T_i \frac{dN_i^g}{dx}\right)\frac{dN_7^g}{dx}dx\Bigg|_{e=3}-\int_0^L QN_7^g\, dx\Bigg|_{e=3}+\gamma\left(T_7-T_0\right)$$

$$0=\int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}T_5+\frac{dN_2}{dx}T_6+\frac{dN_3}{dx}T_7\right)\frac{dN_3}{dx}dx-\int_{e=3} QN_3 dx+\gamma\left(T_7-T_0\right)$$

$$0=\left[k_{31}\quad k_{32}\quad k_{33}\right]^{e=3}\begin{bmatrix}T_5\\ T_6\\ T_7\end{bmatrix}-r_3^{e=3}+\gamma\left(T_7-T_0\right)$$

$$\mathbf{k}_T = k_e\int \mathbf{B}_T^T\mathbf{B}_T dx=\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{k_e}{3l_e}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}$$

Combining all 7 equations, we finally get

$$\begin{bmatrix}(K_{11}+h) & K_{12} & K_{13} & 0 & 0 & 0 & 0\\ K_{21} & K_{22} & K_{23} & 0 & 0 & 0 & 0\\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & 0 & 0\\ 0 & 0 & K_{43} & K_{44} & K_{45} & 0 & 0\\ 0 & 0 & K_{53} & K_{54} & K_{55} & K_{56} & K_{57}\\ 0 & 0 & 0 & 0 & K_{65} & K_{66} & K_{67}\\ 0 & 0 & 0 & 0 & K_{75} & K_{76} & (K_{77}+\gamma)\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\end{bmatrix}=\begin{bmatrix}(R_1+hT_\infty)\\ R_2\\ R_3\\ R_4\\ R_5\\ R_6\\ (R_7+\gamma T_0)\end{bmatrix}$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)}=\frac{20}{0.9}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix},\ \mathbf{k}_T^{(2)}=\frac{30}{0.45}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix},\ \mathbf{k}_T^{(3)}=\frac{50}{0.45}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}$$

Galerkin's method with penalty approach– quadratic shape functions

let γ= max $|K_{ij}|\times 10^4=80\times(200/9)\times 10^4$

$$\frac{200}{9}\begin{bmatrix} 8.125 & -8 & 1 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 \\ 1 & -8 & 28 & -24 & 1 & 0 & 0 \\ 0 & 0 & -24 & 48 & -24 & 0 & 0 \\ 0 & 0 & 3 & -24 & 56 & -40 & 5 \\ 0 & 0 & 0 & 0 & -40 & 80 & -40 \\ 0 & 0 & 0 & 0 & 5 & -40 & 800{,}035 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix}=\begin{bmatrix} (0+20{,}000) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 355.556\times 10^6 \end{bmatrix}$$

Since no heat generation Q occurs in this problem,

we get $\mathbf{r}_Q=[0\ 0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0\ 0\ 0\ 0\ 0]^T$.

Given T_0=20ºC, T_∞=800ºC and h=25 W/m$^2\cdot$ºC,

this linear system can be solved and we get

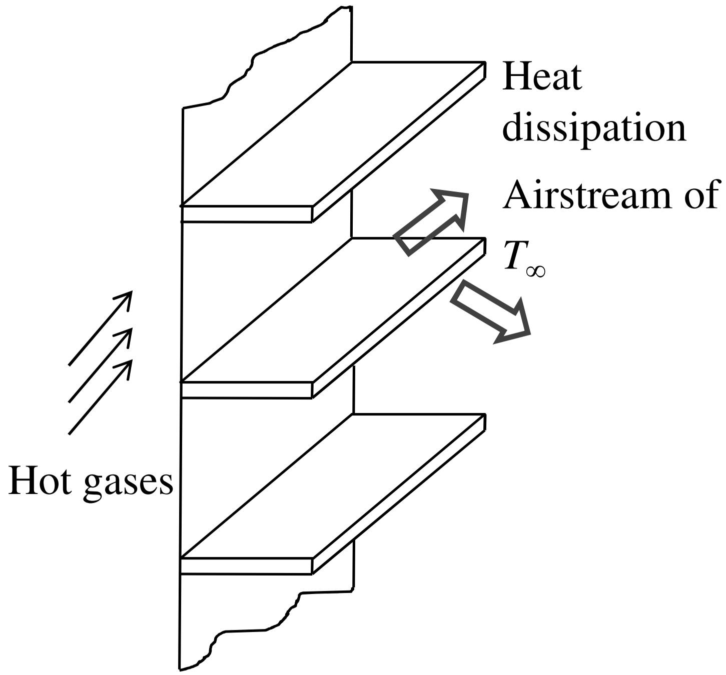
$[T_1, T_2, T_3, T_4, T_5, T_6, T_7]$=**[304.76, 211.91, 119.05, 88.10, 57.14, 38.57, 20.00]** ºC

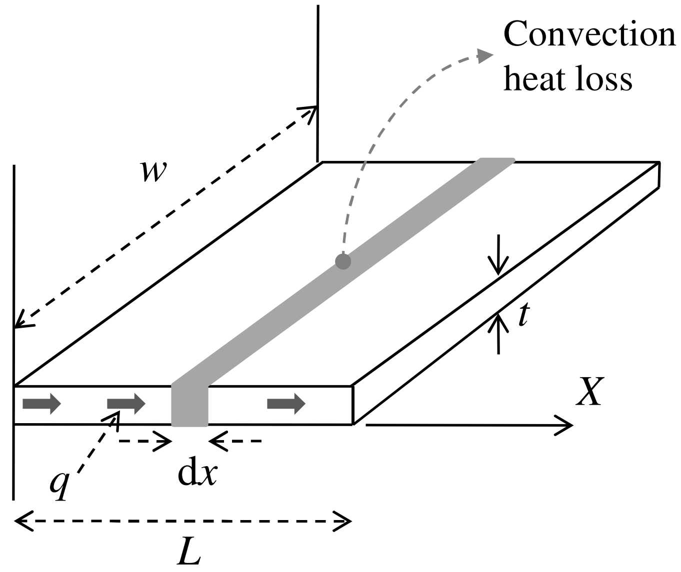
Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

A fin is an extended surface that is added onto a structure to increase the rate of heat removal.

$$\boxed{\begin{aligned} P &= 2(w+t) \\ A_c &= wt \\ \frac{P}{A_c} &\approx \frac{2}{t} \end{aligned}}$$





The governing equation derived from conduction equation with heat source: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0$

The convection heat loss in fin can be considered as a –ve heat source: $Q=-\frac{(Pdx)h(T-T_\infty)}{A_c dx}=-\frac{Ph}{A_c}(T-T_\infty)$

where P = perimeter of fin, A_c=area of cross section.

Finally, we get

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{Ph}{A_c}(T-T_\infty)=0$$

Let the case where the base of fin is held at T_0 and the tip of the fin is insulated (heat going out of the tip is negligible), the boundary conditions: $T=T_0$ [x=0] , $q=0$ [x=L].

Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

Let $\phi(x)$ be any function satisfying $\phi(0)=0$ using the same basis as T, we get $\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{Ph}{A_c}\left(T-T_\infty\right)\right]dx=0$

Integrating the first term by parts, we have

$$\boxed{\phi k\frac{dT}{dx}\bigg|_0^L-\int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx-\frac{Ph}{A_c}\int_0^L \phi T dx+\frac{Ph}{A_c}T_\infty\int_0^L \phi\, dx=0}$$

use $\boxed{q=-k\frac{dT}{dx}}$

Using $\phi(0)=0$, $k(L)[dT(L)/dx]=0$, and the isoparametric relations

$$dx=\frac{l_e}{2}d\xi,\quad T=\mathbf{N}\mathbf{T}^e,\quad \phi=\mathbf{N}\mathbf{\Psi},\quad \frac{dT}{dx}=\left(\frac{d}{dx}\mathbf{N}\right)\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e,\quad \frac{d\phi}{dx}=\left(\frac{d}{dx}\mathbf{N}\right)\mathbf{\Psi}=\mathbf{B}_T\mathbf{\Psi}$$

We get
$$-\sum_e \boldsymbol{\psi}^T\left[\frac{k_e l_e}{2}\int_{-1}^1 \mathbf{B}_T^T\mathbf{B}_T d\xi\right]\mathbf{T}^e-\frac{Ph}{A_c}\sum_e \boldsymbol{\psi}^T\left(\tfrac{l_e}{2}\int_{-1}^1 \mathbf{N}^T\mathbf{N}d\xi\right)\mathbf{T}^e+\frac{PhT_\infty}{A_c}\sum_e \boldsymbol{\psi}^T\left(\frac{l_e}{2}\int_{-1}^1 \mathbf{N}^T d\xi\right)=0 \qquad \text{.. ..... } (a)$$

We define
$$\mathbf{h}_T=\frac{Ph}{A_c}\left(\frac{l_e}{2}\int_{-1}^1 \mathbf{N}^T\mathbf{N}d\xi\right)=\frac{Ph}{A_c}\frac{l_e}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\;\to\;\frac{P}{A_c}\approx\frac{2}{t}\;\to\;\mathbf{h}_T=\frac{hl_e}{3t}\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

and
$$\mathbf{r}_\infty=\frac{Ph}{A_c}T_\infty\left(\frac{l_e}{2}\int_{-1}^1 \mathbf{N}^T d\xi\right)\;\to\;\frac{P}{A_c}\approx\frac{2}{t}\;\to\;\frac{PhT_\infty}{A_c}\frac{l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}\approx\frac{hT_\infty l_e}{t}\begin{bmatrix}1\\1\end{bmatrix}$$

Eq. (a) reduces to
$$\boxed{-\sum_e \boldsymbol{\psi}^T\left(\mathbf{k}_T+\mathbf{h}_T\right)\mathbf{T}^e+\sum_e \boldsymbol{\psi}^T\mathbf{r}_\infty=0}\;\to\;\boxed{\boxed{-\mathbf{\Psi}^T\left(\mathbf{K}_T+\mathbf{H}_T\right)\mathbf{T}+\mathbf{\Psi}^T\mathbf{R}_\infty=0}}$$

Where the above hold for all $\mathbf{\Psi}$ satisfying $\psi_1=0$. Let $\mathbf{\Psi}^T=[\psi_1, \psi_2, \dots, \psi_{NL}]$. We now generate ($NL$-1) equations by letting $\mathbf{\Psi}^T=[0, 1, \dots, 0]$ … $\mathbf{\Psi}^T=[0, 0, \dots, 1]$, and denoting $K_{ij}=(K_T+H_T)_{ij}$, we obtain

$$\begin{bmatrix}K_{22}&K_{23}&\cdots&K_{2,NL}\\K_{32}&K_{33}&\cdots&K_{3,NL}\\\vdots&\vdots&\ddots&\vdots\\K_{NL,2}&K_{NL,3}&\cdots&K_{NL,NL}\end{bmatrix}\cdot\begin{bmatrix}T_2\\T_3\\\vdots\\T_{NL}\end{bmatrix}=\begin{bmatrix}\\\mathbf{R}_\infty\\\\\end{bmatrix}-\begin{bmatrix}K_{21}T_0\\K_{31}T_0\\\vdots\\K_{NL,1}T_0\end{bmatrix}$$

where we let $T_1=T_0$.

Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

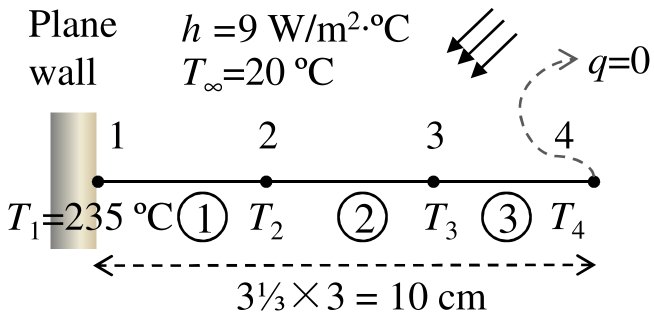
Problem: A metallic fin, with thermal conductivity k=360 W/m·ºC, 0.1 cm thick, and 10 cm long, extends from a plane wall whose temperature is 235 ºC. Determine the temperature distribution and amount of heat Transferred from the fin to the air at 20 ºC with h=9 W/m^2·ºC. Take the width of fin to be 1 m.

Solution:

Let $\phi(x)$ be any function satisfying $\phi(0)=0$ and $P/A_c \approx 2/t$, we get

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{Ph}{A_c}(T-T_\infty)\right]dx = 0 = \int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{2h}{t}(T-T_\infty)\right]dx$$

$$\phi k\frac{dT}{dx}\bigg|_0^L - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx - \frac{2h}{t}\int_0^L \phi T dx + \frac{2h}{t}T_\infty\int_0^L \phi\, dx = 0$$



t=0.1 cm, w=1m, k=360 W/m·ºC

Using $\phi(0)=0$, $k(L)[dT(L)/dx]=0$, and the isoparametric relations

$$dx = \frac{l_e}{2}d\xi,\quad T = \mathbf{N}\mathbf{T}^e,\quad \phi = \mathbf{N}\boldsymbol{\Psi},\quad \frac{dT}{dx} = \left(\frac{d}{dx}\mathbf{N}\right)\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e,\quad \frac{d\phi}{dx} = \left(\frac{d}{dx}\mathbf{N}\right)\boldsymbol{\Psi} = \mathbf{B}_T\boldsymbol{\Psi}$$

We get
$$-\sum_e \boldsymbol{\psi}^T\left[\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi\right]\mathbf{T}^e - \frac{2h}{t}\sum_e \boldsymbol{\psi}^T\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T\mathbf{N}d\xi\right)\mathbf{T}^e + \frac{2hT_\infty}{t}\sum_e \boldsymbol{\psi}^T\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi\right) = 0$$

We define
$$\mathbf{h}_T = \frac{2h}{t}\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T\mathbf{N}d\xi\right) = \frac{2h}{t}\frac{l_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{hl_e}{3t}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\qquad \mathbf{r}_\infty = \frac{2h}{t}T_\infty\left(\frac{l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi\right) = \frac{2hT_\infty}{t}\frac{l_e}{2}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{hT_\infty l_e}{t}\begin{bmatrix}1\\ 1\end{bmatrix}$$

Finally, we get
$$\boxed{-\sum_e \boldsymbol{\psi}^T(\mathbf{k}_T + \mathbf{h}_T)\mathbf{T}^e + \sum_e \boldsymbol{\psi}^T\mathbf{r}_\infty = 0} \rightarrow \boxed{-\boldsymbol{\Psi}^T(\mathbf{K}_T + \mathbf{H}_T)\mathbf{T} + \boldsymbol{\Psi}^T\mathbf{R}_\infty = 0}$$

The element conductivity matrices are $\mathbf{k}_T = \frac{k_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$
and we get

$$\mathbf{k}_T^{(1)} = \frac{360}{3.33\times 10^{-2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \mathbf{k}_T^{(2)} = \mathbf{k}_T^{(3)}$$

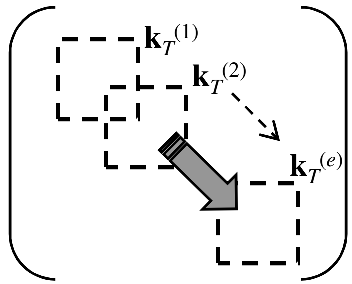
Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

The global $\mathbf{K}_T$=Σ $\mathbf{k}_T$ is obtained

$$\mathbf{K}_T = \frac{360}{3.33\times10^{-2}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = [\mathbf{K}_{ij}]$$

$\mathbf{K}_T =$



Now, we calculate for

$$\mathbf{h}_T = \frac{hl_e}{3t}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \mathbf{h}_T^{(1)} = \frac{9\times3.33\times10^{-2}}{3\times10^{-3}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \mathbf{h}_T^{(2)} = \mathbf{h}_T^{(3)} \rightarrow \mathbf{H}_T = 99.9\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [\mathbf{H}_{ij}]$$

and

$$\mathbf{r}_\infty = \frac{hT_\infty l_e}{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \mathbf{r}_\infty^{(1)} = \frac{9\times20\times3.33\times10^{-2}}{10^{-3}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{r}_\infty^{(2)} = \mathbf{r}_\infty^{(3)} \rightarrow \mathbf{R}_\infty = 5994\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{-\mathbf{\Psi}^T\left(\mathbf{K}_T + \mathbf{H}_T\right)\mathbf{T} + \mathbf{\Psi}^T\mathbf{R}_\infty = 0}$$

letting $\mathbf{\Psi}^T$=[0, 1, 0, 0] and T_1=T_0, we get

$$-\left(\begin{bmatrix} K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} + \begin{bmatrix} H_{21} & H_{22} & H_{23} & H_{24} \end{bmatrix}\right)\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_2 = 0 \rightarrow \left(\begin{bmatrix} K_{22} & K_{23} & K_{24} \end{bmatrix} + \begin{bmatrix} H_{22} & H_{23} & H_{24} \end{bmatrix}\right)\cdot\begin{bmatrix} T_2 \\ T_3 \\ T_4 \end{bmatrix} = R_2 - \left(K_{21} + H_{21}\right)T_0$$

letting $\mathbf{\Psi}^T$=[0, 0, 1, 0] and we get

$$-\left(\begin{bmatrix} K_{31} & K_{32} & K_{33} & K_{34} \end{bmatrix} + \begin{bmatrix} H_{31} & H_{32} & H_{33} & H_{34} \end{bmatrix}\right)\cdot\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} + R_3 = 0 \rightarrow \left(\begin{bmatrix} K_{32} & K_{33} & K_{34} \end{bmatrix} + \begin{bmatrix} H_{32} & H_{33} & H_{34} \end{bmatrix}\right)\cdot\begin{bmatrix} T_2 \\ T_3 \\ T_4 \end{bmatrix} = R_3 - \left(K_{31} + H_{31}\right)T_0$$

Finite Element Method

1-D heat transfer in thin fins –conduction & convection with Galerkin's method

$$\boxed{-\boldsymbol{\Psi}^T\left(\mathbf{K}_T+\mathbf{H}_T\right)\mathbf{T}+\boldsymbol{\Psi}^T\mathbf{R}_\infty=0}$$

letting $\boldsymbol{\Psi}^T$=[0, 0, 0, 1] and we get

$$-\left(\begin{bmatrix}K_{41} & K_{42} & K_{43} & K_{44}\end{bmatrix}+\begin{bmatrix}H_{41} & H_{42} & H_{43} & H_{44}\end{bmatrix}\right)\cdot\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix}+R_4=0 \;\rightarrow\; \left(\begin{bmatrix}K_{42} & K_{43} & K_{44}\end{bmatrix}+\begin{bmatrix}H_{42} & H_{43} & H_{44}\end{bmatrix}\right)\cdot\begin{bmatrix}T_2\\T_3\\T_4\end{bmatrix}=R_4-\left(K_{41}+H_{41}\right)T_0$$

Finally, we get

$$\left(\frac{360}{3.33\times10^{-2}}\begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}+99.9\begin{bmatrix}4 & 1 & 0\\1 & 4 & 1\\0 & 1 & 2\end{bmatrix}\right)\cdot\begin{bmatrix}T_2\\T_3\\T_4\end{bmatrix}=5994\begin{bmatrix}2\\2\\1\end{bmatrix}-\begin{bmatrix}-10711\times235\\0\\0\end{bmatrix}$$

Since this is tridiagonal linear system, it can be solved using Thomas algorithm, we get

[T_2, T_3, T_4]=**[209.8, 195.2, 190.5]ºC**.

$$q=-k\frac{\partial T}{\partial x}=h\left(T_s-T_\infty\right)\;\rightarrow\;k\Delta T\approx h\left(T_s-T_\infty\right)\Delta x$$

vertical horizontal

The total heat loss in the fin can be computed as $H=\Sigma_e H_e$

The loss H_e in each element is

$$H_e=h(T_{av}-T_\infty)A_s$$

$$\boxed{k\Delta T=(\text{W/m}\cdot{}^\circ\text{C})\cdot{}^\circ\text{C}=\text{W/m}}$$

where (area of surface=element length×2[width+thick]), (thick is ignorable)

A_s=2×(1×0.0333)m^2, and T_{av} is the average temperature within the element.

We get H_e^1=9([235+209.8]/2-20)×A_s= 121.3, H_e^2=9([209.8+195.2]/2-20)×A_s=109.4,

H_e^3=9([195.2+190.5]/2-20)×A_s= 103.6.

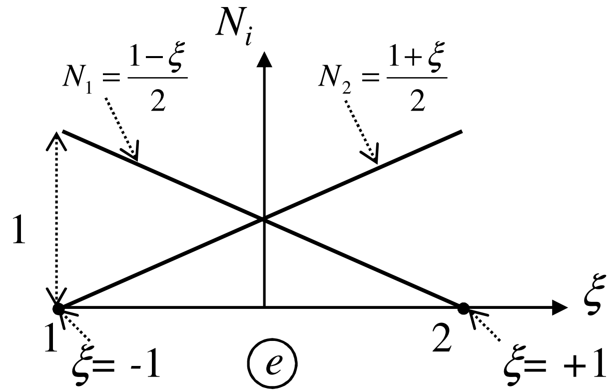
Finally, we get H_{loss}=121.3+109.4+103.6=334.3 W/m.

Finite Element Method

Time-dependent problems – Dynamic considerations – solid body with distributed mass

Let displacement, $u(x)$, the stress-strain and strain-displacement relations are $\boxed{\sigma = E\varepsilon\,,\ \varepsilon = \mathrm{d}u/\mathrm{d}x,\ \sigma = F/A\ (\mathrm{N/m^2})}$ where E is Young's modulus (or modulus of elasticity).

$$\boxed{u(\xi) = N_1 q_1 + N_2 q_2 = \mathbf{Nq}}$$



where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{q}=[q_1, q_2]^T$. (q_i is nodal displacement)

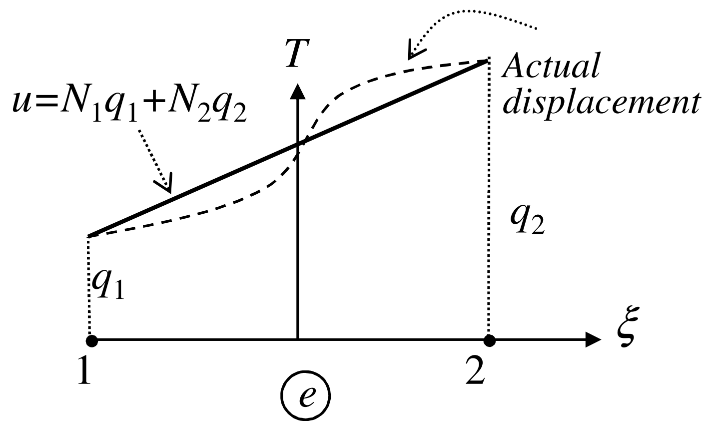
Please note $x=N_1x_1+N_2x_2$, and $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1,\ d\xi = \frac{2}{x_2 - x_1}dx = \frac{2}{l_e}dx.$

Use chain rule, $$\frac{du}{dx} = \frac{du}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{q} = \frac{1}{l_e}[-1,1]\mathbf{q} = \mathbf{Bq} = \varepsilon.$$

where matrix **B** is called element strain-displacement matrix.

The stress, from Hooke's law, is given $\boxed{\sigma = E\mathbf{Bq}}$

The potential-energy approach



The total potential energy, Π of an elastic body is

Π= strain energy (U) + work potential (WP)

where $U = \frac{1}{2}\int_V \sigma^T \varepsilon dV$, $WP = -\int_V u^T f dV - \int_S u^T T dS - \sum_i u_i^T P_i$

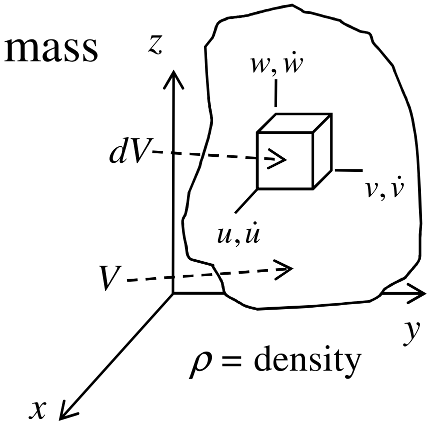
and we get

$$\boxed{\Pi = \frac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T f dV - \int_S u^T T dS - \sum_i u_i^T P_i}$$

where T (distributed force per unit area, traction), f (distributed force per unit volume, e.g. weight per unit volume), P (force, a load acting at point i), u_i (displacement at point i)

Finite Element Method

Time-dependent problems – Dynamic considerations – solid body with distributed mass



The kinetic energy is given by $T = \frac{1}{2}\int_V \dot{u}^T \dot{u} \rho dV$

where the velocity vector is given $\dot{\mathbf{u}} = [\dot{u}, \dot{v}, \dot{w}]^T$

1-D problem

Let **q** is the nodal displacements, we get u=**Nq**

In dynamic analysis, the elements of **q** are dependent on time, **N** represents (spatial) shape function defined on a master element. The velocity vector is $\boxed{\dot{u} = \mathbf{N}\dot{\mathbf{q}}}$

So, we get

$$T_e = \frac{1}{2}\dot{\mathbf{q}}^T \left[\int_e \rho \mathbf{N}^T \mathbf{N} dV\right] \dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{m}^e \dot{\mathbf{q}}$$

where $\mathbf{m}^e$ is the **element mass matrix**. This mass matrix is consistent with the shape functions chosen and is called the **consistent mass matrix**. On summing over all the elements, the kinetic energy is given

$$T = \sum_e T_e = \sum_e \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{m}^e \dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{Q}}^T \mathbf{M}\dot{\mathbf{Q}}$$

The potential energy is given by $\boxed{\Pi = \frac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T f dV - \int_S u^T T dS - \sum_i u_i^T P_i}$

using relation σ=E**Bq**, ε=**Bq**, we get

$$\Pi_e = \frac{1}{2}\int_e \mathbf{q}^T \mathbf{B}^T E \mathbf{B} \mathbf{q} A dx - \mathbf{q}^T A_e f \int_e \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx - \mathbf{q}^T T \int_e \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$\boxed{\Pi = \sum_e \Pi_e - \sum_i q_i P_i = \sum_e \left(\frac{1}{2}\mathbf{q}^T \mathbf{k}^e \mathbf{q} - \mathbf{q}^T \mathbf{f}^e - \mathbf{q}^T \mathbf{T}^e\right) - \mathbf{Q}^T \mathbf{P} = \frac{1}{2}\mathbf{Q}^T \mathbf{K}\mathbf{Q} - \mathbf{Q}^T \mathbf{F}}$$

K (global stiffness matrix),
F (global load vector),
Q (global displacement vector).

where element stiffness matrix, element body force vector, element traction-force vector are:

$$\mathbf{k}^e = \frac{E_e A_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{f}^e = \frac{A_e l_e f}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{T}^e = \frac{T l_e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finite Element Method

Time-dependent problems – Dynamic considerations – solid body with distributed mass

Lagrangean

We define Lagrangean by $\boxed{L = T - \Pi}$

where T (kinetic energy), Π (potential energy).

Hamilton's principle: for an arbitrary time interval from t_1 to t_2, the state of motion of a body extremizes the functional $I = \int_{t_1}^{t_2} L(t, q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$ (Euler eq.→) the equation of motion are: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \cdots, n.$

Now, we get $L = T - \Pi = \frac{1}{2}\dot{\mathbf{Q}}^T \mathbf{M} \dot{\mathbf{Q}} - \frac{1}{2}\mathbf{Q}^T \mathbf{K} \mathbf{Q} + \mathbf{Q}^T \mathbf{F}$

Matrix and vector derivatives

The derivative of the row vector **y** w.r.t scalar x is $\frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \cdots & \frac{\partial y_m}{\partial x} \end{bmatrix}$

The derivative of a scalar y w.r.t. vector **x** is $\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix}^T$

Let **x** be n×1 vector and **y** be m×1 vector, the derivative of **y** w.r.t. **x** is a matrix $\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_m/\partial x_1 \\ \vdots & \ddots & \vdots \\ \partial y_1/\partial x_n & \cdots & \partial y_m/\partial x_n \end{bmatrix}_{n \times m}$

So, we get $\frac{\partial L}{\partial \dot{\mathbf{Q}}} = \frac{1}{2}\left(\mathbf{M}\dot{\mathbf{Q}} + \mathbf{M}^T \dot{\mathbf{Q}}\right)$ $\quad \frac{\partial L}{\partial \mathbf{Q}} = -\frac{1}{2}\left(\mathbf{K}\mathbf{Q} + \mathbf{K}^T \mathbf{Q}\right) + \mathbf{F}$

M and **K** : symmetrical matrix→ $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{Q}}}\right) - \frac{\partial L}{\partial \mathbf{Q}} = \frac{d}{dt}\mathbf{M}\dot{\mathbf{Q}} - (-\mathbf{K}\mathbf{Q} + \mathbf{F}) = 0 \quad \leftrightarrow \quad \mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{F}}$

y (scalar or a vector)	$\partial y/\partial \mathbf{x}$
A**x**	A^T
$\mathbf{x}^T A$	A
$\mathbf{x}^T \mathbf{x}$	2**x**
$\mathbf{x}^T A \mathbf{x}$	$A\mathbf{x} + A^T\mathbf{x}$

For free vibrations, **F**=0. For steady-state condition, starting from the equilibrium state, we set $\mathbf{Q} = \mathbf{U}\sin\omega t$, where **U** is the vector of nodal amplitudes of vibration and ω (rad/s) is the circular frequency (=2πf, f=cycles/s or Hz), we get $\boxed{\mathbf{K}\mathbf{U} = \omega^2 \mathbf{M}\mathbf{U}}$

This is generalized eigenvalue problem $\boxed{\mathbf{K}\mathbf{U} = \lambda \mathbf{M}\mathbf{U}}$

where **U** is eigenvector, representing the vibrating mode, corresponding to eigenvalue λ.

Finite Element Method

Time-dependent problems – Dynamic considerations – solid body with distributed mass

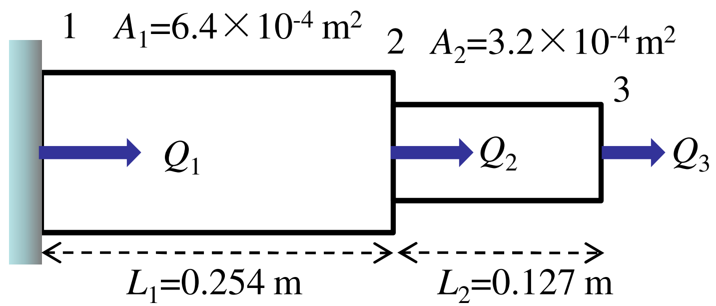
Element mass matrices – 1D bar element

We get, $\mathbf{q}^T=[q_1\ q_2]$, $\mathbf{N}=[N_1\ N_2]$, where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$. Note: $x=N_1x_1+N_2x_2$, $\xi=\frac{2}{x_2-x_1}(x-x_1)-1$, $d\xi=\frac{2}{l_e}dx$.
We get

$$\mathbf{m}^e=\rho\int_e \mathbf{N}^T\mathbf{N}A\,dx=\frac{\rho A_e l_e}{2}\int_{-1}^{+1}\mathbf{N}^T\mathbf{N}\,d\xi=\frac{\rho A_e l_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Problem: Determine the eigenvalues and eigenvectors for the stepped bar below:

Solution: generalized eigenvalue problem is given $\boxed{\mathbf{KU}=\lambda\mathbf{MU}}$



$E=206.8\times10^9$ Pa, specific weight $f=7.68\times10^4$ N/m³.

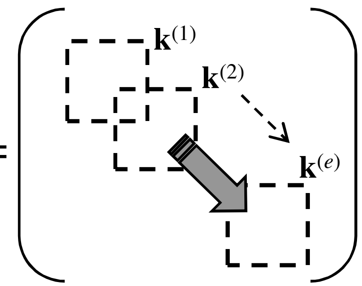
$$L=T-\Pi=\sum_e \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{m}^e\dot{\mathbf{q}}-\sum_e\left(\tfrac{1}{2}\mathbf{q}^T\mathbf{k}^e\mathbf{q}-\mathbf{q}^T\mathbf{f}^e-\mathbf{q}^T\mathbf{T}^e\right)+\mathbf{Q}^T\mathbf{P}$$
$$=\tfrac{1}{2}\dot{\mathbf{Q}}^T\mathbf{M}\dot{\mathbf{Q}}-\tfrac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q}+\mathbf{Q}^T\mathbf{F}$$

So, we get $\mathbf{k}^1=\frac{EA_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\mathbf{k}^2=\frac{EA_2}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$$\mathbf{m}^1=\frac{\rho A_1L_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\quad \mathbf{m}^2=\frac{\rho A_2L_2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$\boxed{\mathbf{Q}=\mathbf{U}\sin\omega t,\ \lambda=\omega^2}$

K=



So, we get

$$\mathbf{K}=\begin{bmatrix}\frac{EA_1}{L_1} & -\frac{EA_1}{L_1} & 0\\ -\frac{EA_1}{L_1} & \frac{EA_1}{L_1} & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{EA_2}{L_2} & -\frac{EA_2}{L_2}\\ 0 & -\frac{EA_2}{L_2} & \frac{EA_2}{L_2}\end{bmatrix}=E\begin{bmatrix}\frac{A_1}{L_1} & -\frac{A_1}{L_1} & 0\\ -\frac{A_1}{L_1} & \left(\frac{A_1}{L_1}+\frac{A_2}{L_2}\right) & -\frac{A_2}{L_2}\\ 0 & -\frac{A_2}{L_2} & \frac{A_2}{L_2}\end{bmatrix}$$

Pa=N/m², N=kg·m/s²

Finite Element Method

Time-dependent problems – Dynamic considerations – solid body with distributed mass

$\boxed{\mathbf{KU}=\lambda\mathbf{MU}\rightarrow}$

$$E\begin{bmatrix} A_1/L_1 & -A_1/L_1 & 0 \\ -A_1/L_1 & \left(A_1/L_1 + A_2/L_2\right) & -A_2/L_2 \\ 0 & -A_2/L_2 & A_2/L_2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \lambda \rho/6 \begin{bmatrix} 2A_1L_1 & A_1L_1 & 0 \\ A_1L_1 & 2(A_1L_1 + A_2L_2) & A_2L_2 \\ 0 & A_2L_2 & 2A_2L_2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

where u_i and U_i are local/global nodal amplitudes of vibration. Q_i and q_i are global/local displacement nodes.

Gathering the stiffness and mass values corresponding to the deg. of freedom Q_2 and Q_3, (delete 1st row, substitute U_1=0 for remaining rows), we get

$$E\begin{bmatrix} \left(A_1/L_1 + A_2/L_2\right) & -A_2/L_2 \\ -A_2/L_2 & A_2/L_2 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \lambda \rho/6 \begin{bmatrix} 2(A_1L_1 + A_2L_2) & A_2L_2 \\ A_2L_2 & 2A_2L_2 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

The density is given $\rho = \frac{f}{g} = \frac{7.68\times10^4}{9.81} = 7828.75\ kg/m^3$

Substitute this values, we get
$$206.8\times10^9\begin{bmatrix} 5\times10^{-3} & -2.5\times10^{-3} \\ -2.5\times10^{-3} & 2.5\times10^{-3} \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \lambda\cdot 7828.75\begin{bmatrix} 2.4\times10^{-4} & 4\times10^{-5} \\ 4\times10^{-5} & 8\times10^{-5} \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

$\boxed{\mathbf{KU}=\lambda\mathbf{MU} \rightarrow (\mathbf{K}-\lambda\mathbf{M})\mathbf{U}=\mathbf{0} \rightarrow \det(\mathbf{K}-\lambda\mathbf{M})=0}$ Get eigenvalue λ_1, λ_2. $(\mathbf{K}-\lambda_i\mathbf{M})\mathbf{U}_i=\mathbf{0}$ get eigenvector U_1, U_2.

We get eigenvalue λ_1, λ_2= 0.1650×10^{10}, 0.1501×10^9; Eigenvectors $\mathbf{U}_1$, $\mathbf{U}_2$=[0.4996, –1]^T, [–0.7484, –1]^T.

For normalization, we use $\boxed{\mathbf{U}_i^T\mathbf{M}\mathbf{U}_i=1}$

Finally, we get

Eigenvectors $\mathbf{U}_1$, $\mathbf{U}_2$=[0.5650, –1.131]^T, [–0.5109, –0.6826]^T.

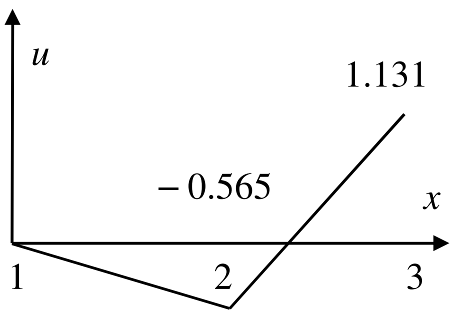
Finite Element Method

Time-dependent problems – Dynamic considerations – solid body with distributed mass

Note that $\lambda=\omega^2$, where ω is the circular frequency ($\omega=2\pi f$), f=frequency in hertz, Hz (cycles/s). Period, T= $2\pi/\omega$

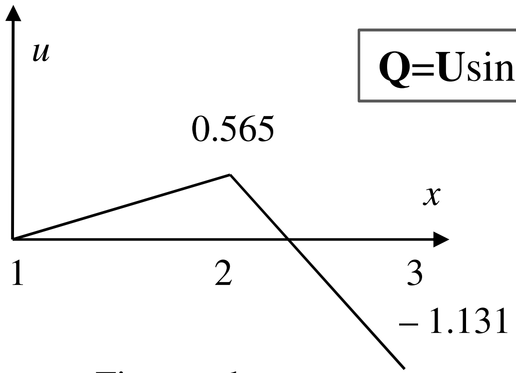
The mode shapes are given

Free vibration at L_1=0.254m, L_2=0.127m! Magnitude depend on elongation.

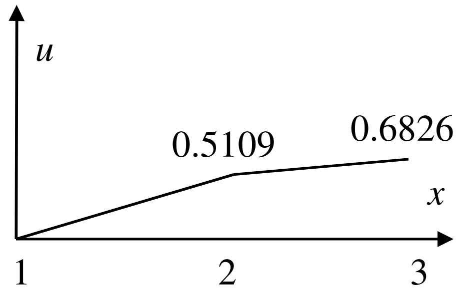


vibration $\Leftrightarrow$

$\mathbf{Q=U}\sin\omega t$



First mode



Second mode

Generalized Eigenvalue problem

The generalized eigenvalue problem is defined by $\boxed{\mathbf{KU}=\lambda\mathbf{MU}}$

There are n (dim. of system) eigenpairs. The i^{th} eigenpair is denoted by $(\lambda_i,\mathbf{U}_i)$, where the eigenvalues are ordered according to their magnitude: $0\le\lambda_1\le\lambda_2\le\ldots\le\lambda_{n-1}\le\lambda_n$.

And the eigenvectors are mass-normalized as:

$$\mathbf{U}_i^T\mathbf{M}\mathbf{U}_j=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}\quad,\quad \mathbf{U}_i^T\mathbf{K}\mathbf{U}_j=\begin{cases}\lambda_i, & i=j\\ 0, & i\neq j\end{cases}$$

or we get $\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}_{n\times n}\quad,\quad \mathbf{U}^T\mathbf{K}\mathbf{U}=\Lambda$

The above properties are hold if **K** is symmetric, **M** is symmetric and positive definite. Using Cholesky factorization, the generalized eigenproblem can be written $\mathbf{KU}=\lambda\mathbf{LL}^T\mathbf{U}\ \rightarrow \mathbf{L}^{-1}\mathbf{KU}=\lambda\mathbf{L}^T\mathbf{U}\rightarrow \mathbf{Cy}=\lambda\mathbf{y}$ (with $\mathbf{y}=\mathbf{L}^T\mathbf{U}$, $\mathbf{C}=\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$, sym.)

If **A** is symmetric, we get (eigenvalues of sym. matrix are real)

$$\begin{bmatrix}2&4&6\\4&7&8\\6&8&5\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\3&4&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&4\\0&0&1\end{bmatrix}$$

If **A** is symmetric and positive definite, then we don't have diagonal matrix, and diagonal entries are not necessary 1 in matrix **L**.

Finite Element Method

Generalized Eigenvalue problem

Let **A** be the matrix representation of any inner product on V. Then **A** is a positive definite matrix.

Let $u_1=(1,1,0)$, $u_2=(1,2,3)$, $u_3=(1,3,5)$ form a basis S for Euclidean space $\mathbf{R}^3$. We get $\langle u_1,u_1\rangle=1+1+0=2$, $\langle u_1,u_2\rangle=1+2+0=3,\ldots$

$$\mathbf{A}=\begin{bmatrix}2 & 3 & 4\\ 3 & 14 & 22\\ 4 & 22 & 35\end{bmatrix}$$

A=positive definite matrix
- $\mathbf{A}^{-1}$ exist
- $a_{ii}>0$
- $(a_{ij})^2<a_{ii}a_{jj}$

Let **A** be a real symmetric matrix, then it is positive definite if, for every nonzero vector **u**, $\langle\mathbf{u},\mathbf{Au}\rangle=\mathbf{u}^T\mathbf{Au}>0$

Inner Product spaces: Let V real vector space. Let $u,v\in V$, there is assigned a real number, denoted by $\langle u,v\rangle$. This function is called a **inner product** on V if it satisfied the following axioms:
- (linear property): $\langle au_1+bu_2,v\rangle= a\langle u_1,v\rangle+ b\langle u_2,v\rangle$
- (symmetric property): $\langle u,v\rangle= \langle v,u\rangle$
- (positive definite property): $\langle u,u\rangle\geq 0$; and $\langle u,u\rangle=0$ if and only if $u=0$.

Examples of inner product spaces:
- Euclidean Space, $\mathbf{R}^n$: $\langle u,v\rangle=u\cdot v=a_1b_1+\ldots a_nb_n$.
- Function Space, $C[a,b]$: $\langle f,g\rangle=\int_a^b f(t)g(t)dt$
- Matrix Space, $\mathbf{M}_{m\times n}$: $\langle\mathbf{A},\mathbf{B}\rangle=\mathrm{tr}(\mathbf{B}^T\mathbf{A})$
- Hilbert Space, l_2-space: $\langle u,v\rangle=u\cdot v=a_1b_1+a_2b_2+\ldots$.

Cholesky factorization
$\mathbf{A}=\mathbf{LL}^T$
The algorithm is given

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}$$

$$\boxed{\begin{aligned}l_{kk}&=\sqrt{a_{kk}-\sum_{s=1}^{k-1}l_{ks}^2}\\ l_{ik}&=\left(a_{ik}-\sum_{s=1}^{k-1}l_{is}l_{ks}\right)\Big/l_{kk},\quad i=k,k+1,\cdots,n.\end{aligned}}$$

Finite Element Method

Generalized Eigenvalue problem

Inverse Iteration Method

Let $u_1=\Sigma_{i=1}^{n}\, a_i\phi_i$ (ϕ_i are normalized eigenvector)
use $\mathbf{K}u_{k+1}=\mathbf{M}u_k$, we get
$u_2=\mathbf{K}^{-1}\mathbf{M}u_1=\Sigma_{i=1}^{n}\, a_i\mathbf{K}^{-1}\mathbf{M}\phi_i$
Since $\mathbf{K}\phi_i=\lambda_i\mathbf{M}\phi_i$, we get
$u_2=\Sigma_{k=1}^{n}\, a_k\mathbf{K}^{-1}\mathbf{M}\phi_k=\Sigma_{k=1}^{n}\, a_k\mathbf{K}^{-1}(\lambda_k^{-1}\mathbf{K}\phi_k)=\Sigma_{k=1}^{n}\, a_k\lambda_k^{-1}\phi_k$.
Similarly,
$u_3=\mathbf{K}^{-1}\mathbf{M}u_2=\Sigma_{k=1}^{n}\, a_k\lambda_k^{-1}\mathbf{K}^{-1}\mathbf{M}\phi_k=\Sigma_{k=1}^{n}\, a_k\lambda_k^{-2}\phi_k$.
And generally,
$u_{k+1}=\Sigma_{i=1}^{n}\, a_i\lambda_i^{-k}\phi_i=\lambda_1^{-k}\left(a_i\phi_1+\sum_{i=2}^{n} a_i\left[\lambda_1/\lambda_i\right]^k\phi_i\right)$
When k becomes large, $(\lambda_1/\lambda_i)^k\to 0$ (since $\lambda_1<\lambda_i$) and we get
$u_{k+1}=\lambda_1^{-k}\, a_1\phi_1 \propto \phi_1$ (proportional to), which after normalization, becomes the 1st eigenvector (λ_1 is eigenvalue).
Smallest eigenvalue/eigenvector are: $(\boldsymbol{\lambda_1}, \boldsymbol{u_{k+1}})$.

Find the lowest eigenvalue (with eigenvector) for $\mathbf{KU}=\lambda\mathbf{MU}$

The procedure are:

1.Choose a guess starting vector u_1,
2.Solve w_{k+1} from: $\mathbf{K}w_{k+1}=\mathbf{M}u_k$,
3.Normalize w_{k+1} as: $u_{k+1}=w_{k+1}\Big/\sqrt{w_{k+1}^T\mathbf{M}w_{k+1}}$
4.Compute λ_{k+1} using: $\lambda_{k+1}=u_{k+1}^{T}\mathbf{K}u_{k+1}$,
5.Check the convergence: $|\lambda_{k+1}-\lambda_k|/\lambda_{k+1}\le\varepsilon$. ($\varepsilon$ is error tolerance and stop iteration when satisfied)

Finite Element Method

Generalized Eigenvalue problem

Problem: Find the lowest eigenvalue and eigenvector of the system with following mass and stiffness matrices.

$$\mathbf{K}=\begin{bmatrix}3 & -2 & 0 & 0\\ -2 & 3 & -1 & 0\\ 0 & -1 & 3 & -2\\ 0 & 0 & -2 & 2\end{bmatrix},\quad \mathbf{M}=\begin{bmatrix}3 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}$$

Solution: Start with our first guess eigenvector, u_1=[1 1 1 1]^T.

$\mathbf{M}u_1$=[3 3 3 2]^T.

$$\mathbf{K}w_2=\mathbf{M}u_1 \rightarrow \begin{bmatrix}3 & -2 & 0 & 0\\ -2 & 3 & -1 & 0\\ 0 & -1 & 3 & -2\\ 0 & 0 & -2 & 2\end{bmatrix} w_2=\begin{bmatrix}3\\3\\3\\2\end{bmatrix}$$

Using Thomas algorithm, we get w_2=[11 15 20 21]^T. $w_2^T\mathbf{M}w_2$=3120.

So, $u_2=w_2/\sqrt{(3120)}$=[0.1969 0.2685 0.3580 0.3760]^T.

$\lambda_2=u_2^T\mathbf{K}u_2$=0.05768

$\mathbf{M}u_2$=[0.5907 0.8055 1.0740 0.7520]^T

Solve $\mathbf{K}w_3=\mathbf{M}u_2$, we get w_3=[3.2224 4.5381 6.3641 6.7401]^T

$w_3^T\mathbf{M}w_3$=305.30, and we get $u_3=w_3/\sqrt{(305.3)}$=[0.1844 0.2597 0.3642 0.3857]^T.

$\lambda_3=u_3^T\mathbf{K}u_3$=0.05719

So, after 2 iterations, we get

λ_3=0.05719, u_3=[0.1844 0.2597 0.3642 0.3857]^T.

The exact eigensolution is:

$\boldsymbol{\lambda_1}$**=0.05719096**, λ_2=0.42264973

λ_3=1.57735027, λ_4=1.94280904

$$[\phi]=\begin{bmatrix}\mathbf{0.182574} & 0.365148 & 0.365148 & -0.182574\\ \mathbf{0.258198} & 0.316227 & -0.316227 & 0.258198\\ \mathbf{0.365148} & -0.182574 & -0.182574 & -0.365148\\ \mathbf{0.387298} & -0.316227 & 0.316227 & 0.387298\end{bmatrix}$$

The convergence rate is **very fast** since $\boldsymbol{\lambda_1}$ is **far smaller** than λ_2 (and other eigenvalues).

Finite Element Method

Generalized Eigenvalue problem

Forward Iteration Method

Find the **largest** eigenvalue (with eigenvector) for **KU**=λ**MU**

The procedure are:

1.Choose a guess starting vector u_1,

2.Solve w_{k+1} from: $\mathbf{M}w_{k+1}=\mathbf{K}u_k$,

3.Normalize w_{k+1} as: $u_{k+1} = w_{k+1} \Big/ \sqrt{w_{k+1}^T \mathbf{M} w_{k+1}}$

4.Compute λ_{k+1} using: $\lambda_{k+1}=u_{k+1}^T\mathbf{K}u_{k+1}$,

5.Check the convergence: $|\lambda_{k+1}-\lambda_k|/\ \lambda_{k+1} \le \varepsilon$. ($\varepsilon$ is error tolerance and stop iteration when satisfied)

Problem: Find the largest eigenvalue and eigenvector of the system with following mass and stiffness matrices.

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Start with our first guess eigenvector, $u_1=[1\ 1\ 1\ 1]^T$.

$\mathbf{K}u_1=[2\ {-1}\ {-1}\ 2]^T$.

$$\mathbf{M}w_2 = \mathbf{K}u_1 \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

So, we get $w_2=[1\ {-0.5}\ {-1}\ 2]^T$. $w_2^T\mathbf{M}w_2=7.5$

So, $u_2=w_2/\sqrt{(7.5)}=[0.3651\ \ {-0.1826}\ \ {-0.3651}\ \ 0.7303]^T$.

$\lambda_2=u_2^T\mathbf{K}u_2=5.9333$

Finite Element Method

Generalized Eigenvalue problem

$\mathbf{K}u_2$=[2.1909 -0.3652 -4.0166 4.9295]^T
Solve $\mathbf{M}w_3=\mathbf{K}u_2$, we get w_3=[1.0955 -0.1826 -4.0166 4.9295]^T
$w_3^T\mathbf{M}w_3$=42.900, and we get $u_3=w_3/\sqrt{(42.9)}$=[0.1673 -0.0279 -0.0557 0.1115]^T.
$\lambda_3=u_3^T\mathbf{K}u_3$=8.5788
u_4=[0.0184 0.1306 -0.7068 0.6823]^T
$\lambda_4=u_4^T\mathbf{K}u_4$=10.1597

$\Downarrow$

u_{11}=**[-0.1073 0.2554 -0.7283 0.5623]^T**
$\lambda_{11}=u_{11}^T\mathbf{K}u_{11}$=**10.6384**

The exact eigensolution is:
λ_1=0.09653732, λ_2=1.39146545
λ_3=4.37354955, $\boldsymbol{\lambda_4}$=**10.63844766**

$$[\phi]=\begin{bmatrix} 0.31262952 & 0.44526615 & -0.43866985 & -\mathbf{0.1075620} \\ 0.49547585 & 0.12443600 & 0.41674029 & \mathbf{0.25563036} \\ 0.47911662 & -0.4894418 & 0.02322175 & -\mathbf{0.72825457} \\ 0.28979330 & -0.5770218 & -0.51696549 & \mathbf{0.56197181} \end{bmatrix}$$

The convergence rate is **not fast** since $\boldsymbol{\lambda_4}$ is **not far bigger** than λ_3 (and other eigenvalues).

Problem: Find the largest/smallest eigenpairs of the system with following mass and stiffness matrices (in 3 D.P.).

$$\mathbf{K}=\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{M}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

The exact eigensolution is:
$\lambda_1=-0.1679$, λ_2=1.1675
λ_3=10.20

$$\phi=\begin{bmatrix} -0.8706 & 1.000 & 0.0022 \\ 0.5084 & 0.0838 & 0.0100 \\ -1.000 & -0.1897 & 1.000 \end{bmatrix}$$

Some notes

How to create Generalized Eigenvalue problem?

$$\boxed{\mathbf{KU}=\lambda\mathbf{MU}}$$

The above properties are hold if **K** is symmetric, **M** is symmetric and positive definite. Using Cholesky factorization, the generalized eigenproblem can be written $\mathbf{KU}=\lambda\mathbf{LL}^T\mathbf{U} \rightarrow \mathbf{L}^{-1}\mathbf{KU}=\lambda\mathbf{L}^T\mathbf{U} \rightarrow \mathbf{Cy}=\lambda\mathbf{y}$ (with $\mathbf{y}=\mathbf{L}^T\mathbf{U}$, $\mathbf{C}=\mathbf{L}^{-1}\mathbf{KL}^{-T}$, sym.) So, with given example of **C** (give simple eigenpairs), we can get **K** with $\mathbf{K}=\mathbf{LCL}^T$. We also get **M** as $\mathbf{M}=\mathbf{LL}^T$.

Problem: Solve the generalized eigenproblem using Cholesky factorization.

$$\mathbf{KU}=\lambda\mathbf{MU} \quad \rightarrow \quad \begin{bmatrix} 1 & 5 \\ 5 & 25 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Answer: eigenvalues are $(\lambda_1,\lambda_2)=(0,5)$, $[u_1,u_2]=[(1,-0.2)^T, (0,-1)^T]$.

$$\mathbf{L}=\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad , \quad \mathbf{C}=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

ξ=xi, ϕ=phi, η=eta, δ=delta, ε=epsilon, ψ=psi.