Finite Element Method
MSM 1333

Chapter 3.1

Element & Shape function

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Finite Element Method

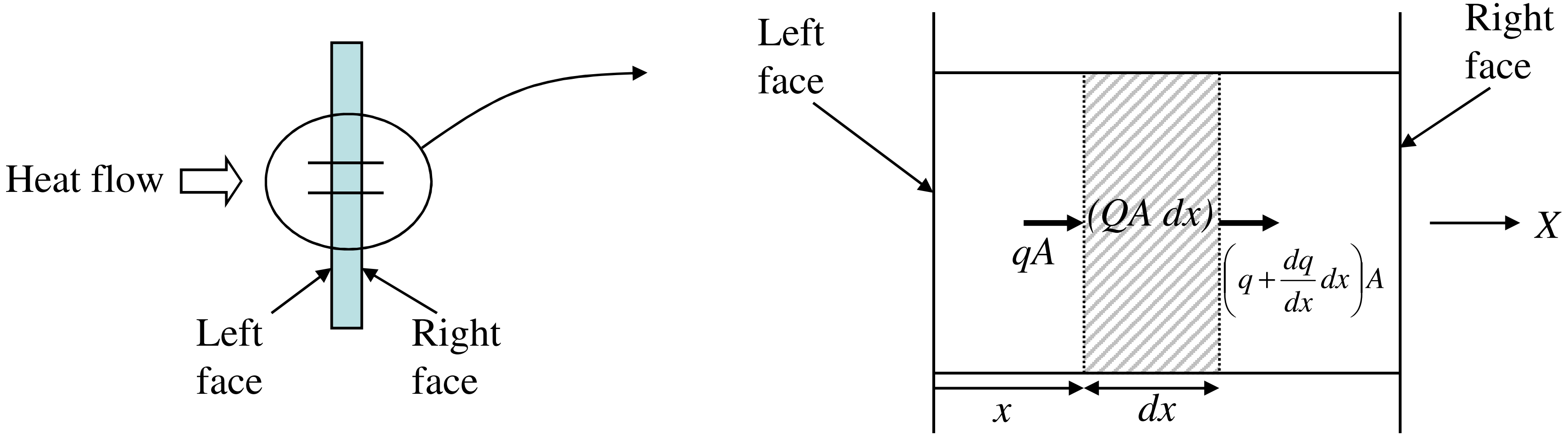
Common 1D problems

Problems	Strong form	Constitutive law (flux)
Heat distribution	$\frac{d}{dx}\left(A(x)k(x)\frac{dT}{dx}\right)+Q(x)=0$	Fourier's law: $q=-k\frac{dT}{dx}$
Diffusion problem	$\frac{d}{dx}\left(A(x)D(x)\frac{dc}{dx}\right)+Q(x)=0$	Fick's law: $q=-D\frac{dc}{dx}$
Axially loaded elastic bar	$\frac{d}{dx}\left(A(x)E(x)\frac{du}{dx}\right)+Q(x)=0$	Hooke's law: $\sigma=E\frac{du}{dx}$

Finite Element Method

Steady-state 1-D heat conduction

Governing equation (heat conduction in plane wall with uniform heat generation)



Let A =area normal to direction of heat flow,

Q(W/m^3) =internal heat generated per unit volume.

Heat rate (heat flux × area) enter the control volume **+** heat rate generated **=** Heat rate leaving control volume.

$$qA + QAdx = \left(q + \frac{dq}{dx}dx\right)A \quad \xrightarrow{\text{simplify}} \quad Q = \frac{dq}{dx}$$

$$q = -k\frac{small - big}{dx} = +ve$$

+ve=heat flux same direction with x-axis

Substitute Fourier's law $q = -k\dfrac{dT}{dx}$ $\Longrightarrow$ $\dfrac{d}{dx}\left(k\dfrac{dT}{dx}\right) + Q = 0$

Q is called source when +ve (heat is generated) and is called sink when –ve (heat is consumed)

Here, Q is referred as source.

Finite Element Method

Steady-state 2-D heat conduction

The heat flow through the wall of a heated room on a winter day is an example of conduction. In a thermally isotropic medium, Fourier's law for 2-D heat flow is:

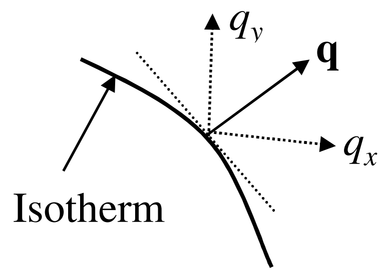
Index, not partial derivative!

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}$$

$T=T(x,y)$=temperature, q_x and q_y are components of heat flux (W/m^2), k is thermal conductivity (W/m. °C). (1W=1J/s=1Nm/s). Minus sign: heat is transferred in direction of decreasing temperature. k is material property.

$\mathbf{q}=q_x\mathbf{i}+q_y\mathbf{j}$, resultant heat flux (at right angles to an isotherm or a line of constant temperature.

$\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ are temperature gradients along x and y.



Constitutive relation- contains a material property.

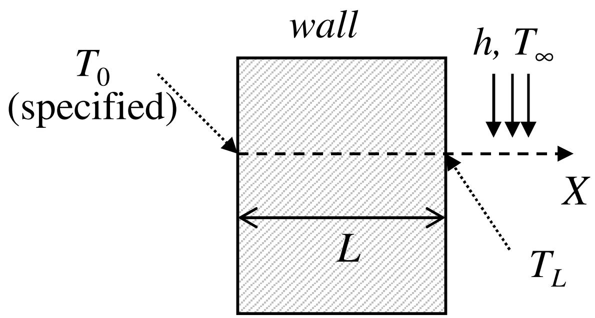
Convection-the flow of heat through a gas or a liquid

$q=h(T_s-T_\infty)$, q is convective heat flux (W/m^2), h is convection heat transfer coefficient or film coef (W/m^2·°C), T_s and T_∞ are surface and fluid temperature.

Finite Element Method

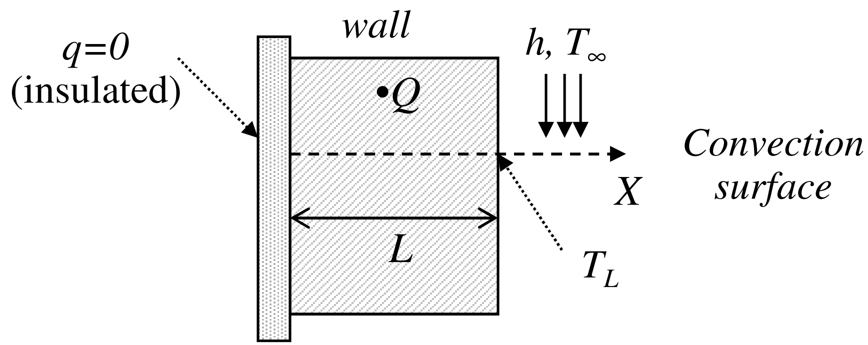
Steady-state 1-D heat conduction, Boundary conditions

Specified temperature



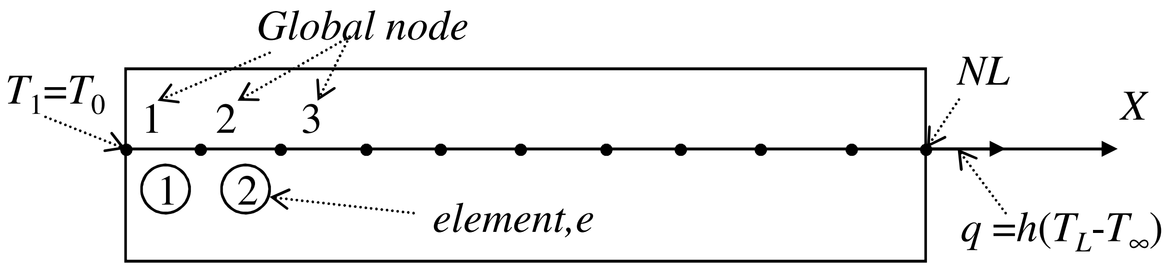
Wall of tank contain hot liquid at T_0,
airstream of T_∞ passed on outside,
maintain T_L at boundary.
$T|_{x=0} = T_0$, $q|_{x=L} = h(T_L - T_\infty)$. [note: $T_L > T_\infty$]

Specified heat flux

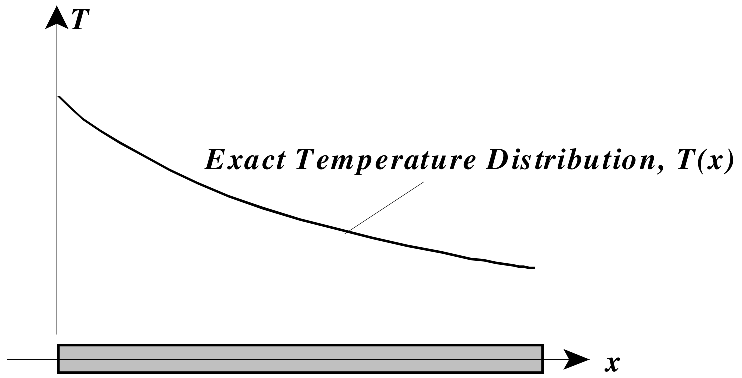


A wall where the inside surface is insulated
And outside is convection surface.
$q|_{x=0} = 0$, $q|_{x=L} = h(T_L - T_\infty)$.

1-D element : two-node element with linear shape functions

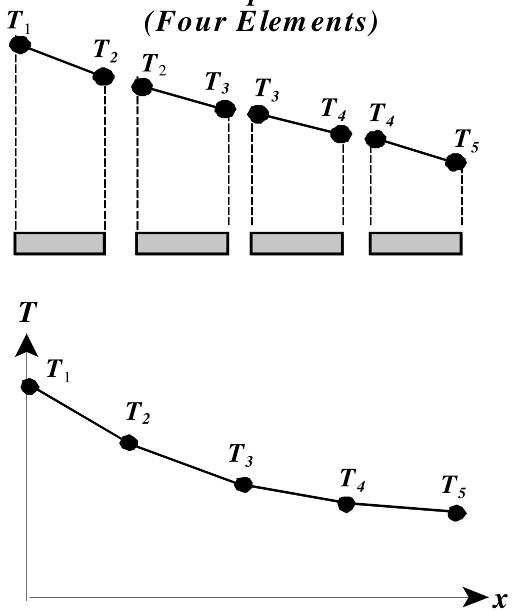


Discretization Concepts



Finite Element Discretization

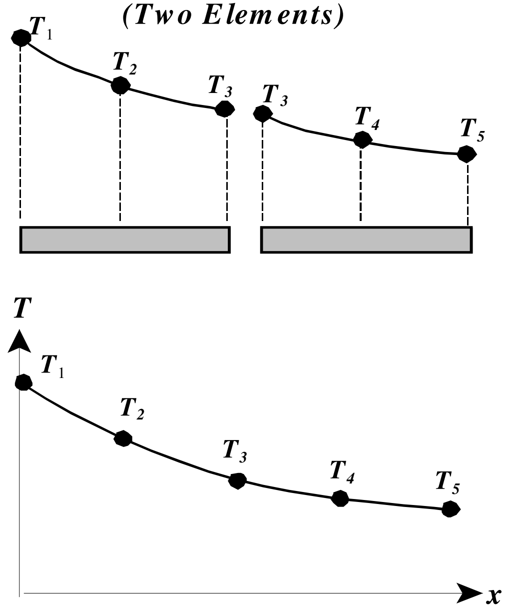
Linear Interpolation Model (Four Elements)



Piecewise Linear Approximation

Temperature Continuous but with Discontinuous Temperature Gradients

Quadratic Interpolation Model (Two Elements)

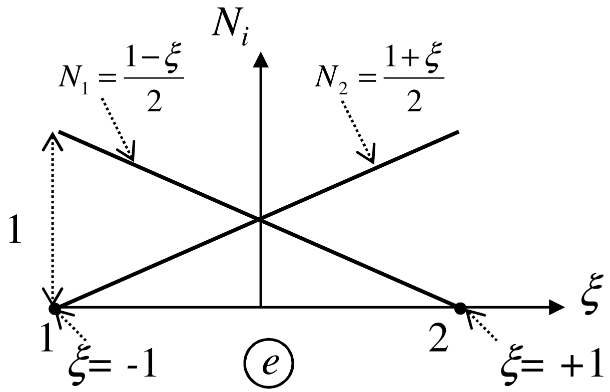


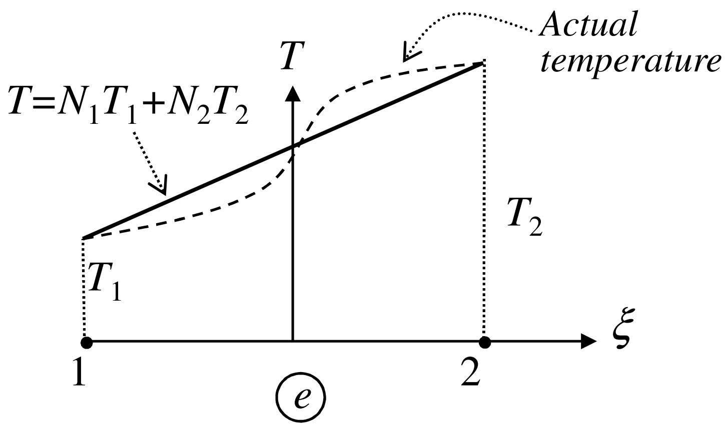
Piecewise Quadratic Approximation

Temperature and Temperature Gradients Continuous

Finite Element Method

1-D element





$$T(\xi)=N_1T_1+N_2T_2 = \mathbf{N}\mathbf{T}^e$$

where N_1=(1-ξ)/2, N_2=(1+ ξ)/2, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{T}^e=[T_1, T_2]^T$.

Please note $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1,\ d\xi = \frac{2}{x_2 - x_1} dx = \frac{2}{l_e} dx.$

$$x=N_1x_1+N_2x_2$$

$$x = \frac{(1-\xi)}{2} x_1 + \frac{(1+\xi)}{2} x_2$$

Use chain rule, $\frac{dT}{dx} = \frac{dT}{d\xi} \cdot \frac{d\xi}{dx} = \frac{2}{x_2 - x_1} \frac{d\mathbf{N}}{d\xi} \cdot \mathbf{T}^e = \frac{1}{x_2 - x_1}[-1,1]\mathbf{T}^e = \mathbf{B}_T \mathbf{T}^e.$

$$\int_e f dx = \int_{-1}^{1} f\ J d\xi\ ,\quad J = \frac{l_e}{2} = Jacobian$$

where $\mathbf{B}_T = \frac{d}{dx}\mathbf{N} = \frac{1}{x_2 - x_1}[-1, 1] = \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}$

$$N_1 = \frac{x - x_2}{x_1 - x_2},\ \rightarrow N_1 = \frac{\xi - 1}{-1-1}$$

Lagrange interpolation $P(x) = \sum_1^N L_i(x) f_i,\ L_i(x) = \Pi_{\substack{j=0\\ j\neq i}}^{N} \frac{(x - x_j)}{(x_i - x_j)}$

$$N_2 = \frac{x - x_1}{x_2 - x_1},\ \rightarrow N_2 = \frac{\xi - (-1)}{1-(-1)}$$

Finite Element Method

1-D element

Lagrange's interpolation

$$P(x)=\sum_{1}^{N} L_i(x) f_i,\; L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{N} \frac{(x-x_j)}{(x_i-x_j)}$$

$$N_1(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

Now we require that our function $u(x)$ be approximated locally by the quadratic function

$$u(\xi)=c_1+c_2\xi+c_3\xi^2$$

Our node points are defined at $\xi_{1,2,3}=-1,0,1$ and we require that

Vandermonde system

$$u_1=c_1-c_2+c_3$$
$$u_2=c_1$$
$$u_3=c_1+c_2+c_3$$

$$\begin{bmatrix} u_1\\ u_2\\ u_3 \end{bmatrix}=\begin{bmatrix} 1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}$$

$\Rightarrow$ $\mathbf{c}=\mathbf{Au}$

$$\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0\\ -0.5 & 0 & 0.5\\ 0.5 & -1 & 0.5 \end{bmatrix}\begin{bmatrix} u_1\\ u_2\\ u_3 \end{bmatrix}$$

$$u(\xi)=(u_2)+(-0.5u_1+0.5u_3)\xi+(0.5u_1-u_2+0.5u_3)\xi^2$$
$$=(-0.5\xi+0.5\xi^2)u_1+(1-\xi^2)u_2+(0.5\xi+0.5\xi^2)u_3$$

The temperature field within the element is written in terms of the nodal temperature as

$$\boxed{T(\xi)=N_1T_1+N_2T_2+N_3T_3=\mathbf{NT}^e}$$

Where $N_1(\xi)=-\frac{1}{2}\xi(1-\xi)$, $N_2(\xi)=(1+\xi)(1-\xi)$, $N_3(\xi)=\frac{1}{2}\xi(1+\xi)$, ξ varies from -1 to $+1$, $\mathbf{N}=[N_1, N_2, N_3]$, $\mathbf{T}^e=[T_1, T_2, T_3]^T$.

Finite Element Method 1-D element

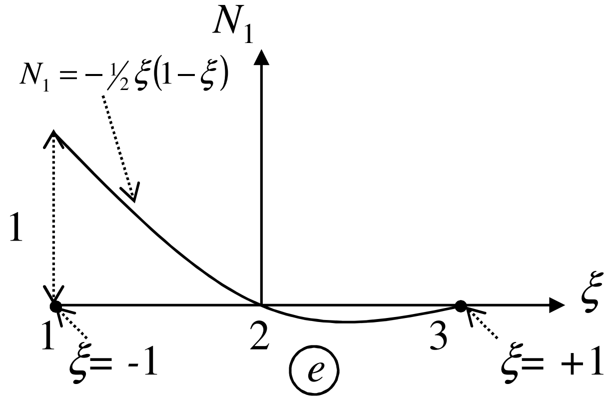
The temperature field within the element is written in terms of the nodal temperature as

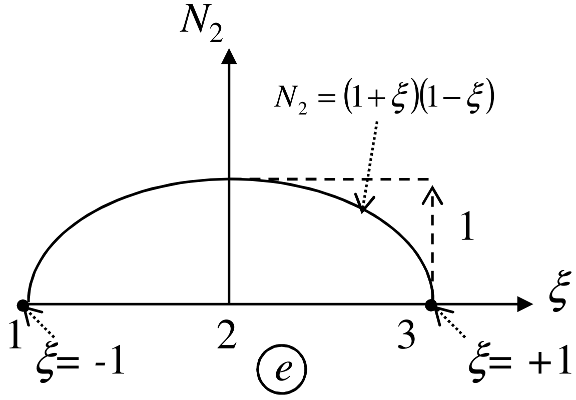
$$P(x)=\sum_{1}^{N} L_i(x) f_i,\; L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{N}\frac{(x-x_j)}{(x_i-x_j)}$$

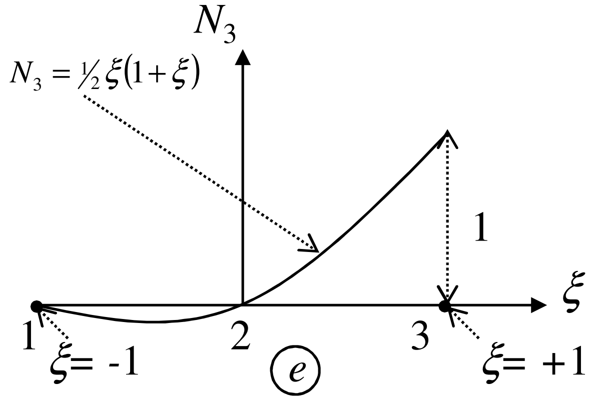
$$\boxed{T(\xi)=N_1T_1+N_2T_2+N_3T_3=\mathbf{N}\mathbf{T}^e}$$ Lagrange's interpolation

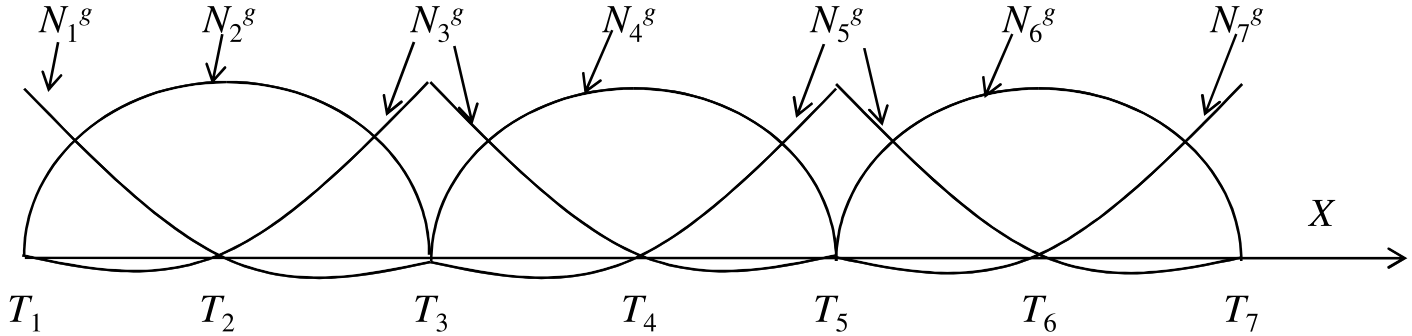
$$N_1(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

Where $N_1(\xi)= -½\xi(1-\xi)$, $N_2(\xi)=(1+\xi)(1-\xi)$, $N_3(\xi)=½\xi(1+\xi)$, ξ varies from – 1 to +1, $\mathbf{N}=[N_1, N_2, N_3]$, $\mathbf{T}^e=[T_1, T_2, T_3]^T$.









Please note

$$\xi=\frac{2(x-x_2)}{x_3-x_1},\; d\xi=\frac{2}{x_3-x_1}dx=\frac{2}{l_e}dx.$$

$$\int_{-1}^{1}\mathbf{N}^T d\xi=\int_{-1}^{1}\begin{bmatrix}-½\xi(1-\xi)\\(1+\xi)(1-\xi)\\½\xi(1+\xi)\end{bmatrix}d\xi=\begin{bmatrix}1/3\\4/3\\1/3\end{bmatrix}$$

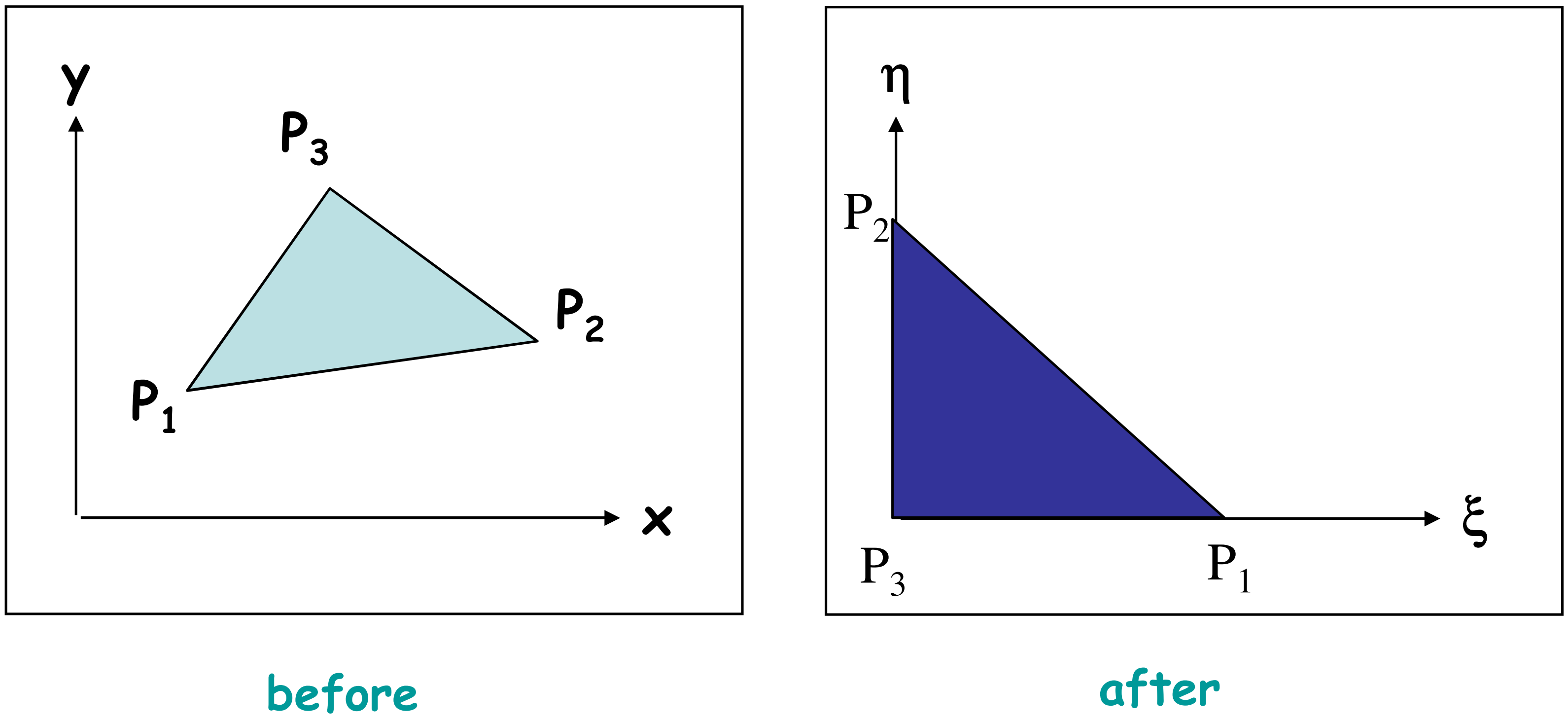
Use chain rule, $$\frac{dT}{dx}=\frac{dT}{d\xi}\cdot\frac{d\xi}{dx}=\frac{2}{x_3-x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e=\frac{2}{x_3-x_1}[-{}^{1-2\xi}\!/_2,-2\xi,{}^{1+2\xi}\!/_2]\mathbf{T}^e=\mathbf{B}_T\mathbf{T}^e.$$

$$\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{2}{3l_e^2}\begin{bmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{bmatrix}$$

$$\mathbf{B}_T=\frac{d}{dx}\mathbf{N}=\frac{2}{x_3-x_1}[-{}^{1-2\xi}\!/_2,-2\xi,{}^{1+2\xi}\!/_2]=\frac{2}{l_e}[-{}^{1-2\xi}\!/_2,-2\xi,{}^{1+2\xi}\!/_2]$$

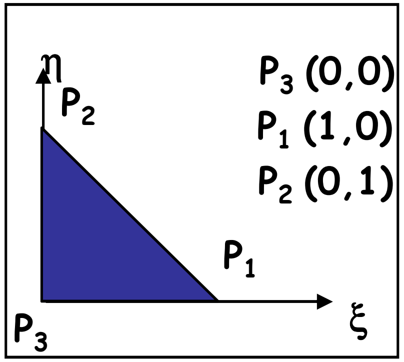
Finite Element Method

2-D element – triangle element



Finite Element Method

2-D element – triangle element



Any triangle with corners $P_i(x_i,y_i)$, i=1,2,3 can be transformed into a rectangular, equilateral triangle with

$$x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta$$

$$y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta$$

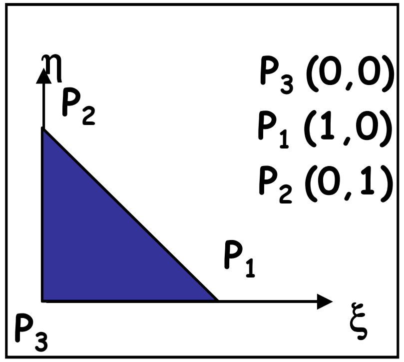
using counterclockwise numbering. Note that if η=0, then these equations are equivalent to the 1-D tranformations. We seek to approximate a function by the linear form

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta$$

we proceed in the same way as in the 1-D case

Finite Element Method

2-D element – triangle element



... and we obtain

$$u_3 = u(0,0) = c_1$$
$$u_1 = u(1,0) = c_1 + c_2$$
$$u_2 = u(0,1) = c_1 + c_3$$

Vandermonde system

... and we obtain the coefficients as a function of the values at the grid nodes by matrix inversion

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

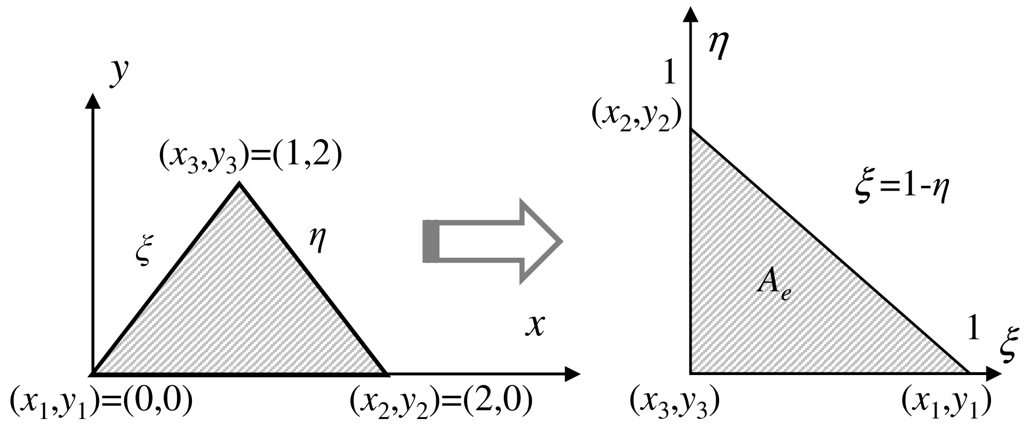
$$\mathbf{c} = \mathbf{A}\mathbf{u}$$

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta = (u_3) + (u_1 - u_3)\xi + (u_2 - u_3)\eta$$
$$= (\xi)u_1 + (\eta)u_2 + (1-\xi-\eta)u_3 = N_1u_1 + N_2u_2 + N_3u_3$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Finite Element Method

2-D element – triangle element - linear



Triangular element:

The temperature field within an element is

$T=N_1T_1+N_2T_2+N_3T_3$

or $T=\mathbf{N}\mathbf{T}^e$

where $\mathbf{N}=[\xi, \eta, 1-\xi-\eta]$ are element shape function

$\mathbf{T}^e=[T_1,T_2,T_3]^T$.

We have

$x=N_1x_1+N_2x_2+N_3x_3=\xi x_1+\eta x_2+(1-\xi-\eta)x_3$

$y=N_1y_1+N_2y_2+N_3y_3=\xi y_1+\eta y_2+(1-\xi-\eta)y_3$

Using chain rule, we get

or

where

$x_{ij}=x_i-x_j$, $y_{ij}=y_i-y_j$, and $|\det \mathbf{J}|=2A_e$,

where A_e is the area of the triangular.

Finite Element Method

2-D element – triangle element - quadratic

Any function defined on a triangle can be approximated by the quadratic function

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

and in the transformed system we obtain

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi^2 + c_5\xi\eta + c_6\eta^2$$

To determine the coefficients we calculate the function u at each grid point to obtain

$$u_1 = c_1 + c_2 + c_4$$

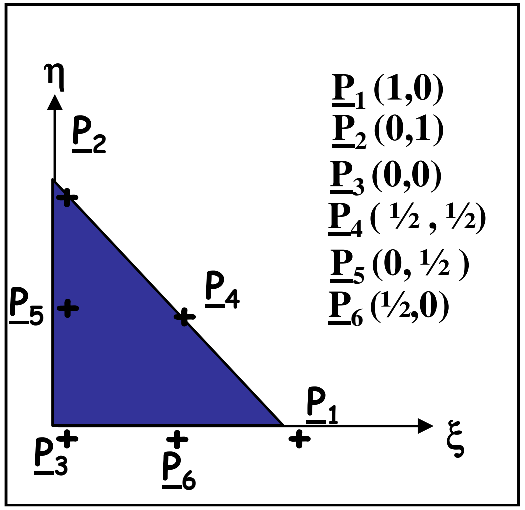
$$u_2 = c_1 + c_3 + c_6$$

$$u_3 = c_1$$

$$u_4 = c_1 + \tfrac{1}{2}c_2 + \tfrac{1}{2}c_3 + \tfrac{1}{4}c_4 + \tfrac{1}{4}c_5 + \tfrac{1}{4}c_6$$

$$u_5 = c_1 + \tfrac{1}{2}c_3 + \tfrac{1}{4}c_6$$

$$u_6 = c_1 + \tfrac{1}{2}c_2 + \tfrac{1}{4}c_4$$

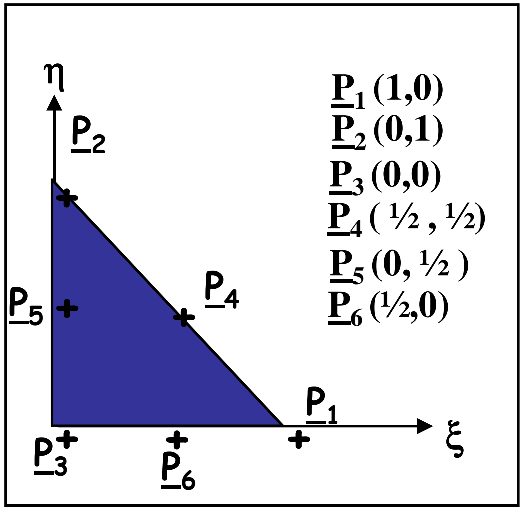


... and by matrix inversion we can calculate the coefficients as a function of the values at P_i

$$\mathbf{A}^{-1}\mathbf{c} = \mathbf{u} \quad \rightarrow \quad \mathbf{c} = \mathbf{A}\mathbf{u}$$

Finite Element Method

2-D element – triangle element - quadratic



c = Au

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 & 0 & 4 \\ 0 & -1 & -3 & 0 & 4 & 0 \\ 2 & 0 & 2 & 0 & 0 & -4 \\ 0 & 0 & 4 & 4 & -4 & -4 \\ 0 & 2 & 2 & 0 & -4 & 0 \end{bmatrix}$$

Vandermonde system

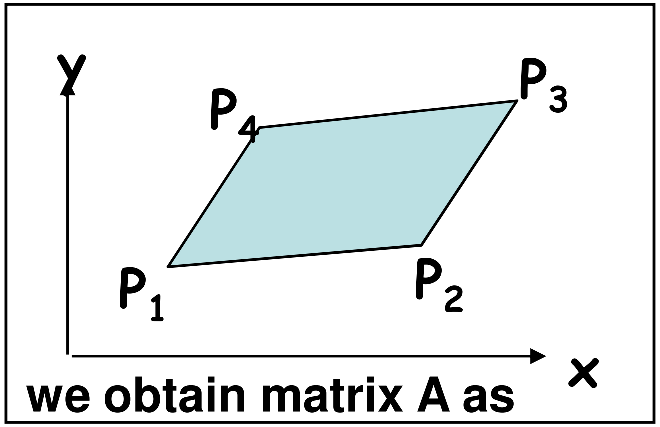
... to obtain the basis functions

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi^2 + c_5\xi\eta + c_6\eta^2$$
$$= u_3 + (-u_1 - 3u_3 + 4u_6)\xi + (-u_2 - 3u_3 + 4u_5)\eta$$
$$+ (2u_1 + 2u_3 - 4u_6)\xi^2 + (4u_3 + 4u_4 - 4u_5 - 4u_6)\xi\eta$$
$$+ (2u_2 + 2u_3 - 4u_5)\eta^2.$$

[N_1=$\xi(2\xi-1)$, N_2=$\eta(2\eta-1)$, N_3=$\tau(2\tau-1)$,
N_4=$4\xi\eta$, N_5=$4\tau\eta$, N_6=$4\xi\tau$] are

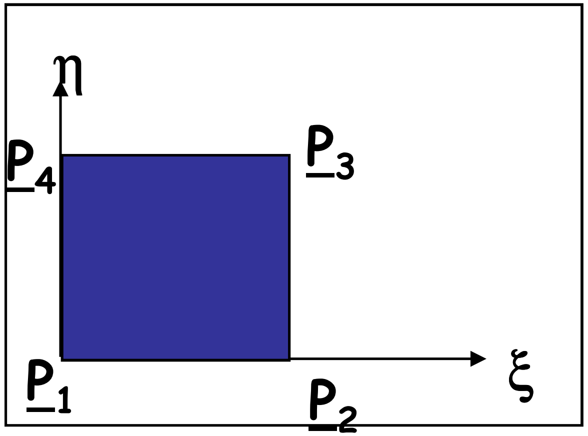
Finite Element Method

2-D element – rectangle element - linear



we obtain matrix A as

before



after

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta$$

$$N_1(\xi,\eta) = (1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \xi(1-\eta)$$
$$N_3(\xi,\eta) = \xi\eta$$
$$N_4(\xi,\eta) = (1-\xi)\eta$$

and the basis functions

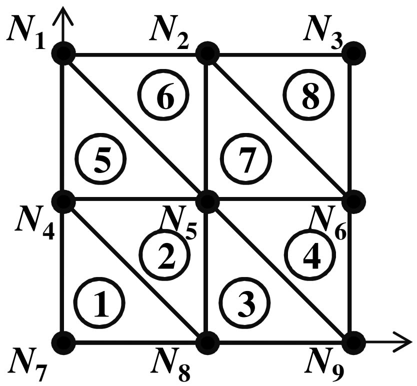
Finite Element Method

Galerkin's approach for heat conduction

Preprocessing

Preprocessing of the problem includes one or more of the following tasks:

- Read geometry and material data (E), and boundary and initial conditions of the problem.
- Mesh generation.
- Generation of node numbers.
- Generation of coordinates and connectivity.



element	1	2	3	← local
1	7	8	4	
2	8	5	4	
3	8	9	5	↑
4	9	6	5	Global
5	4	5	1	
6	5	2	1	↓
7	5	6	2	
8	6	3	2	

Linear triangular element

Processing of FEM

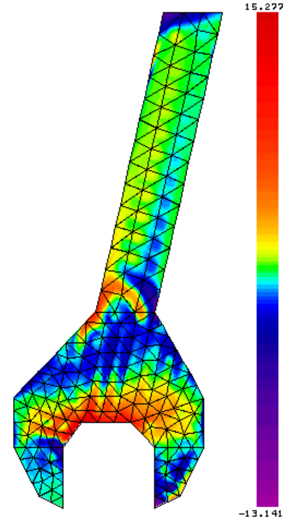
Processing of the FEM includes one or more of the following tasks:

- Calculate element matrices.
- Assemble element equations.
- Solve the system of equations.

Finite Element Method

Galerkin's approach for heat conduction

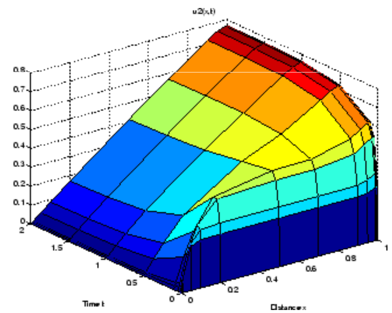
Postprocessing



Contour plot for stress

Postprocessing of the FEM includes one or more of the following tasks:

- Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
- Interpretation of the results to check whether the solution makes sense (based on physical Process and experience when other solutions are not available.
- Tabular and/or graphical presentation of the results. Contour plotting uses $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$



Contour plot for $u_2(x,t)$

Interpolation of temperature within each element is given

$$T(\xi) = N_1 T_1 + N_2 T_2 = \mathbf{N}\mathbf{T}^e$$

where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{T}^e=[T_1, T_2]^T$.

The derivative of the solution is obtained by differentiation

Use chain rule, $$\frac{dT}{dx} = \frac{dT}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e = \frac{1}{l_e}[-1,1]\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e.$$

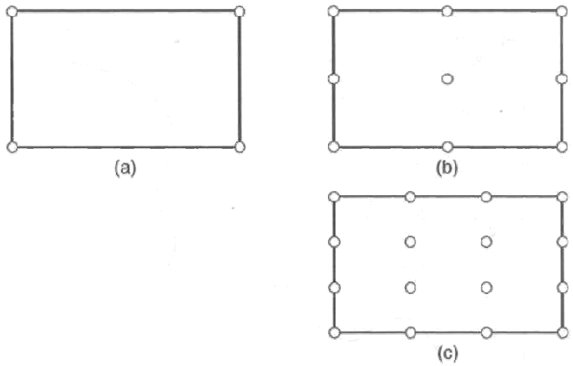
For element 1, we get $$\frac{dT^{e=1}}{dx} = \mathbf{B}_T\mathbf{T}^{e=1} = \frac{1}{l_e}[-1,1]\mathbf{T}^{e=1} = \frac{1}{0.3}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}304.6\\119.0\end{bmatrix} = -618.67$$

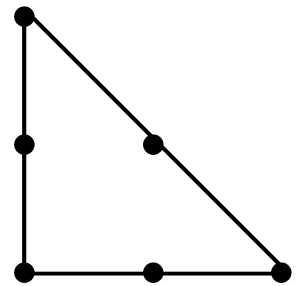
For element 2, we get $$\frac{dT^{e=2}}{dx} = \mathbf{B}_T\mathbf{T}^{e=2} = \frac{1}{l_e}[-1,1]\mathbf{T}^{e=2} = \frac{1}{0.15}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}119.0\\57.1\end{bmatrix} = -412.67$$

Note that the derivative above is discontinuous, for any order element, at the nodes connecting the different elements because the continuity of the derivative of FE solution at the connecting nodes is not imposed.

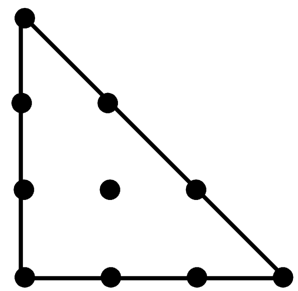
Lagrangian Elements:

- Order *n* element has $(n+1)^2$ nodes arranged in square-symmetric pattern – requires <u>internal nodes</u>.



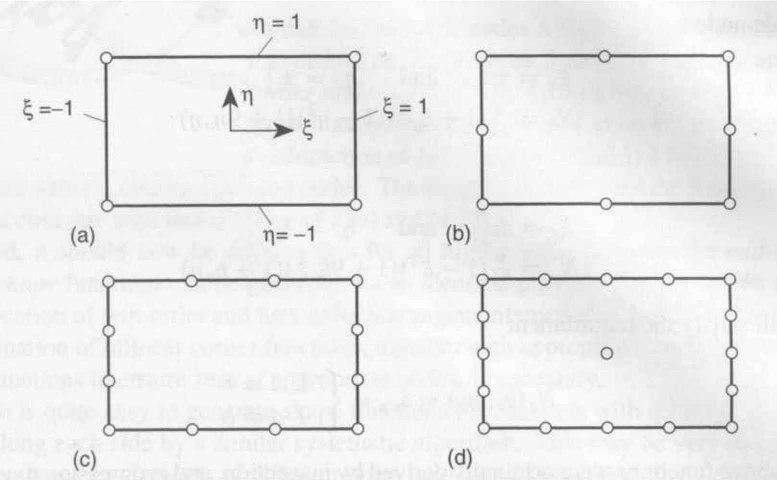


- Shape functions are products of *n*th order polynomials in each direction. (“biquadratic”, “bicubic”, …)
- Bilinear quad is a Lagrangian element of order $n = 1$.



Serendipity Elements:

- In general, only boundary nodes – avoids internal ones.

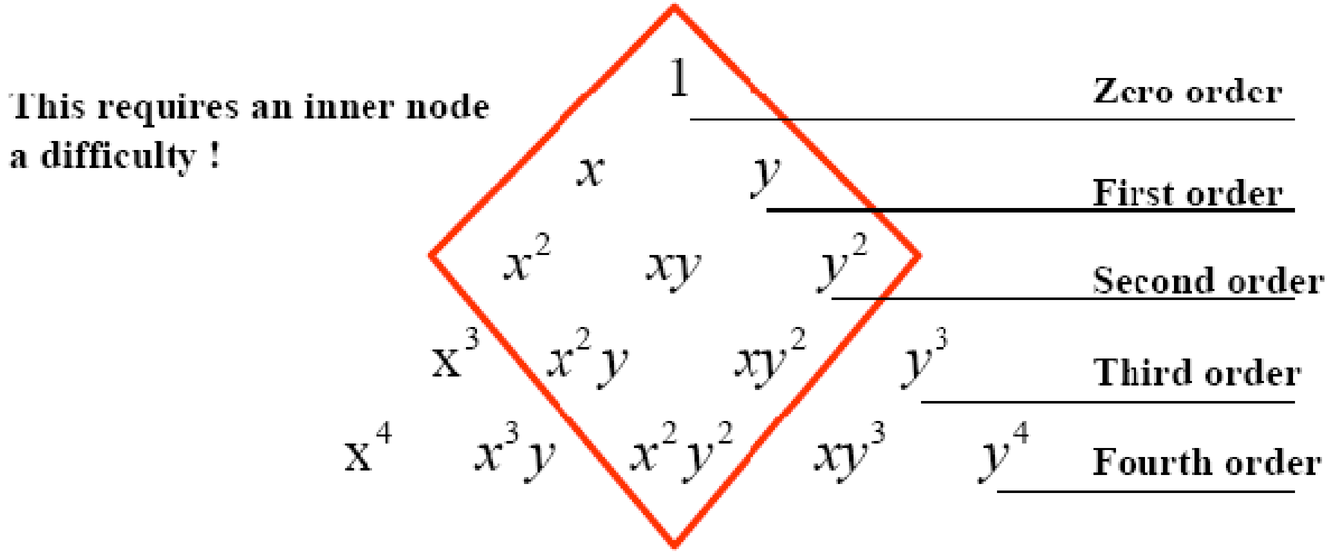


Lagrange polynomials – complete poly expansion
Serendipity poly – incomplete polynomial expansion
Hermitian poly – Polynomials including derivative

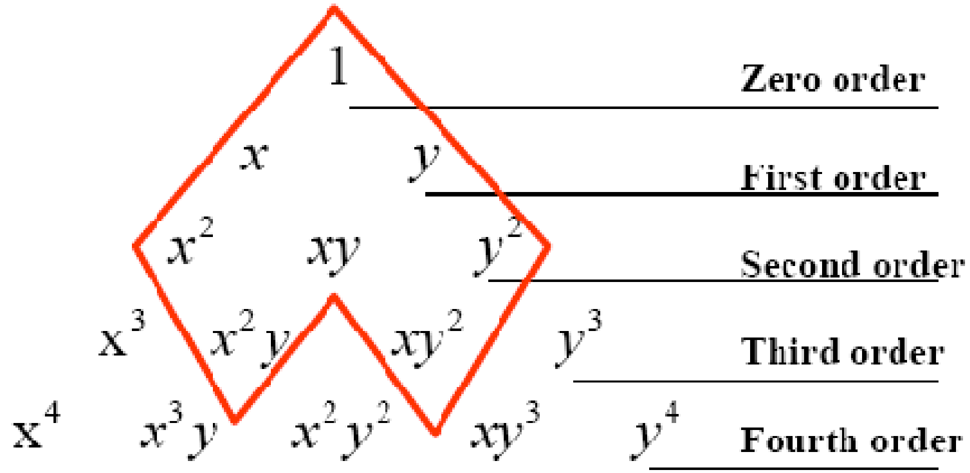
- Not as accurate as Lagrangian elements.
- However, more efficient than Lagrangian elements and avoids certain types of instabilities.

Finite Element Method

Lagrange Element

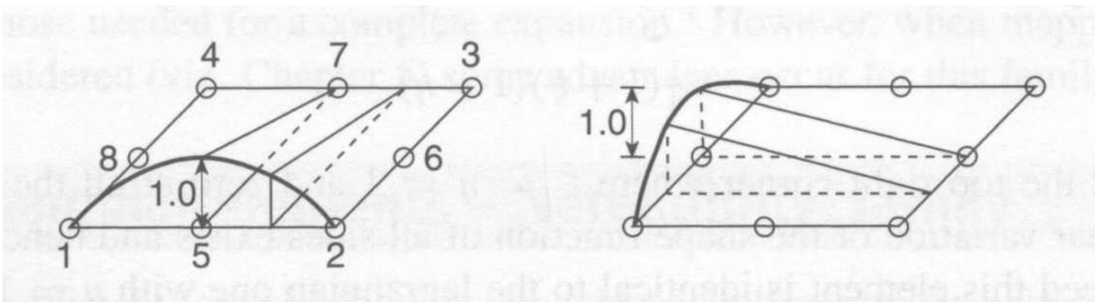


Serendipity shape functions are constructed by incomplete polynomials – avoiding inner nodes



Serendipity Shape Functions:

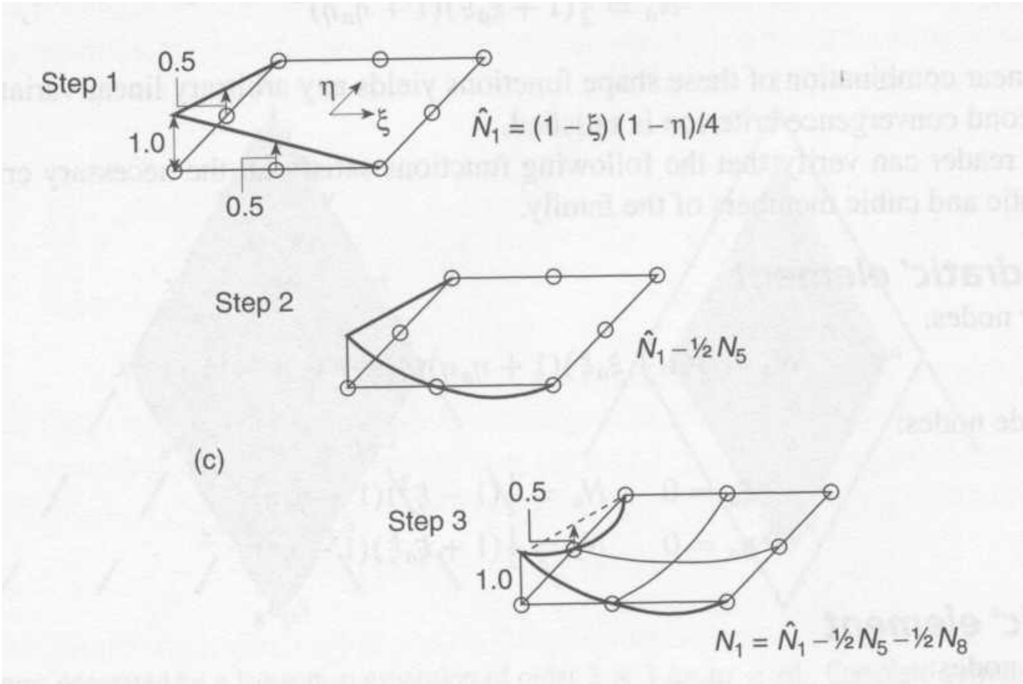
- Shape functions for **mid-side nodes** are products of an *n*th order polynomial parallel to side and a linear function perpendicular to the side.
  - E.g., quadratic serendipity element:



(a) $N_5 = ½(1-\xi^2)(1-\eta)$ (b) $N_8 = ½(1-\xi)(1-\eta^2)$

$$N_6 = \tfrac{1}{2}(1+\xi)(1-\eta^2); N_7 = \tfrac{1}{2}(1-\xi^2)(1+\eta).$$

- Shape functions for **corner nodes** are modifications of the shape functions of the bilinear quad.
  - Step #1: start with appropriate bilinear quad shape function, .
  - Step #2: subtract out mid-side shape function N_5 with appropriate weight
  - Step #3: repeat Step #2 using mid-side shape function N_8 and weight



$$N_k = \tfrac{1}{4}\left(1+\xi_k\xi\right)\left(1+\eta_k\eta\right)\left(\xi_k\xi+\eta_k\eta-1\right); k=1,2,3,4.$$