Finite Element Method

MSM 1333

Chapter 3.2

spring & truss

Prepared by:

Dr. Yeak Su Hoe
C22-432, U.T.M. Skudai
s.h.yeak@utm.my
012-7116604

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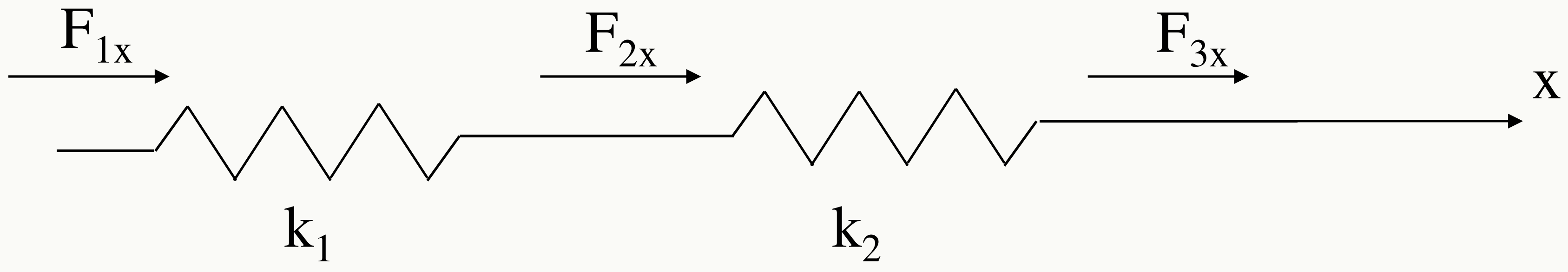
Spring theory



Problem

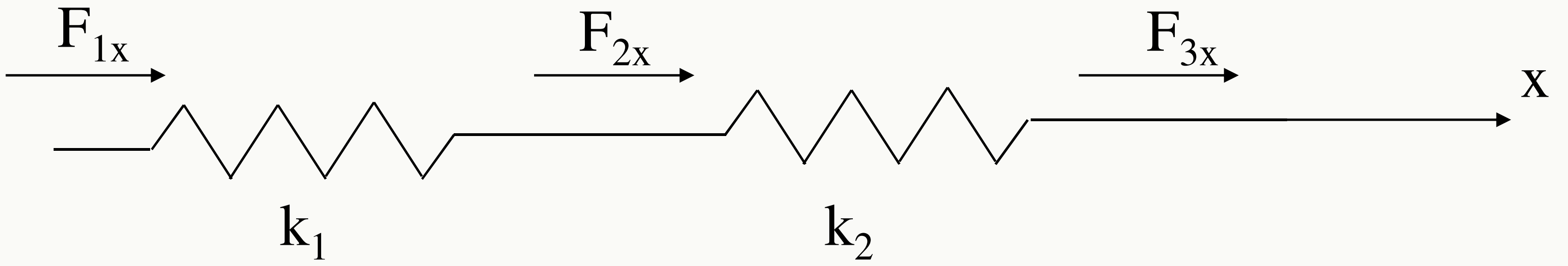
Analyze the behavior of the system composed of the two springs loaded by external forces as shown above

Given

F_{1x} , F_{2x} ,F_{3x} are external loads. Positive directions of the forces are along the positive x-axis

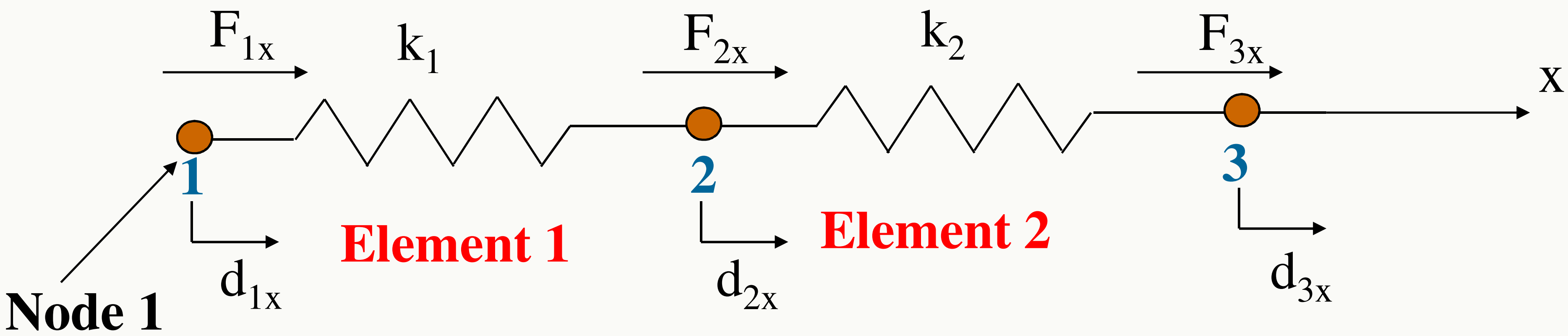
k_1 and k_2 are the stiffnesses of the two springs

Spring theory



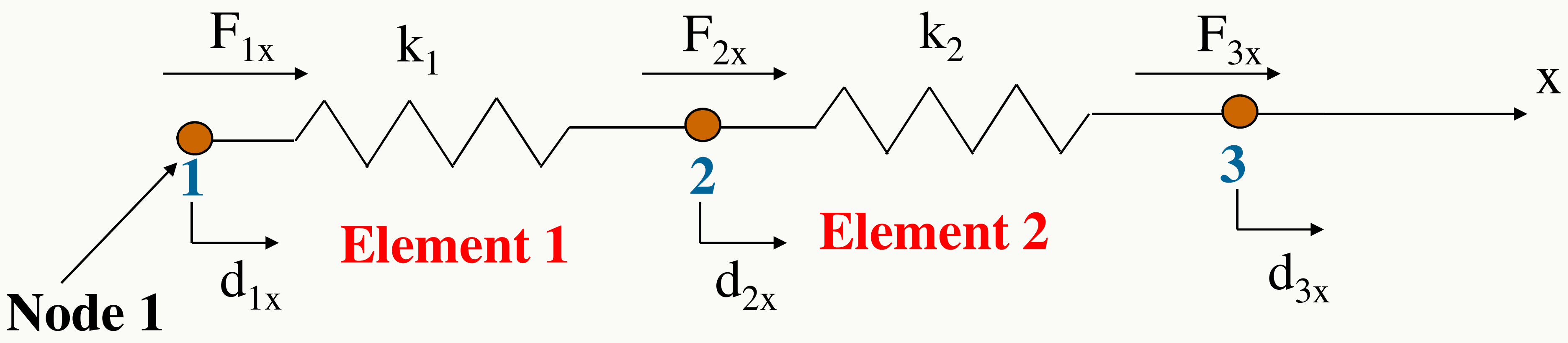
Solution

Step 1: In order to analyze the system we break it up into smaller parts, i.e., “elements” connected to each other through “nodes”



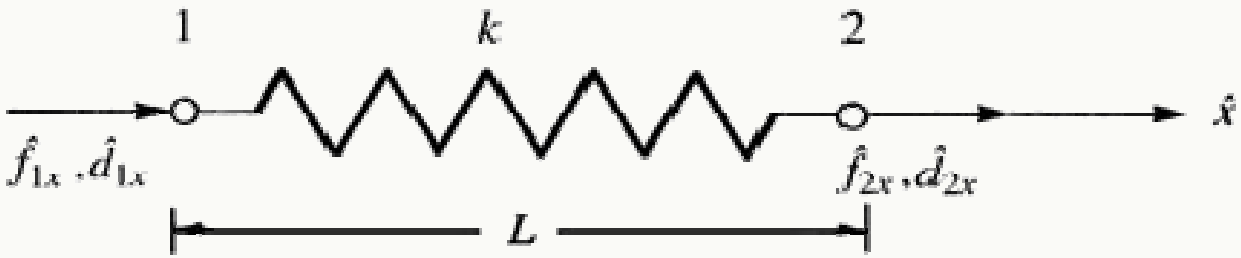
Unknowns: nodal displacements d_{1x}, d_{2x}, d_{3x},

Spring theory



Solution

Step 2: Analyze the behavior of a single element (spring)



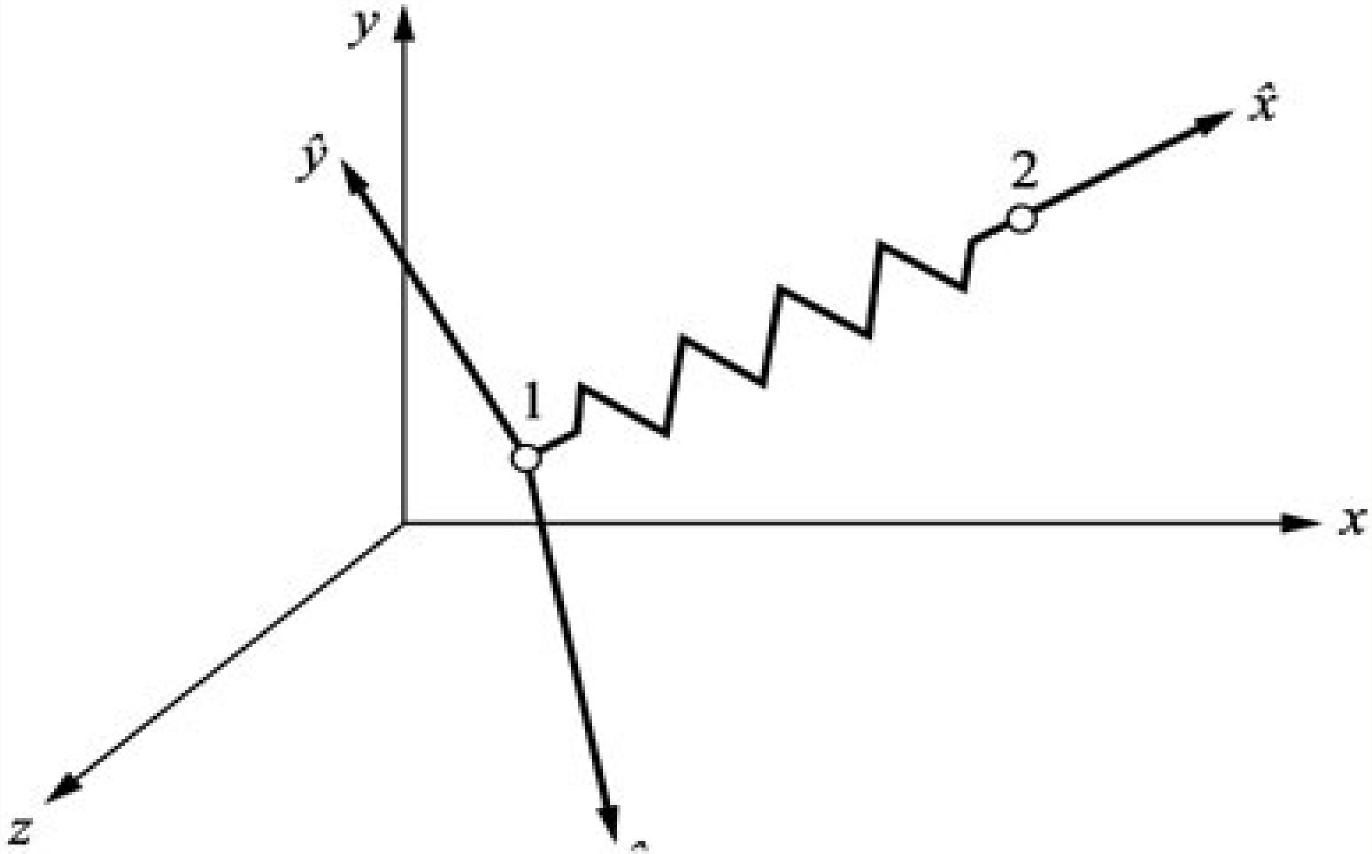
Two nodes: 1, 2

Nodal displacements: $\hat{d}_{1x}$ $\hat{d}_{2x}$

Nodal forces: $\hat{f}_{1x}$ $\hat{f}_{2x}$

Spring constant: k

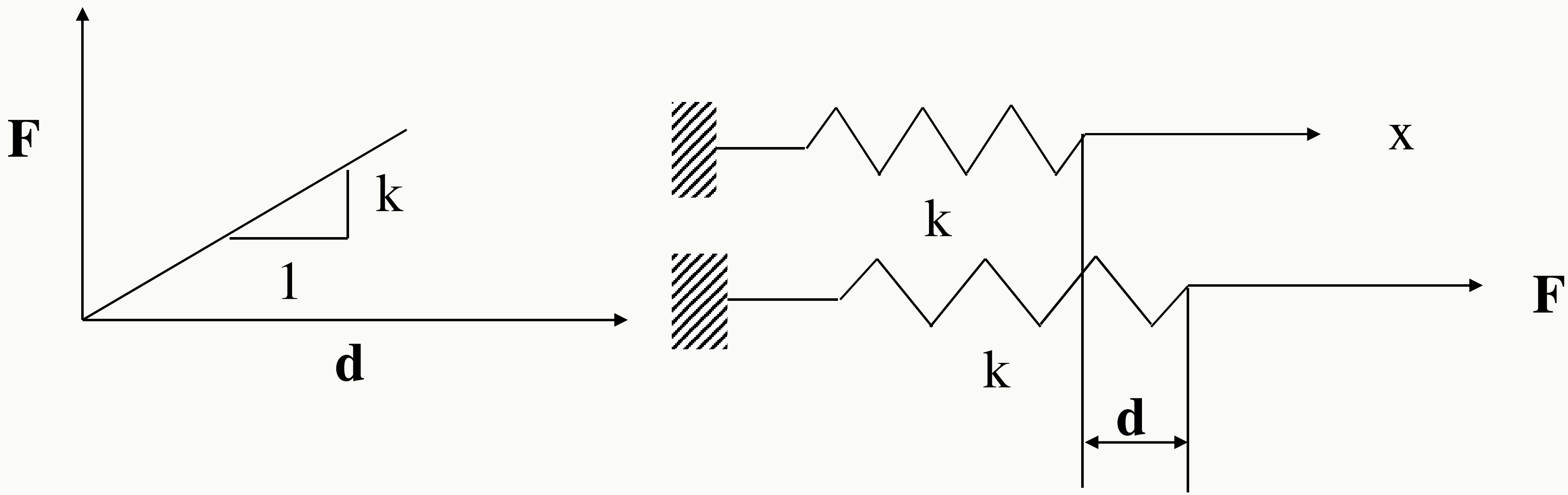
Spring theory



Local ($\hat{x}$, $\hat{y}$, $\hat{z}$) and global (x,y,z) coordinate systems

Spring theory

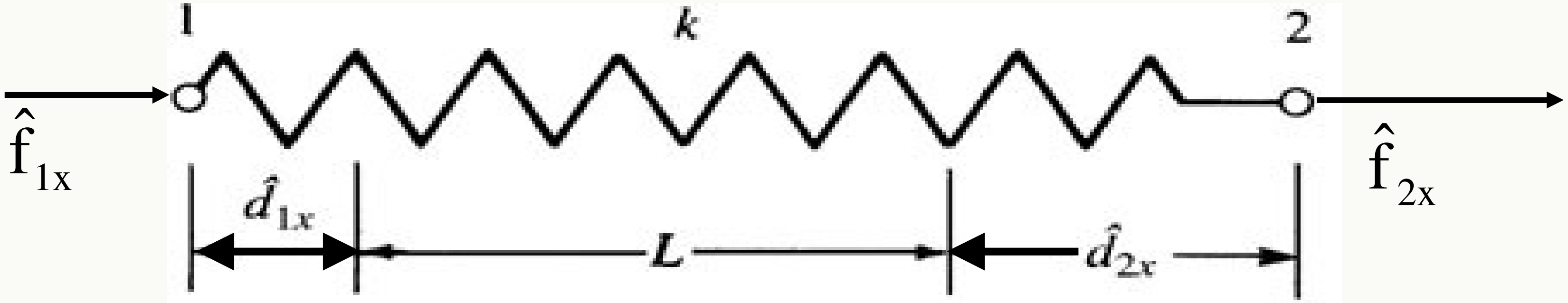
Behavior of a linear spring (recap)



F = Force in the spring
d = deflection of the spring
k = "stiffness" of the spring

Hooke's Law
F = kd

Spring theory



Hooke's law for our spring element

$$\hat{f}_{2x} = k\,(\hat{d}_{2x} - \hat{d}_{1x}) \qquad \textbf{Eq (1)}$$

Force equilibrium for our spring element (recap free body diagrams)

$$\hat{f}_{1x} + \hat{f}_{2x} = 0$$

$$\Rightarrow \hat{f}_{1x} = -\hat{f}_{2x} = -k\,(\hat{d}_{2x} - \hat{d}_{1x}) \qquad \textbf{Eq (2)}$$

Collect Eq (1) and (2) in matrix form

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}}\ \underline{\hat{d}}}$$

Element force vector

Element **stiffness matrix**

Element nodal displacement vector

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\underline{\hat{k}}} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\underline{\hat{d}}}$$

Note

1. The element stiffness matrix is "**symmetric**", i.e. $\underline{\hat{k}}^T = \underline{\hat{k}}$
2. The element stiffness matrix is **singular**, i.e.,

$$\det(\underline{\hat{k}}) = k^2 - k^2 = 0$$

The consequence is that the matrix is NOT invertible. It is not possible to invert it to obtain the displacements. Why?

The spring is not constrained in space and hence it can attain multiple positions in space for the same nodal forces e.g.,

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

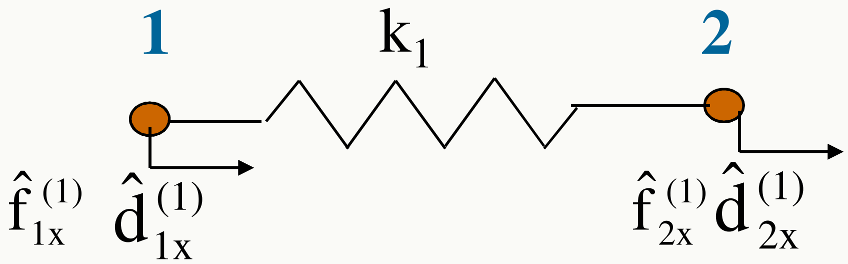
Spring theory

Solution

Step 3: Now that we have been able to describe the behavior of each spring element, lets try to obtain the behavior of the original structure by assembly

Split the original structure into component elements

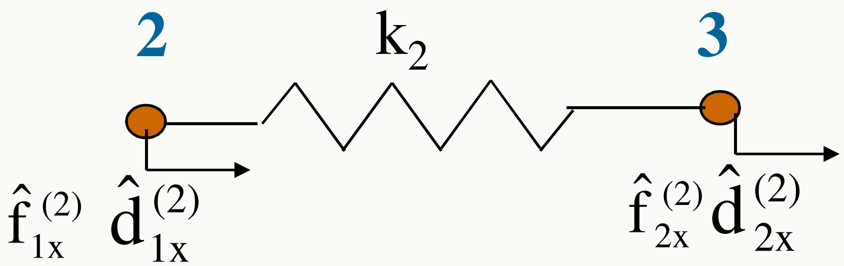
Element 1



$$\underbrace{\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix}}_{\underline{\hat{f}}^{(1)}} = \underbrace{\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}}_{\underline{\hat{k}}^{(1)}} \underbrace{\begin{Bmatrix} \hat{d}_{1x}^{(1)} \\ \hat{d}_{2x}^{(1)} \end{Bmatrix}}_{\underline{\hat{d}}^{(1)}}$$

Eq (3)

Element 2



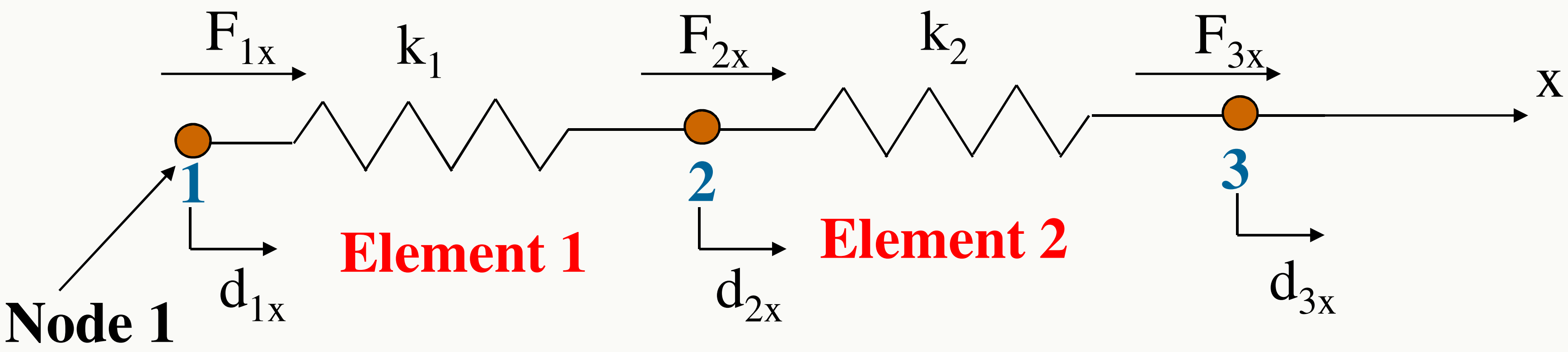
$$\underbrace{\begin{Bmatrix} \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}}_{\underline{\hat{f}}^{(2)}} = \underbrace{\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\underline{\hat{k}}^{(2)}} \underbrace{\begin{Bmatrix} \hat{d}_{1x}^{(2)} \\ \hat{d}_{2x}^{(2)} \end{Bmatrix}}_{\underline{\hat{d}}^{(2)}}$$

Eq (4)

Spring theory

To assemble these two results into a single description of the response of the entire structure we need to link between the **local** and **global** variables.

Question 1: How do we relate the **local** (element) **displacements** back to the **global** (structure) displacements?



$$\boxed{\begin{aligned} \hat{d}_{1x}^{(1)} &= d_{1x} \\ \hat{d}_{2x}^{(1)} &= \hat{d}_{1x}^{(2)} = d_{2x} \\ \hat{d}_{2x}^{(2)} &= d_{3x} \end{aligned}}$$

Eq (5)

Spring theory

Hence, equations (3) and (4) may be rewritten as

$$\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} \qquad \begin{Bmatrix} \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

Or, we may **expand** the matrices and vectors to obtain

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \\ 0 \end{Bmatrix}}_{\underline{\hat{f}}^{(1)e}} = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{\hat{k}}^{(1)e}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}}_{\underline{d}}$$

Eq (6)

$$\underbrace{\begin{Bmatrix} 0 \\ \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}}_{\underline{\hat{f}}^{(2)e}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\underline{\hat{k}}^{(2)e}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}}_{\underline{d}}$$

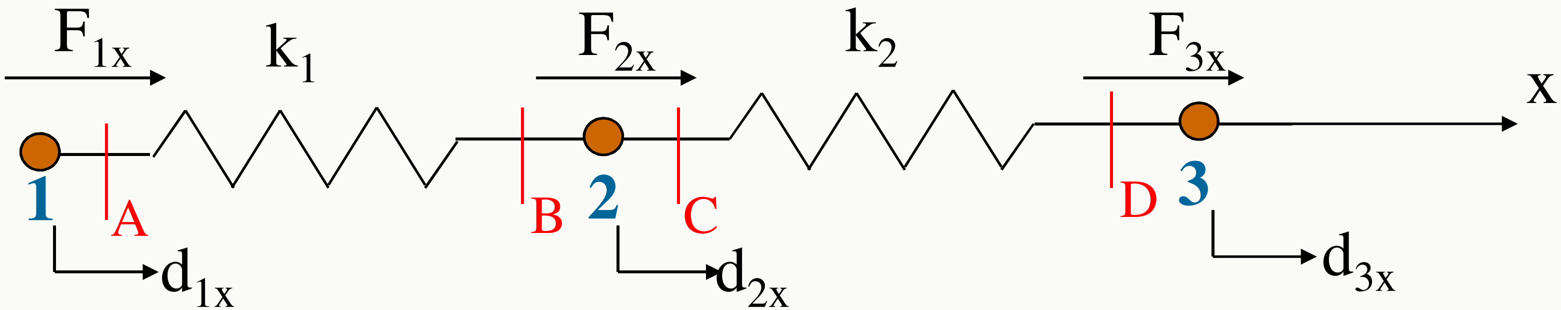
Eq (7)

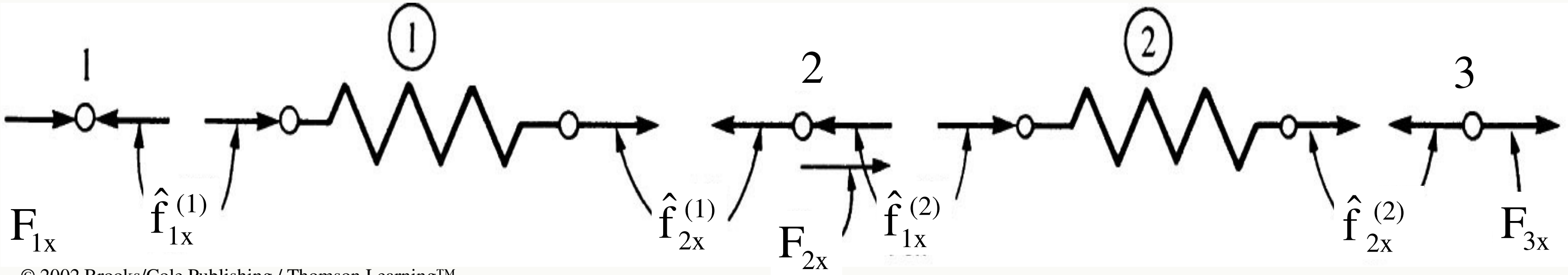
$\underline{\hat{k}}^{(1)e}$ Expanded element stiffness matrix of element 1 (local)

$\underline{\hat{f}}^{(1)e}$ Expanded nodal force vector for element 1 (local)

$\underline{d}$ Nodal load vector for the entire structure (global)

Question 2: How do we relate the **local** (element) **nodal forces** back to the **global** (structure) forces? Draw 5 FBDs





$$\text{At node 1:} \quad F_{1x} - \hat{f}_{1x}^{(1)} = 0$$

$$\text{At node 2:} \quad F_{2x} - \hat{f}_{2x}^{(1)} - \hat{f}_{1x}^{(2)} = 0$$

$$\text{At node 3:} \quad F_{3x} - \hat{f}_{2x}^{(2)} = 0$$

Spring theory

In vector form, the nodal force vector (global)

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} + \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}$$

Recall that the expanded element force vectors were

$$\underline{\hat{f}}^{(1)e} = \begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \\ 0 \end{Bmatrix} \quad \text{and} \quad \underline{\hat{f}}^{(2)e} = \begin{Bmatrix} 0 \\ \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}$$

Hence, the global force vector is simply the sum of the **expand** element nodal force vectors

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \underline{\hat{f}}^{(1)e} + \underline{\hat{f}}^{(2)e}$$

Spring theory

But we know the expressions for the expanded local force vectors from Eqs (6) and (7)

$$\underline{\hat{f}}^{(1)e} = \underline{\hat{k}}^{(1)e}\,\underline{d} \quad \text{and} \quad \underline{\hat{f}}^{(2)e} = \underline{\hat{k}}^{(2)e}\,\underline{d}$$

Hence

$$\underline{F} = \underline{\hat{f}}^{(1)e} + \underline{\hat{f}}^{(2)e} = \underline{\hat{k}}^{(1)e}\,\underline{d} + \underline{\hat{k}}^{(2)e}\,\underline{d} = \left(\underline{\hat{k}}^{(1)e} + \underline{\hat{k}}^{(2)e}\right)\underline{d}$$

$$\boxed{\underline{F} = \underline{K}\,\underline{d}}$$

$\underline{F} =$ Global nodal force vector

$\underline{d} =$ Global nodal displacement vector

$\underline{K} =$ Global stiffness matrix

$=$ sum of expanded element stiffness matrices

For our original structure with two springs, the **<u>global stiffness matrix</u>** is

$$\underline{\mathrm{K}} = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{\underline{k}}^{(1)e}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\hat{\underline{k}}^{(2)e}}$$

$$= \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

NOTE

1. The global stiffness matrix is **symmetric**
2. The global stiffness matrix is **singular**

Spring theory

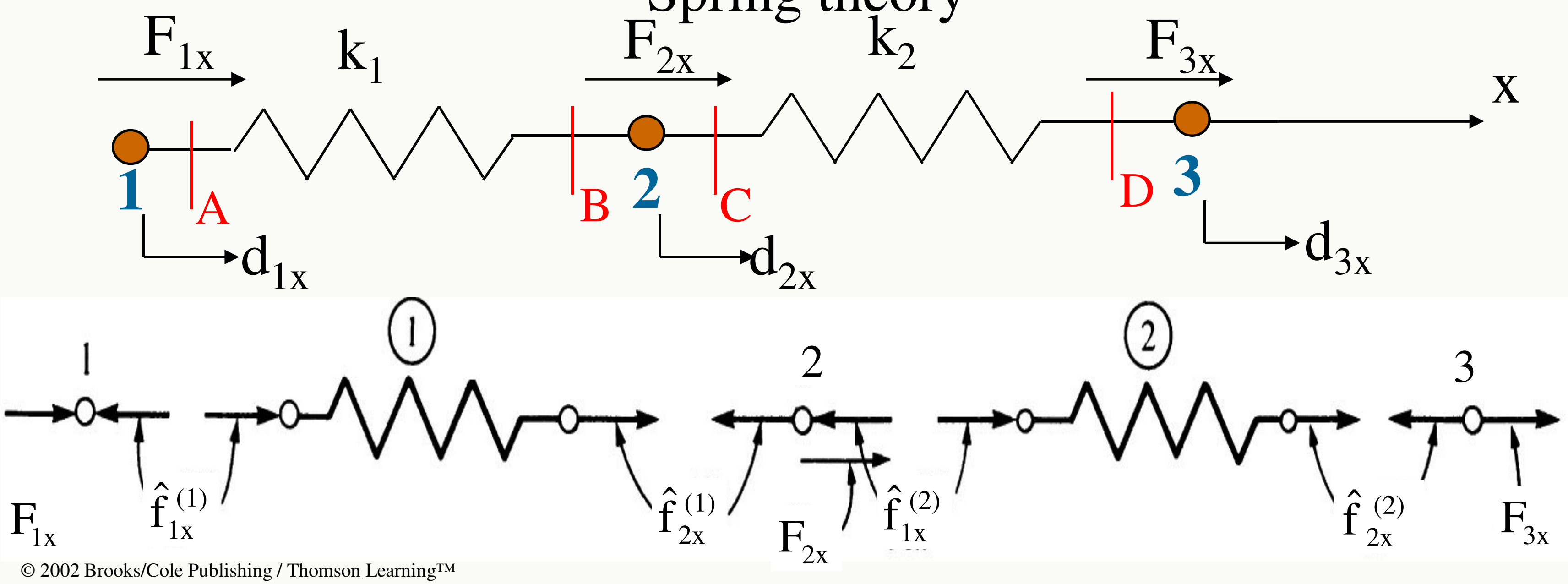
The system equations $\boxed{\underline{F} = \underline{K}\ \underline{d}}$ imply

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$\begin{aligned} & F_{1x} = k_1 d_{1x} - k_1 d_{2x} \\ \Rightarrow\ & F_{2x} = -k_1 d_{1x} + (k_1 + k_2) d_{2x} - k_2 d_{3x} \\ & F_{3x} = -k_2 d_{2x} + k_2 d_{3x} \end{aligned}$$

These are the 3 equilibrium equations at the 3 nodes.

Spring theory



At node 1: $F_{1x} - \hat{f}_{1x}^{(1)} = 0$

At node 2: $F_{2x} - \hat{f}_{2x}^{(1)} - \hat{f}_{1x}^{(2)} = 0$

At node 3: $F_{3x} - \hat{f}_{2x}^{(2)} = 0$

$$F_{1x} = k_1(d_{1x} - d_{2x}) = \hat{f}_{1x}^{(1)}$$

$$\begin{aligned} F_{2x} &= -k_1 d_{1x} + (k_1 + k_2) d_{2x} - k_2 d_{3x} \\ &= -k_1(d_{1x} - d_{2x}) + k_2(d_{2x} - d_{3x}) \\ &= \hat{f}_{2x}^{(1)} + \hat{f}_{1x}^{(2)} \end{aligned}$$

$$F_{3x} = -k_2(d_{2x} - d_{3x}) = \hat{f}_{2x}^{(2)}$$

Spring theory

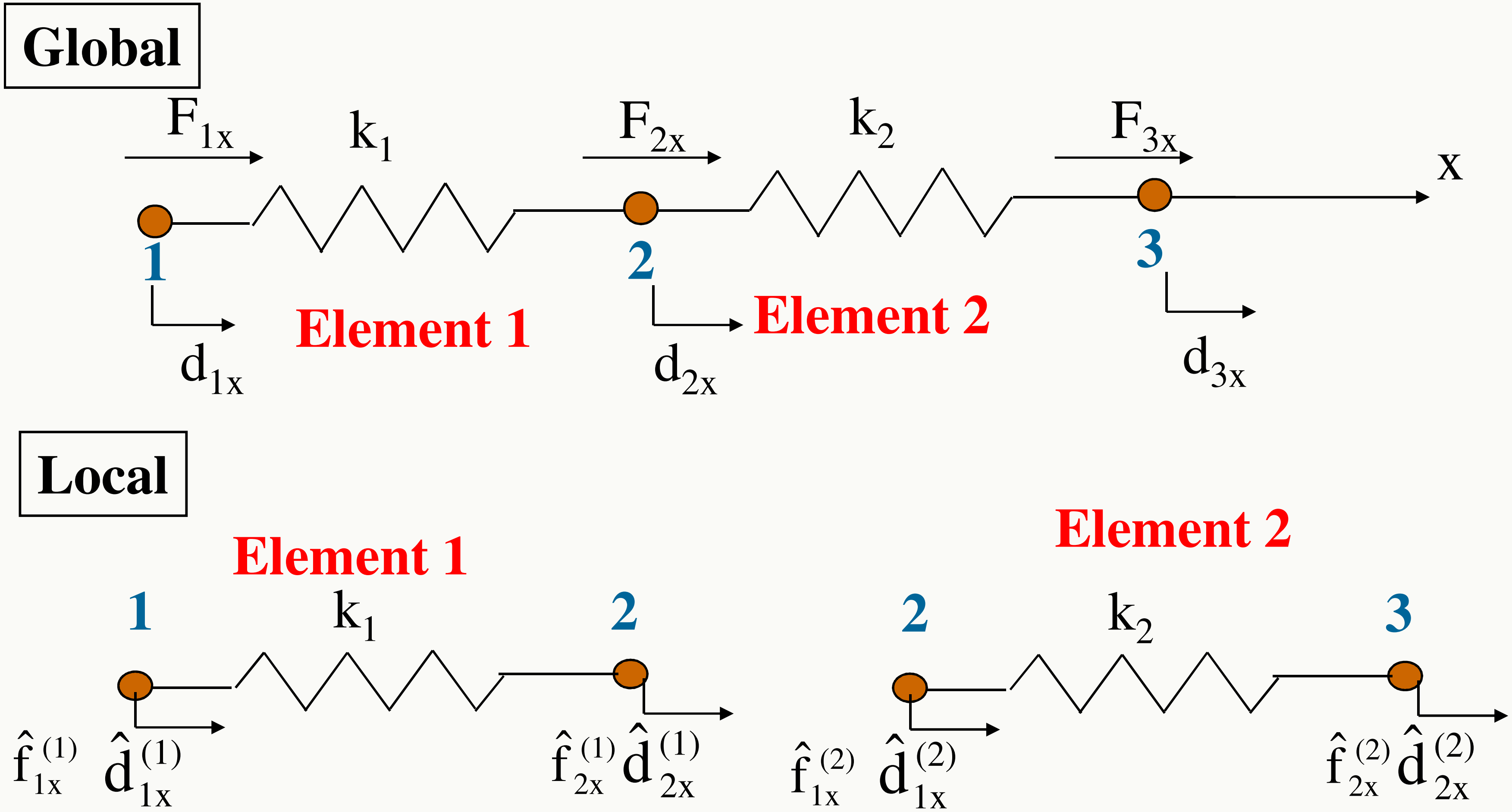
Notice that the sum of the forces equal zero, i.e., the structure is in static equilibrium.

$$F_{1x} + F_{2x} + F_{3x} = 0$$

Given the nodal forces, can we solve for the displacements?

To obtain unique values of the displacements, **at least one of the nodal displacements must be specified**.

Direct assembly of the global stiffness matrix



Node element connectivity chart : Specifies the global node number corresponding to the local (element) node numbers

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Local node number

Global node number

Stiffness matrix of element 1

$$\underline{\hat{k}}^{(1)} = \begin{array}{cc} \begin{array}{cc} d_{1x} & d_{2x} \end{array} & \\ \left[\begin{array}{c|c} k_1 & -k_1 \\ \hline -k_1 & k_1 \end{array}\right] & \begin{array}{c} d_{1x} \\ d_{2x} \end{array} \end{array}$$

Stiffness matrix of element 2

$$\underline{\hat{k}}^{(2)} = \begin{array}{cc} \begin{array}{cc} d_{2x} & d_{3x} \end{array} & \\ \left[\begin{array}{c|c} k_2 & -k_2 \\ \hline -k_2 & k_2 \end{array}\right] & \begin{array}{c} d_{2x} \\ d_{3x} \end{array} \end{array}$$

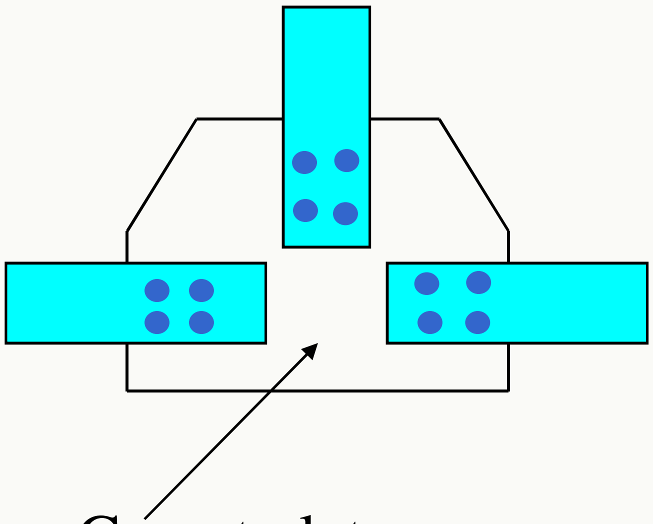
Global stiffness matrix

$$\underline{K} = \begin{array}{cc} \begin{array}{ccc} d_{1x} & d_{2x} & d_{3x} \end{array} & \\ \left[\begin{array}{c|c|c} k_1 & -k_1 & 0 \\ \hline -k_1 & k_1 + k_2 & -k_2 \\ \hline 0 & -k_2 & k_2 \end{array}\right] & \begin{array}{c} d_{1x} \\ d_{2x} \\ d_{3x} \end{array} \end{array}$$

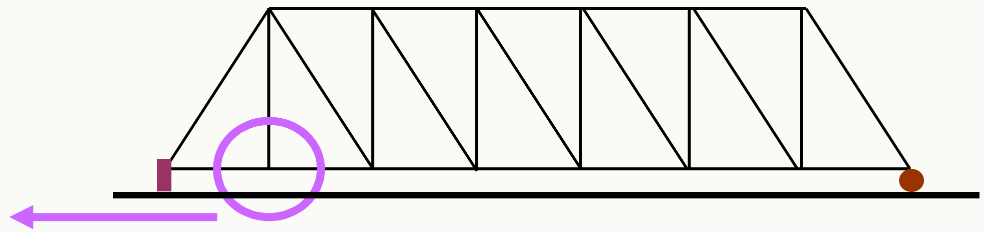
Examples: Problems 2.1 and 2.3 of Logan

Trusses: Engineering structures that are composed only of *two-force members. e.g., bridges, roof supports*

Actual trusses: Airy structures composed of slender members (I-beams, channels, angles, bars etc) joined together at their ends by welding, riveted connections or large bolts and pins



Gusset plate

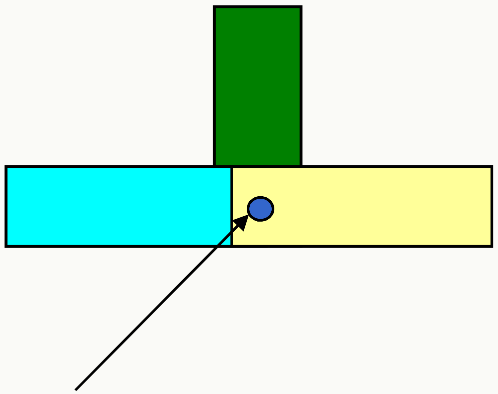


A typical truss structure

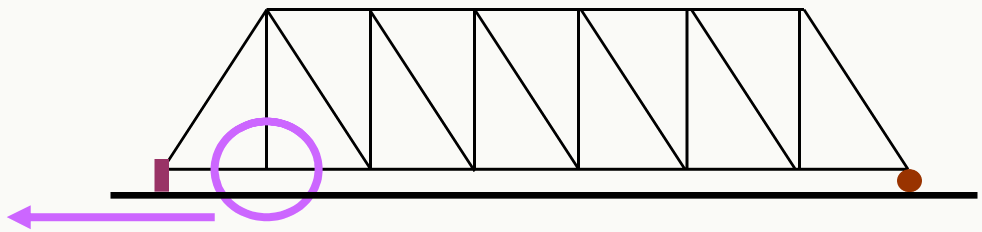
Ideal trusses:

Assumptions

- Ideal truss members are connected only at their ends.
- Ideal truss members are connected by frictionless pins (no moments)
- The truss structure is loaded only at the pins
- Weights of the members are neglected

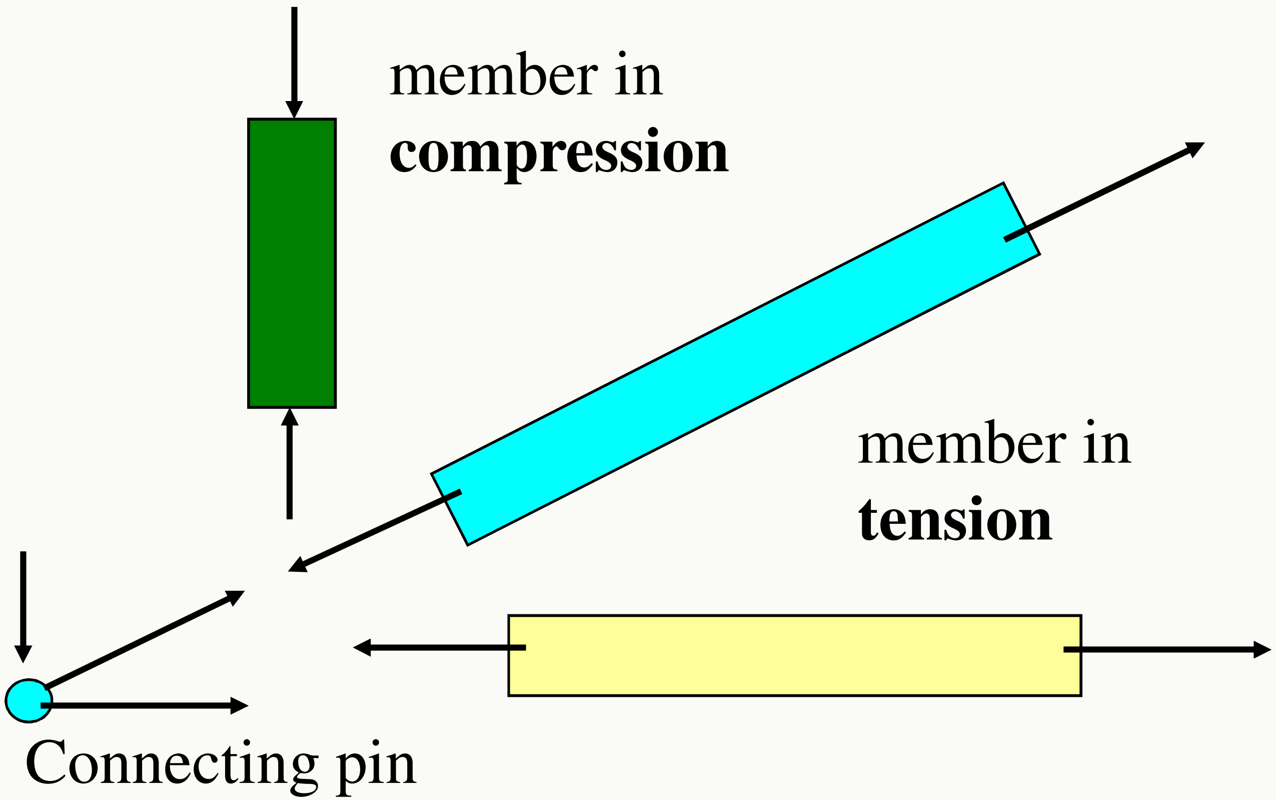


Frictionless pin



A typical truss structure

These assumptions allow us to idealize each truss member as a two-force member (members loaded **only** at their extremities by equal opposite and collinear forces)



FEM analysis scheme

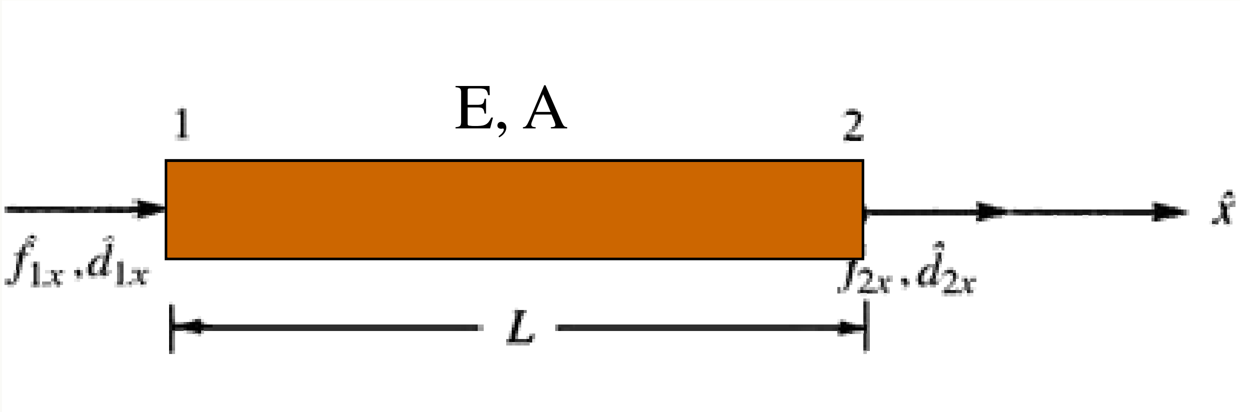
Step 1: Divide the truss into **bar/truss elements** connected to each other through special points ("**nodes**")

Step 2: Describe the behavior of each bar element (i.e. derive its **stiffness matrix** and **load vector** in local AND global coordinate system)

Step 3: Describe the behavior of the entire truss by putting together the behavior of each of the bar elements (by **assembling** their stiffness matrices and load vectors)

Step 4: Apply appropriate boundary conditions and solve

Stiffness matrix of bar element



L: Length of bar

A: Cross sectional area of bar

E: Elastic (Young's) modulus of bar

$\hat{u}(\hat{x})$:displacement of bar as a function of local coordinate bar $\hat{x}$

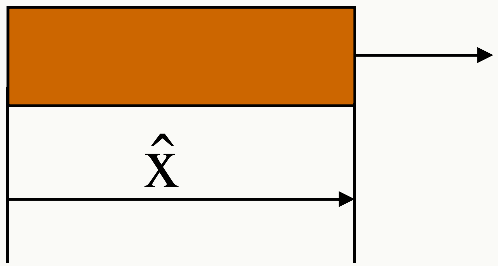
The **strain** in the bar at $\hat{x}$

$$\varepsilon(\hat{x}) = \frac{d\hat{u}}{d\hat{x}}$$

The **stress** in the bar (Hooke's law)

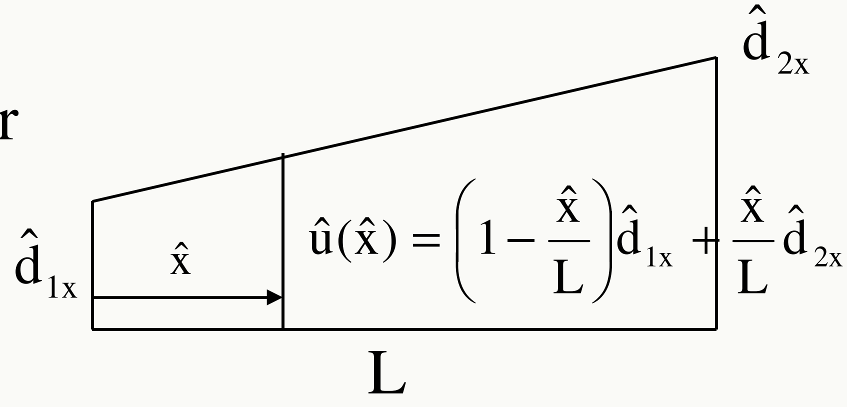
$$\sigma(\hat{x}) = E\,\varepsilon(\hat{x})$$

Trusses



Tension in the bar

$$T(\hat{x}) = EA\varepsilon$$



Assume that the displacement $\hat{u}(\hat{x})$ is varying **linearly** along the bar

$$\hat{u}(\hat{x}) = \left(1 - \frac{\hat{x}}{L}\right)\hat{d}_{1x} + \frac{\hat{x}}{L}\hat{d}_{2x}$$

Then, strain is **constant** along the bar:

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

Stress is also **constant** along the bar:

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

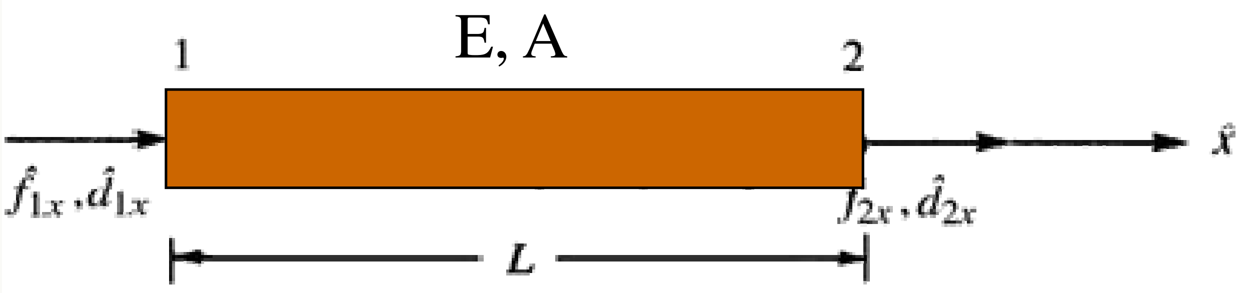
Tension is **constant** along the bar:

$$T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

<u>The bar is acting like a spring with stiffness</u>

$$k = \frac{EA}{L}$$

Springs theory



$\underline{\hat{d}}$ = Element /local nodal displacement vector

Two nodes: 1, 2

Nodal displacements: $\hat{d}_{1x} \quad \hat{d}_{2x}$

Nodal forces: $\hat{f}_{1x} \quad \hat{f}_{2x}$

Spring constant: $k = \dfrac{EA}{L}$

Element stiffness matrix in local coordinates

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}}\ \underline{\hat{d}}}$$

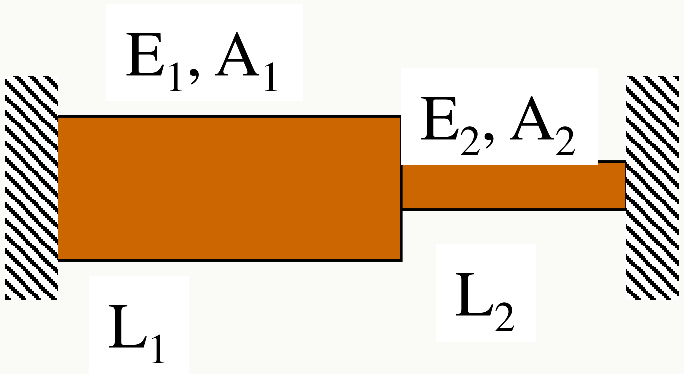
Element force vector

Element **stiffness matrix**

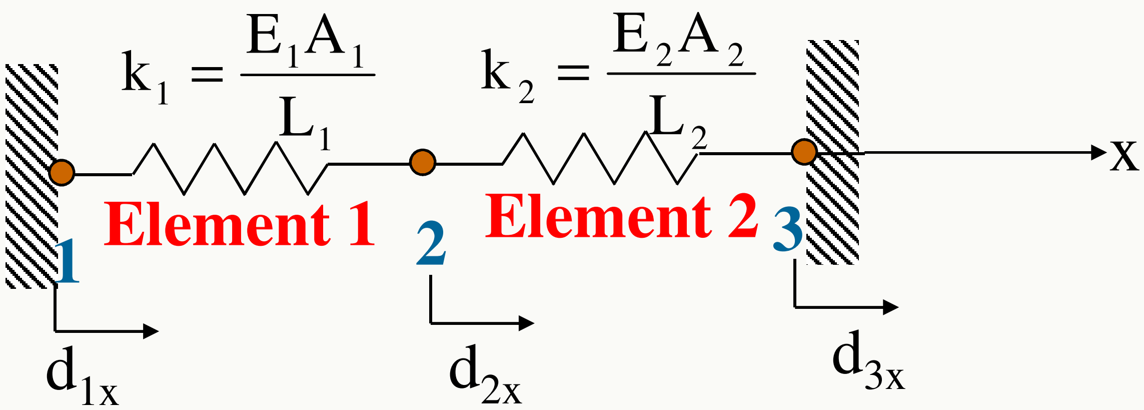
Element nodal displacement vector

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\underline{\hat{k}}} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\underline{\hat{d}}}$$

2 bars cases

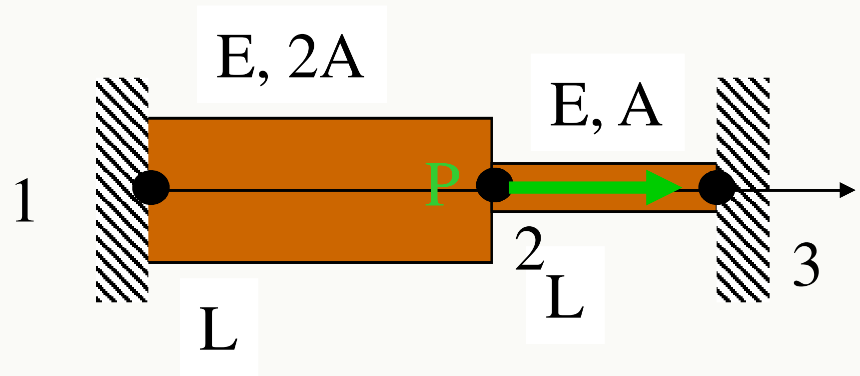


This is equivalent to the following system of springs

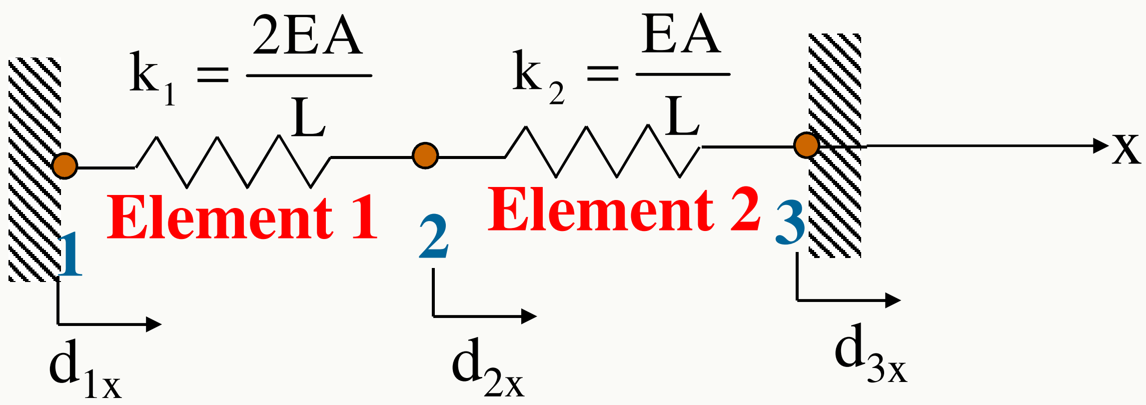


PROBLEM

Problem 1: Find the stresses in the two-bar assembly loaded as shown below



Solution: This is equivalent to the following system of springs



We will first compute the displacement at node 2 and then the stresses within each element

Trusses

The global set of equations can be generated using the technique developed in the springs theory

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \mathrm{d}_{1\mathrm{x}} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} \mathrm{F}_{1\mathrm{x}} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

here $\quad d_{1x} = d_{3x} = 0 \quad and \quad F_{2x} = P$

Hence, the above set of equations may be explicitly written as

$$-k_1 d_{2x} = F_{1x} \qquad (1)$$

$$(k_1 + k_2) d_{2x} = P \qquad (2)$$

$$-k_2 d_{2x} = F_{3x} \qquad (3)$$

From equation (2) $\quad d_{2x} = \dfrac{P}{k_1 + k_2} = \dfrac{PL}{3EA}$

Trusses

To calculate the **stresses**:

For element #1 first compute the element strain

$$\varepsilon^{(1)} = \frac{d_{2x} - d_{1x}}{L} = \frac{d_{2x}}{L} = \frac{P}{3EA}$$

and then the stress as

$$\sigma^{(1)} = E\varepsilon^{(1)} = \frac{P}{3A}$$

(element in tension)

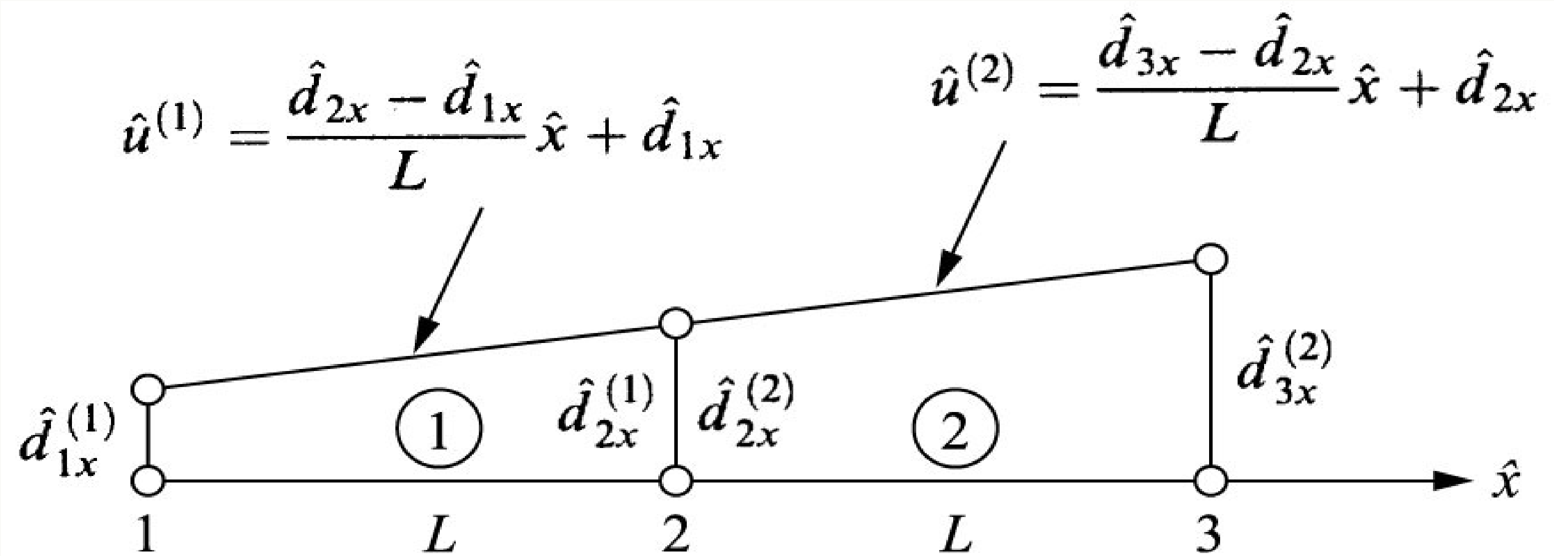
Similarly, in element # 2

$$\varepsilon^{(2)} = \frac{d_{3x} - d_{2x}}{L} = -\frac{d_{2x}}{L} = -\frac{P}{3EA}$$

$$\sigma^{(2)} = E\varepsilon^{(2)} = -\frac{P}{3A}$$

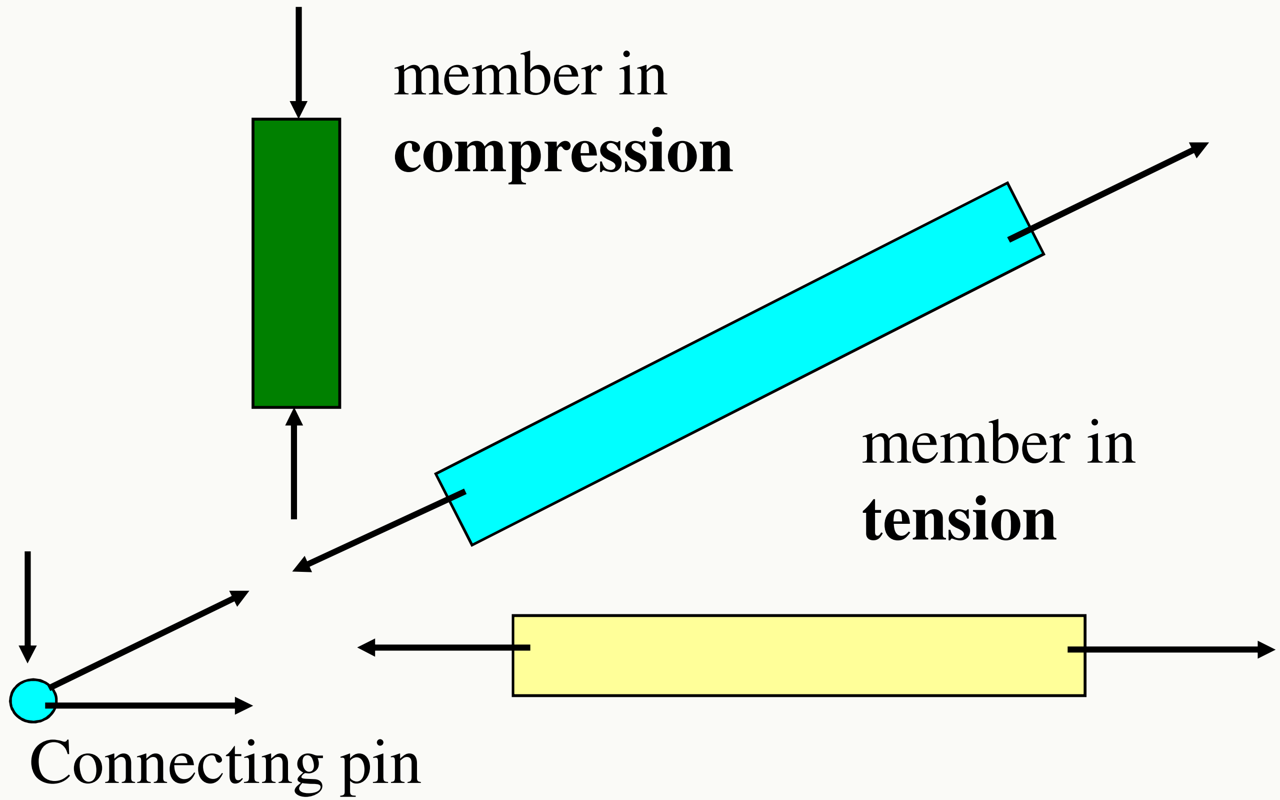
(element in compression)

Trusses

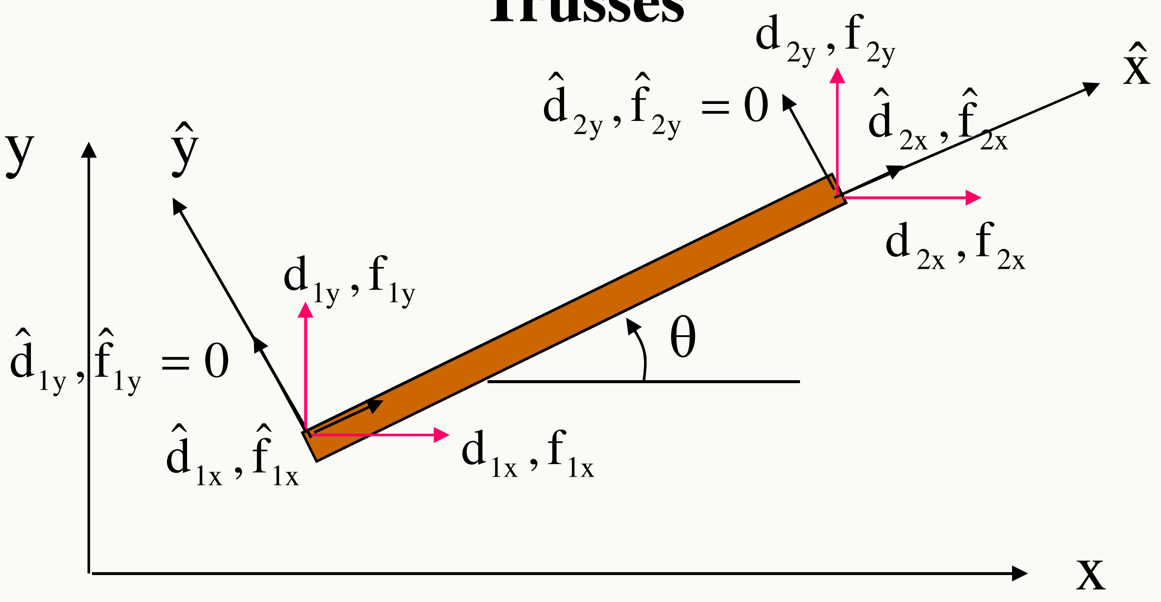


Inter-element continuity of a two-bar structure

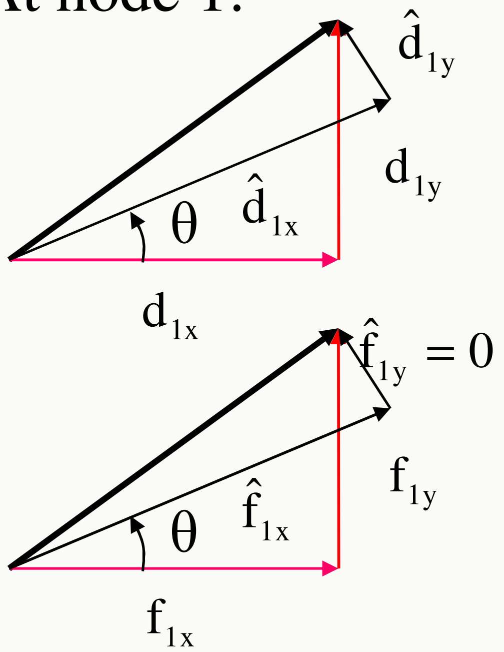
Bars in a truss have various orientations



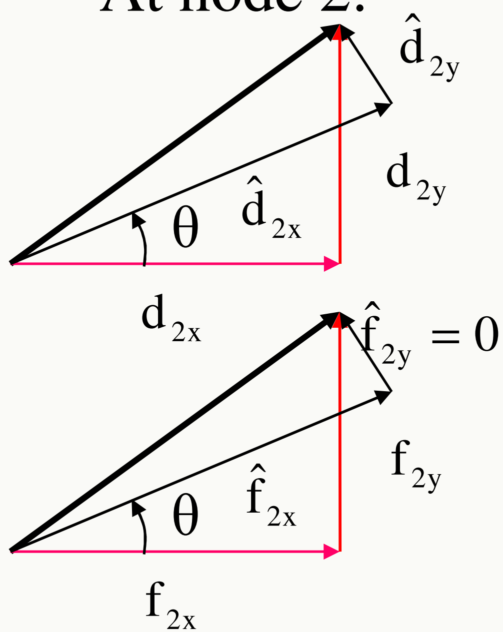
Trusses



At node 1:



At node 2:



Trusses

In the **global coordinate system**, the vector of nodal displacements and loads

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}; \qquad \underline{f} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

$\underline{\hat{d}}$ = Element /local nodal displacement vector

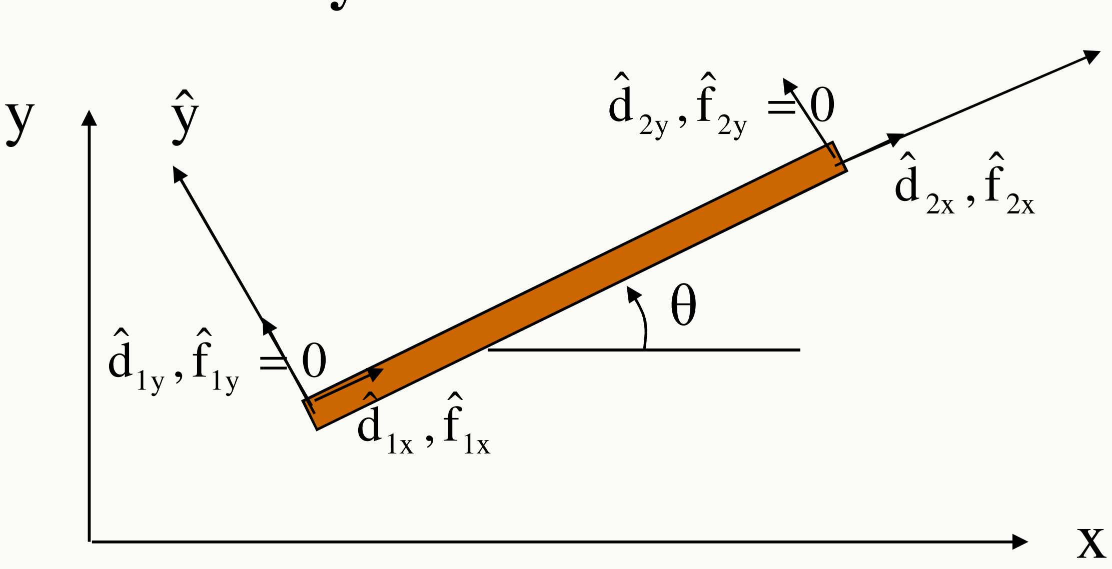
$\underline{d}$ = Global nodal displacement vector

Our objective is to obtain a relation of the form

$$\underset{4\times 1}{\underline{f}} = \underset{4\times 4}{\underline{k}} \; \underset{4\times 1}{\underline{d}}$$

Where $\underline{k}$ is the 4x4 element stiffness matrix in global coordinate system

Trusses
The key is to look at the local coordinates



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$k = \frac{EA}{L}$$

Rewrite as

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}}\ \underline{\hat{d}}}$$

NOTES

1. **Assume** that there is **no stiffness** in the local $\hat{y}$ direction.

2. If you consider the displacement at a point along the local x direction as a vector, then the components of that vector along the global x and y directions are the global x and y displacements.

3. The expanded stiffness matrix in the local coordinates is symmetric and singular.

NOTES

5. In local coordinates we have

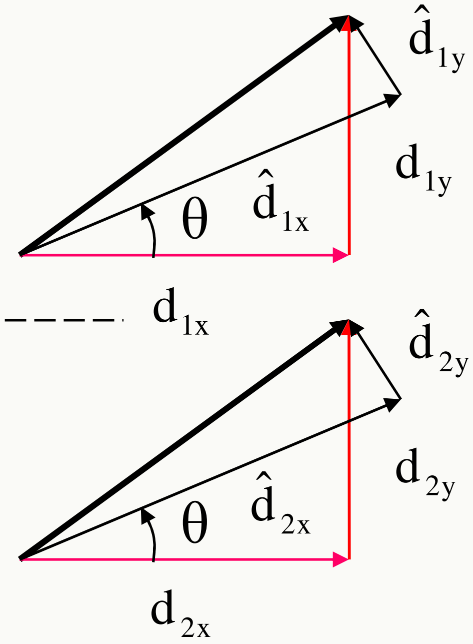
$$\boxed{\underset{4\times1}{\underline{\hat{f}}} = \underset{4\times4}{\underline{\hat{k}}}\ \underset{4\times1}{\underline{\hat{d}}}}$$

But or **goal** is to obtain the following relationship

$$\boxed{\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}}\ \underset{4\times1}{\underline{d}}}$$

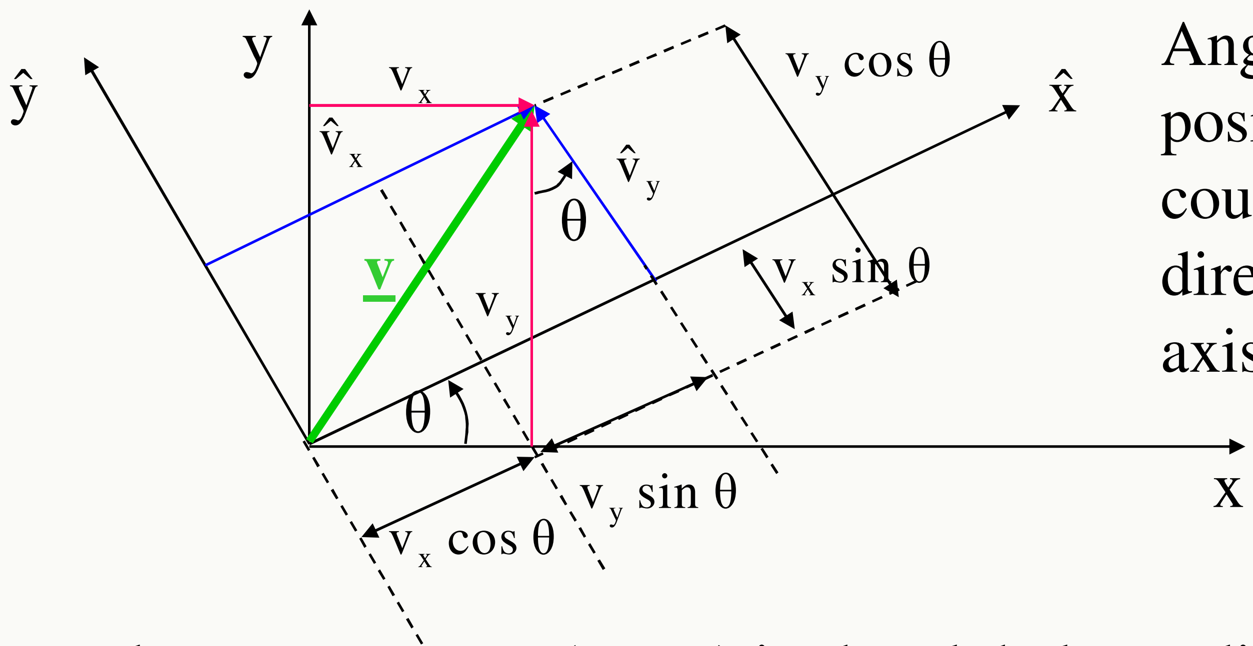
Hence, need a relationship between $\underline{\hat{d}}$ and $\underline{d}$ and between $\underline{\hat{f}}$ and $\underline{f}$

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \quad \underline{\hat{d}} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$



Need to understand how the components of a vector change with coordinate transformation

Transformation of a vector in two dimensions



Angle θ is measured positive in the counter clockwise direction from the +x axis)

The vector $\underline{\mathbf{v}}$ has components (v_x, v_y) in the global coordinate system and $(\hat{v}_x, \hat{v}_y)$ in the local coordinate system. From geometry

$$\hat{v}_x = v_x \cos\theta + v_y \sin\theta$$

$$\hat{v}_y = -v_x \sin\theta + v_y \cos\theta$$

In matrix form

$$\begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$$

Or

$$\begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$$

where

Direction cosines

$$l = \cos\theta$$
$$m = \sin\theta$$

Transformation matrix for a single vector in 2D

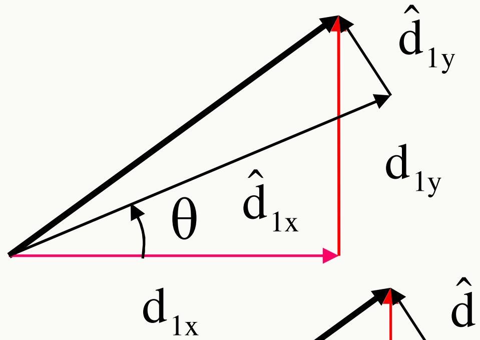
$$\underline{T}^* = \begin{bmatrix} l & m \\ -m & l \end{bmatrix}$$ relates $\boxed{\underline{\hat{v}} = \underline{T}^* \underline{v}}$

where $\underline{\hat{v}} = \begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix}$ and $\underline{v} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$ are components of the **same vector** in local and global coordinates, respectively.

Relationship between $\underline{\hat{d}}$ and $\underline{d}$ for the truss element

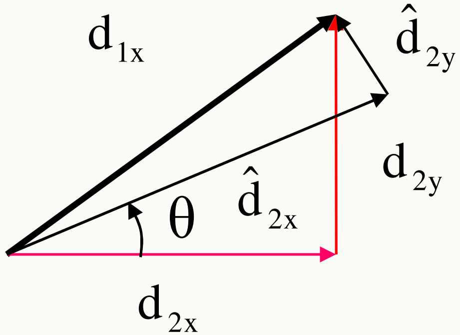
At node 1

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$$



At node 2

$$\begin{Bmatrix} \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix}$$



Putting these together $\boxed{\underline{\hat{d}} = \underline{T}\,\underline{d}}$

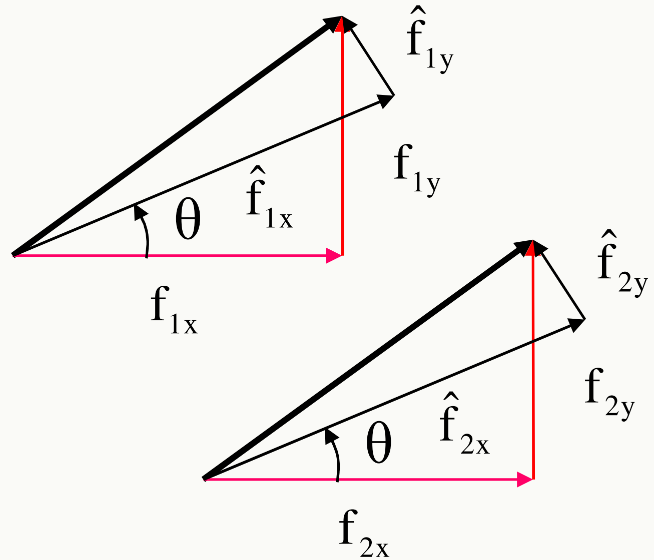
$$\underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}}_{\underline{\hat{d}}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}}_{\underline{T}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}}_{\underline{d}}$$

$$\underset{4\times4}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

Relationship between $\underline{\hat{f}}$ and $\underline{f}$ for the truss element

At node 1

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix}$$



At node 2

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix}$$

Putting these together $\boxed{\underline{\hat{f}} = \underline{T}\,\underline{f}}$

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}}_{\underline{T}} \underbrace{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}}_{\underline{f}}$$

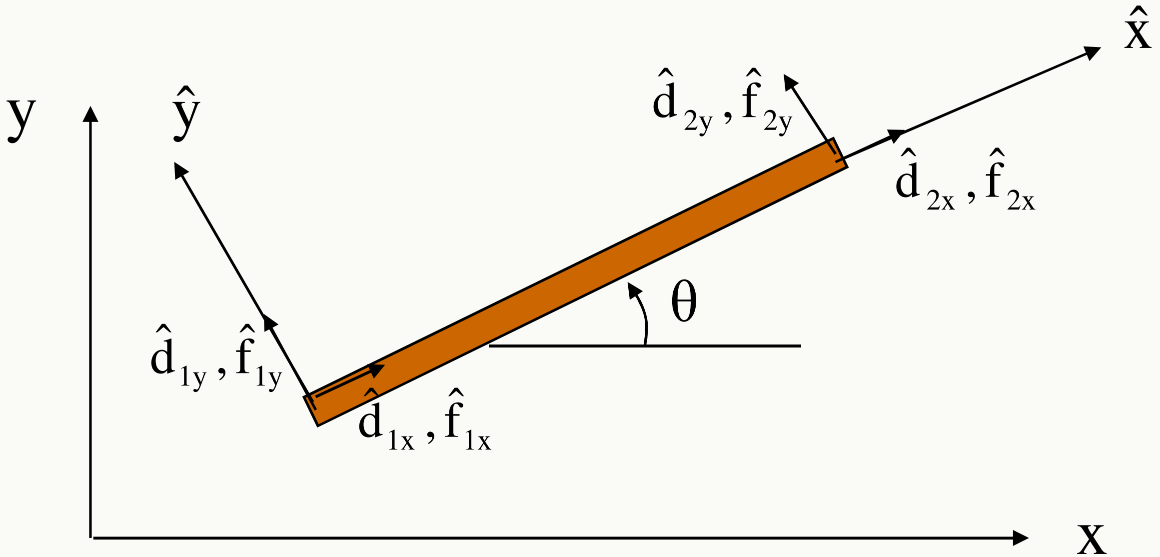
$$\underset{4\times 4}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

transformation matrix $\underline{T}$ is *orthogonal,*

$$\boxed{\underline{T}^{-1} = \underline{T}^T}$$

Use the property that $l^2+m^2=1$

Putting all the pieces together



$$\boxed{\underline{\hat{f}} = \underline{T}\,\underline{f}}$$

$$\boxed{\underline{\hat{d}} = \underline{T}\,\underline{d}}$$

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$$

$$\Rightarrow \underline{T}\,\underline{f} = \underline{\hat{k}}\,\underline{T}\,\underline{d}$$

$$\Rightarrow \underline{f} = \underbrace{\left(\underline{T}^{-1}\,\underline{\hat{k}}\,\underline{T}\right)}_{\underline{k}}\underline{d}$$

The desired relationship is $\boxed{\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}}\;\underset{4\times1}{\underline{d}}}$

Where $\boxed{\underset{4\times4}{\underline{k}} = \underset{4\times4}{\underline{T}^{T}}\;\underset{4\times4}{\underline{\hat{k}}}\;\underset{4\times4}{\underline{T}}}$ is the **element stiffness matrix in the global coordinate system**

Trusses

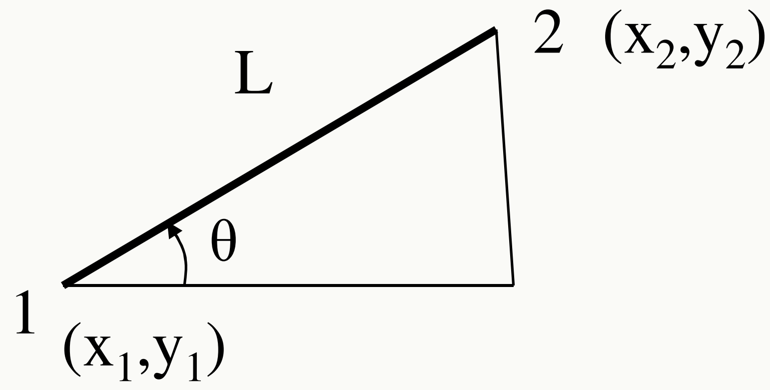
$$\underline{\mathrm{T}} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \qquad \hat{\underline{\mathrm{k}}} = \begin{bmatrix} \mathrm{k} & 0 & -\mathrm{k} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{k} & 0 & \mathrm{k} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathrm{k}} = \underline{\mathrm{T}}^{T}\,\hat{\underline{\mathrm{k}}}\,\underline{\mathrm{T}} = \frac{\mathrm{EA}}{\mathrm{L}} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Computation of the direction cosines

$$l = \cos\theta = \frac{x_2 - x_1}{L}$$

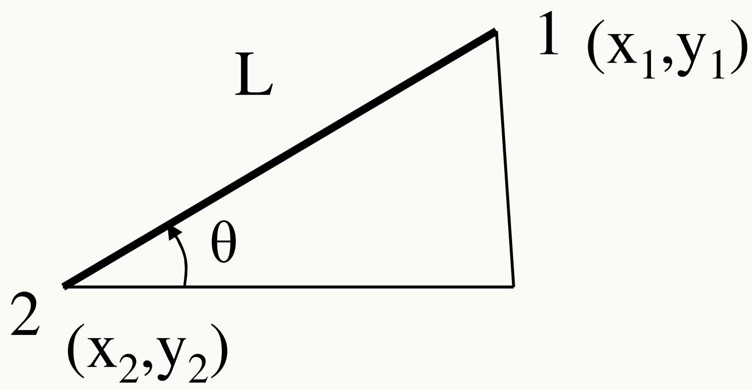
$$m = \sin\theta = \frac{y_2 - y_1}{L}$$



What happens if I reverse the node numbers?

$$l' = \cos\theta = \frac{x_1 - x_2}{L} = -l$$

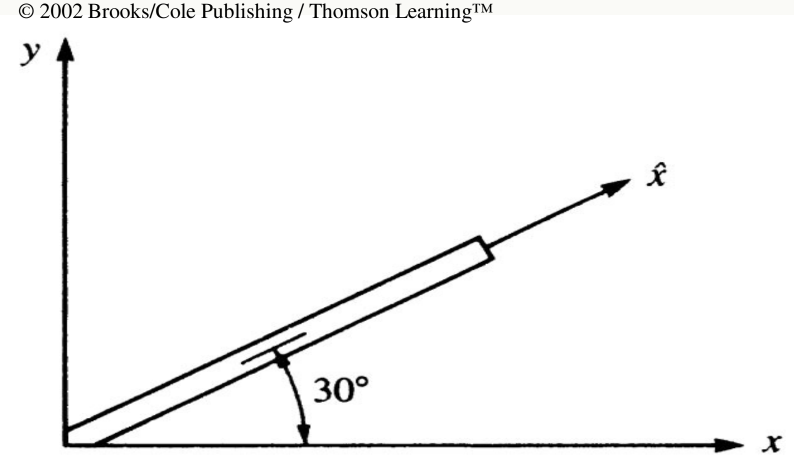
$$m' = \sin\theta = \frac{y_1 - y_2}{L} = -m$$



Question: Does the stiffness matrix change?

Trusses

Example Bar element for stiffness matrix evaluation



$$E = 30\times10^{6} \ psi$$

$$A = 2 \ in^{2}$$

$$L = 60 \ \ in$$

$$\theta = 30^{\circ}$$

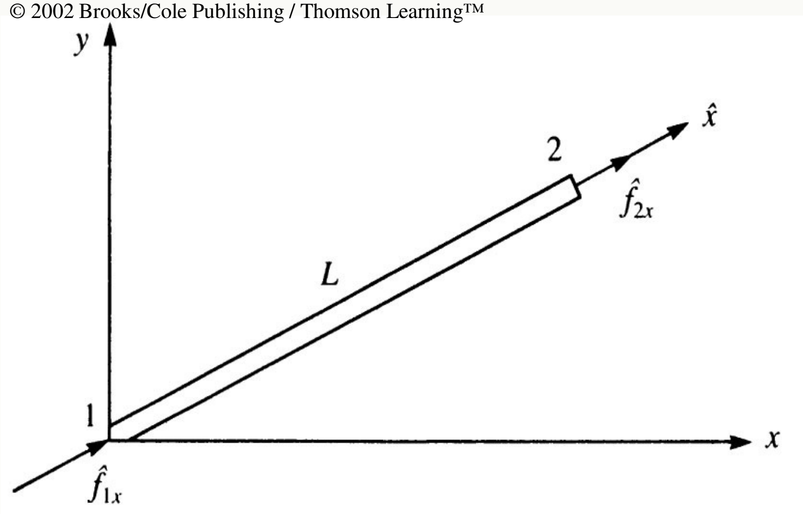
$$l = \cos 30 = \frac{\sqrt{3}}{2}$$

$$m = \sin 30 = \frac{1}{2}$$

$$\underline{\text{k}} = \frac{(30\times10^{6})(2)}{60}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}\frac{lb}{in}$$

Trusses

Computation of element strains



Recall that the element strain is

$$\varepsilon = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} = \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{1y}\\ \hat{d}_{2x}\\ \hat{d}_{2y}\end{Bmatrix}$$

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$= \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\ \underline{\hat{d}}$$

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$= \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\underline{T}\,\underline{d}$$

$$T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$k = \frac{EA}{L}$$

Trusses

$$\varepsilon = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}\underline{d}$$

$$= \frac{1}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

$$= \frac{1}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

Computation of element stresses stress and tension

Recall that the element **stress** is

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right) = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

Recall that the element **tension** is

$$T = EA\varepsilon = \frac{EA}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$k = \frac{EA}{L}$$

Trusses

Steps in solving a problem

Step 1: Write down the **node-element connectivity table** linking local and global nodes; also form the **table of direction cosines** (l, m)

Step 2: Write down the **stiffness matrix of each element in global coordinate system with global numbering**

Step 3: **Assemble** the element stiffness matrices to form the global stiffness matrix for the entire structure using the node element connectivity table

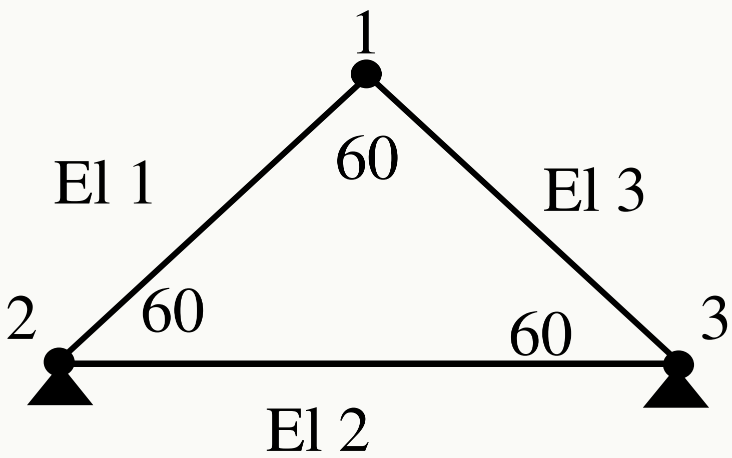
Step 4: Incorporate appropriate **boundary conditions**

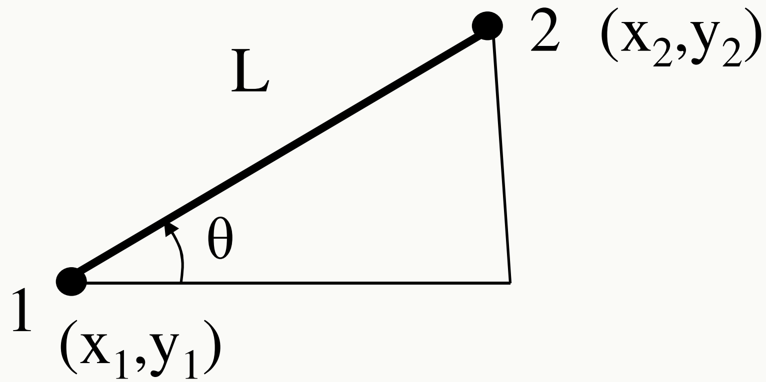
Step 5: Solve resulting set of reduced equations for the unknown displacements

Step 6: Compute the unknown nodal forces

Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3
3	3	1





Stiffness matrix of element 1

$$\underline{k}^{(1)} = \begin{array}{c c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} & \\ \left[\begin{array}{c:c:c:c} \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \end{array}\right] & \begin{array}{c} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{array} \end{array}$$

Stiffness matrix of element 2

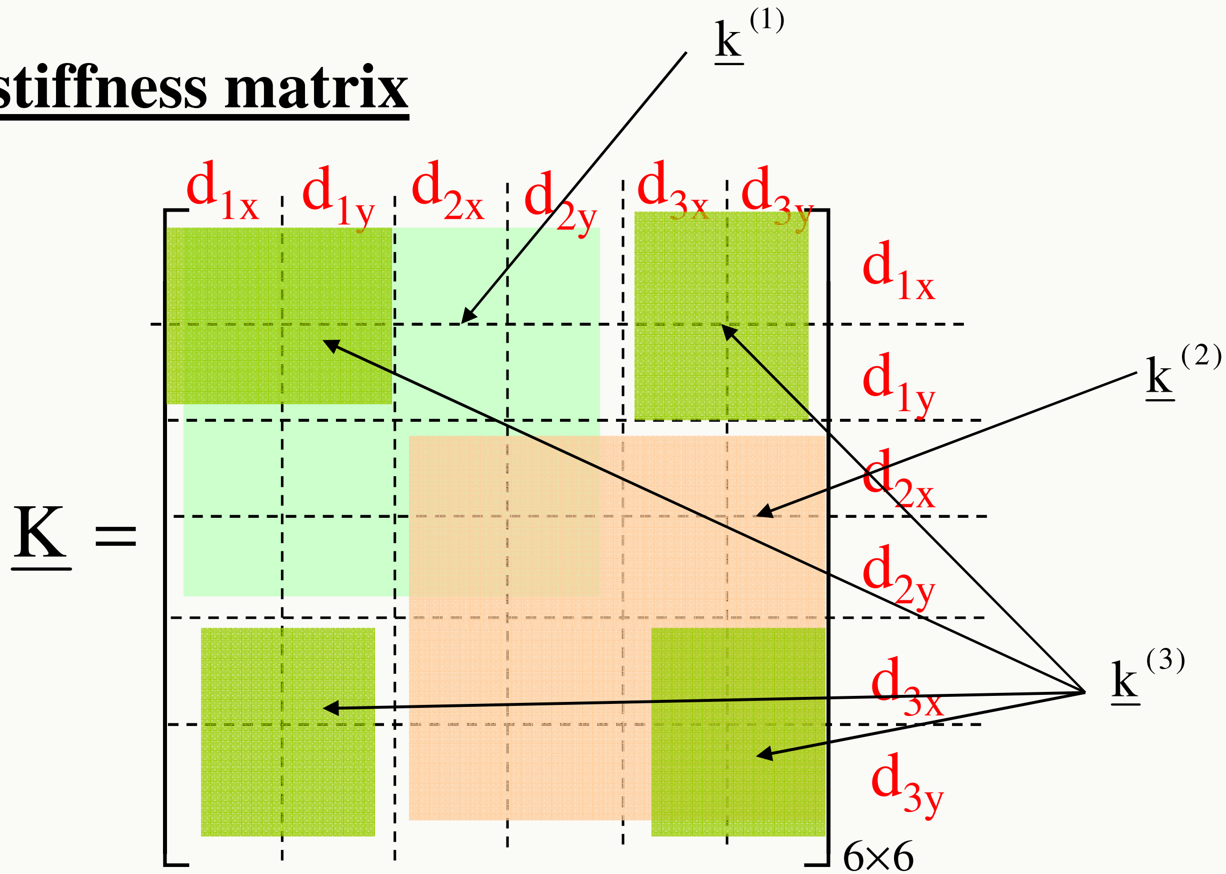
$$\underline{k}^{(2)} = \begin{array}{c c} \begin{array}{cccc} d_{2x} & d_{2y} & d_{3x} & d_{3y} \end{array} & \\ \left[\begin{array}{c:c:c:c} \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \end{array}\right] & \begin{array}{c} d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{array} \end{array}$$

Stiffness matrix of element 3

$$\underline{k}^{(3)} = \begin{array}{c c} \begin{array}{cccc} d_{3x} & d_{3y} & d_{1x} & d_{1y} \end{array} & \\ \left[\begin{array}{c:c:c:c} \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \\ \hdashline \; & \; & \; & \; \end{array}\right] & \begin{array}{c} d_{3x} \\ d_{3y} \\ d_{1x} \\ d_{1y} \end{array} \end{array}$$

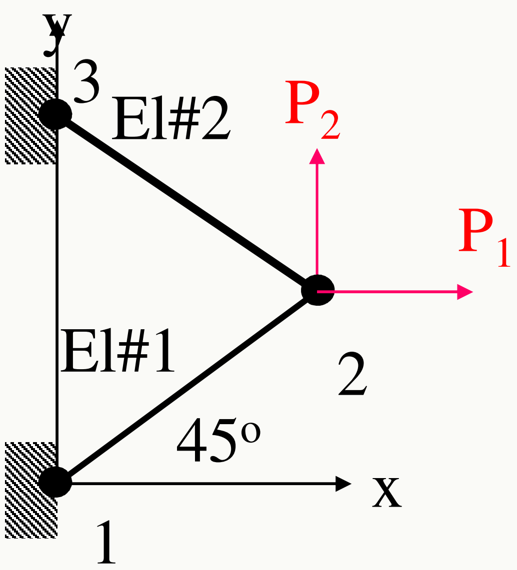
There are 4 **degrees of freedom** (**dof**) per element (2 per node)

Global stiffness matrix



How do you incorporate **boundary conditions**?

Example 2



The length of bars 12 and 23 are equal (L)
E: Young's modulus
A: Cross sectional area of each bar
Solve for
(1) d_{2x} and d_{2y}
(2) Stresses in each bar

Solution

Step 1: Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Trusses

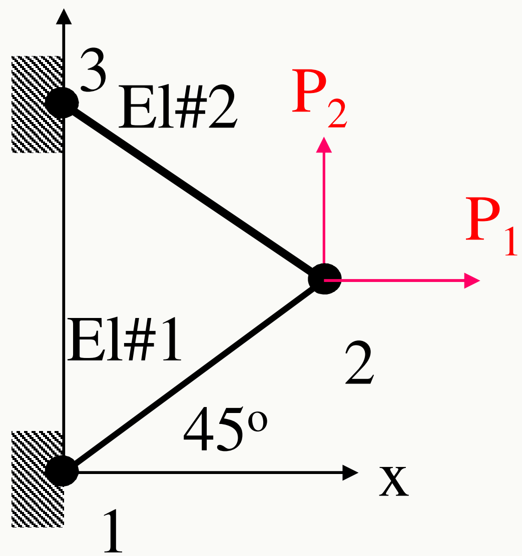


Table of nodal coordinates

Node	x	y
1	0	0
2	Lcos45	Lsin45
3	0	2Lsin45

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Table of direction cosines

ELEMENT	Length	$l = \dfrac{x_2 - x_1}{length}$	$m = \dfrac{y_2 - y_1}{length}$
1	L	cos45	sin45
2	L	-cos45	sin45

Trusses

Step 2: Stiffness matrix of each element in global coordinates with global numbering

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$k = \frac{EA}{L}$$

Stiffness matrix of element 1

$$\underline{k}^{(1)} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{EA}{2L}\begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} \\ \left[\begin{array}{c|c|c|c} 1 & 1 & -1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 \\ \hline -1 & -1 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{array}$$

Trusses

Stiffness matrix of element 2

$$
\underline{k}^{(2)} = \frac{EA}{2L}
\begin{array}{c}
\begin{array}{cccc} d_{2x} & d_{2y} & d_{3x} & d_{3y} \end{array} \\
\left[\begin{array}{cccc}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{array}\right]
\end{array}
\begin{array}{c} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{array}
$$

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Step 3: Assemble the global stiffness matrix

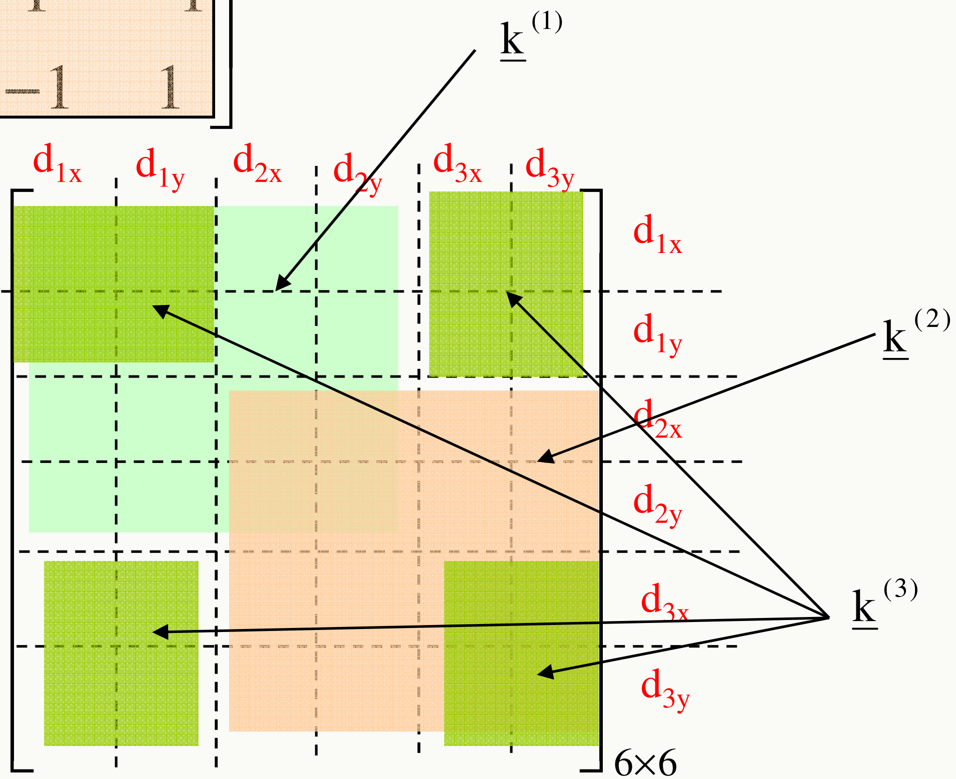
$$\underline{K} = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The final set of equations is

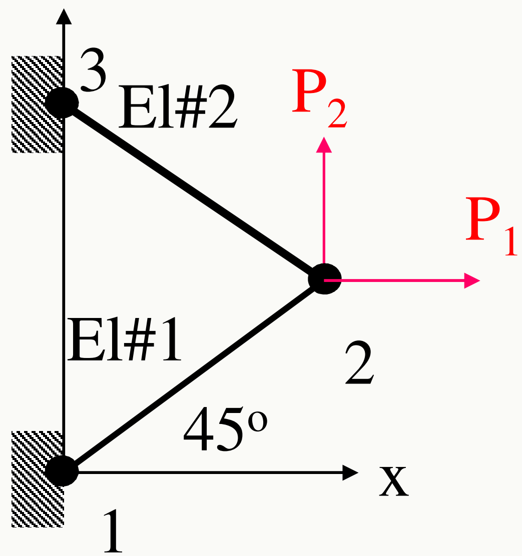
$$\underline{K}\,\underline{d} = \underline{F}$$

$$\underline{d} = \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ P_1 \\ P_2 \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

$\underline{K} =$



Trusses



Step 4: Incorporate boundary conditions

$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ d_{2y} \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{d} = \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ P_1 \\ P_2 \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Hence reduced set of equations to solve for unknown displacements at node 2

$$\underline{K}\,\underline{d} = \underline{F} \quad \Rightarrow \quad \frac{EA}{2L}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Trusses

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

$$k = \frac{EA}{L}$$

Step 5: Solve for unknown displacements

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{P_1 L}{EA} \\ \dfrac{P_2 L}{EA} \end{Bmatrix}$$

Step 6: Obtain stresses in the elements

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right) = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

For element #1:

$$\sigma^{(1)} = \frac{E}{L}\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \qquad \begin{matrix} 0 \\ 0 \end{matrix}$$

$$= \frac{E}{\sqrt{2}L}(d_{2x} + d_{2y}) = \frac{P_1 + P_2}{A\sqrt{2}}$$

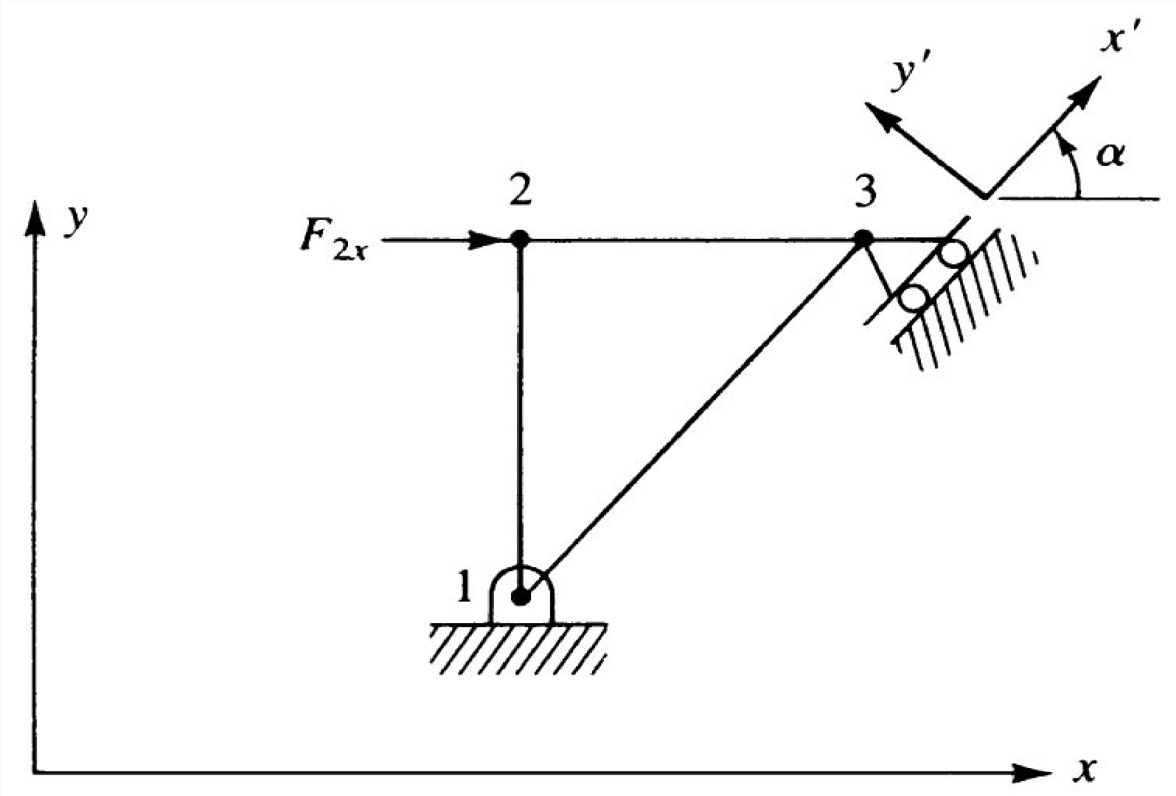
For element #2:

$$\sigma^{(2)} = \frac{\text{E}}{\text{L}}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \cancelto{0}{d_{3x}} \\ \cancelto{0}{d_{3y}} \end{Bmatrix}$$

$$= \frac{\text{E}}{\sqrt{2}\text{L}}(d_{2x} - d_{2y}) = \frac{P_1 - P_2}{A\sqrt{2}}$$

$$\sigma = \text{E}\varepsilon = \frac{\text{E}}{\text{L}}\left(\hat{\text{d}}_{2\text{x}} - \hat{\text{d}}_{1\text{x}}\right) = \frac{\text{E}}{\text{L}}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{\text{d}}$$

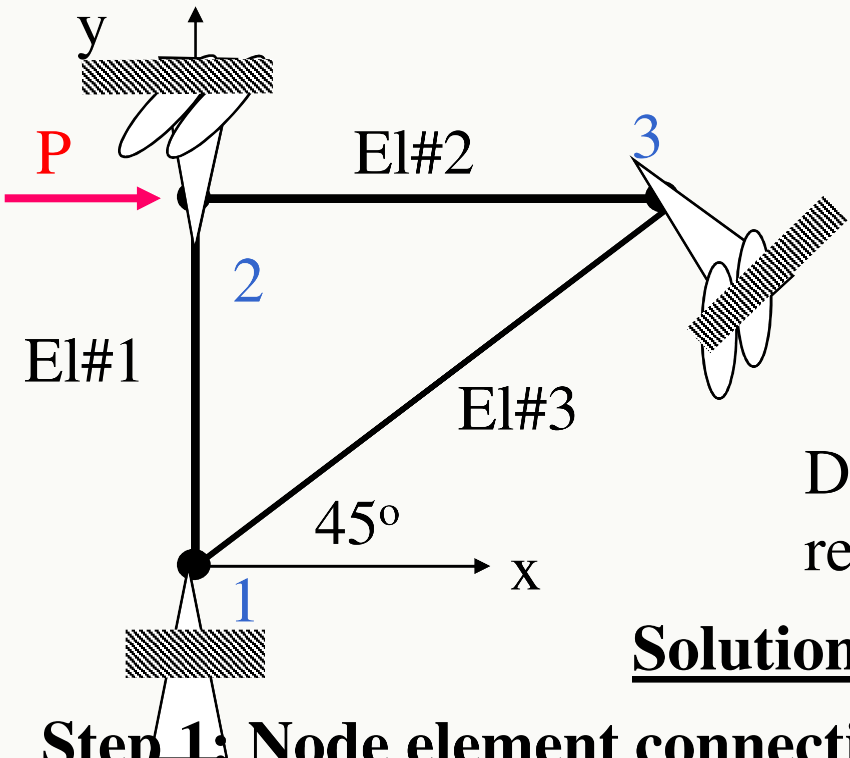
Multi-point constraints



$$\underline{d} = \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Figure 3-19 Plane truss with inclined boundary conditions at node 3

Problem 3: For the plane truss



$P = 1000$ kN,

L = length of elements 1 and 2 = 1m

$E = 210$ GPa

$A = 6\times10^{-4}\text{m}^2$ for elements 1 and 2

$= 6\sqrt{2}\times10^{-4}$ m2 for element 3

Determine the unknown displacements and reaction forces.

Solution

Step 1: Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3
3	1	3

Trusses

Table of nodal coordinates

Node	x	y
1	0	0
2	0	L
3	L	L

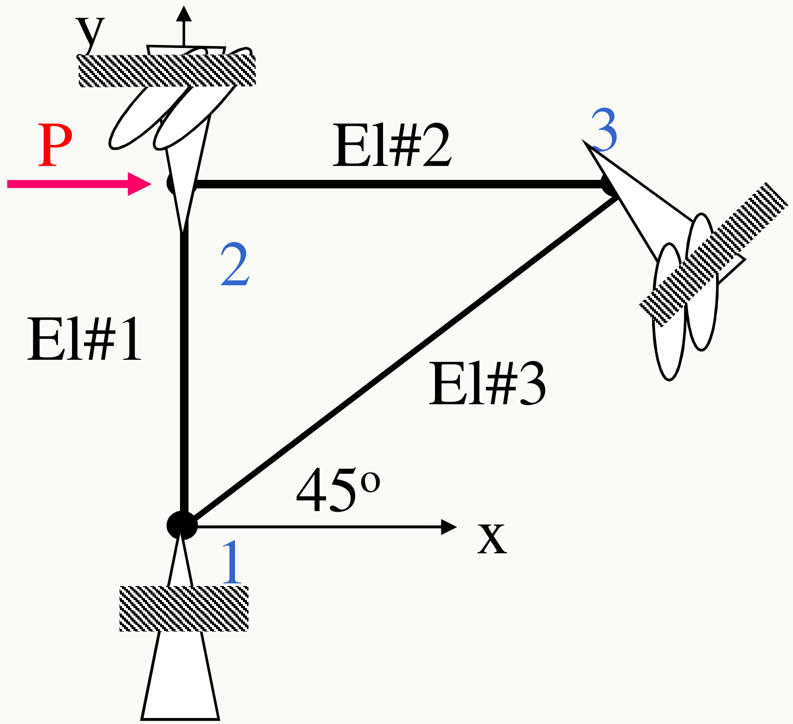


Table of direction cosines

ELEMENT	Length	$l = \dfrac{x_2 - x_1}{length}$	$m = \dfrac{y_2 - y_1}{length}$
1	L	0	1
2	L	1	0
3	$L\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$$l = \cos\theta = \frac{x_2 - x_1}{L}$$

$$m = \sin\theta = \frac{y_2 - y_1}{L}$$

Step 2: Stiffness matrix of each element in global coordinates with global numbering

Stiffness matrix of element 1

$$\underline{k}^{(1)} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{(210\times 10^9)(6\times 10^{-4})}{1}\begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{array}$$

Trusses

Stiffness matrix of element 2

$$
\underline{k}^{(2)} = \frac{(210\times10^{9})(6\times10^{-4})}{1}
\begin{array}{c}
\begin{array}{cccc} d_{2x} & d_{2y} & d_{3x} & d_{3y} \end{array} \\
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{array}
$$

Stiffness matrix of element 3

$$
\underline{k}^{(3)} = \frac{(210\times10^{9})(6\sqrt{2}\times10^{-4})}{\sqrt{2}}
\begin{array}{c}
\begin{array}{cccc} d_{1x} & d_{1y} & d_{3x} & d_{3y} \end{array} \\
\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}
\end{array}
\begin{array}{c} d_{1x} \\ d_{1y} \\ d_{3x} \\ d_{3y} \end{array}
$$

Trusses

Step 3: Assemble the global stiffness matrix

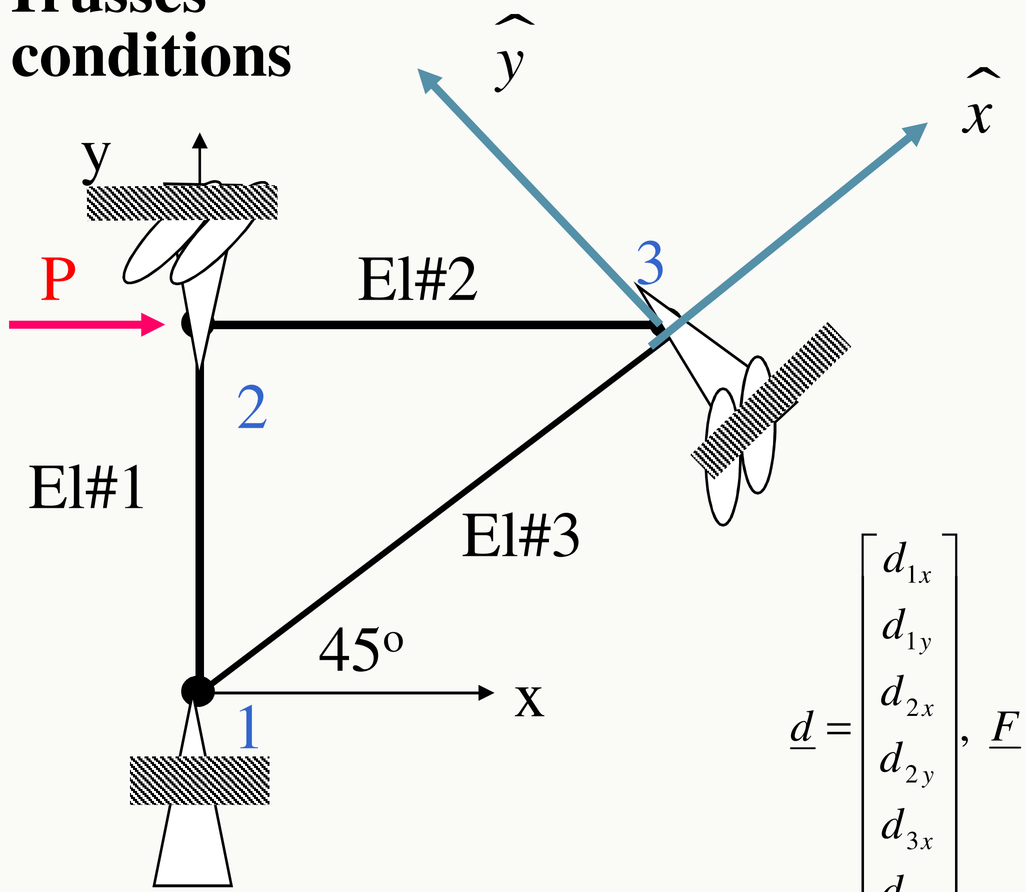
$$\underline{K} = 1260 \times 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix} \quad \text{N/m}$$

The final set of equations is $\underline{K}\,\underline{d} = \underline{F}$ Eq(1)

Trusses

Step 4: Incorporate boundary conditions

$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ 0 \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$



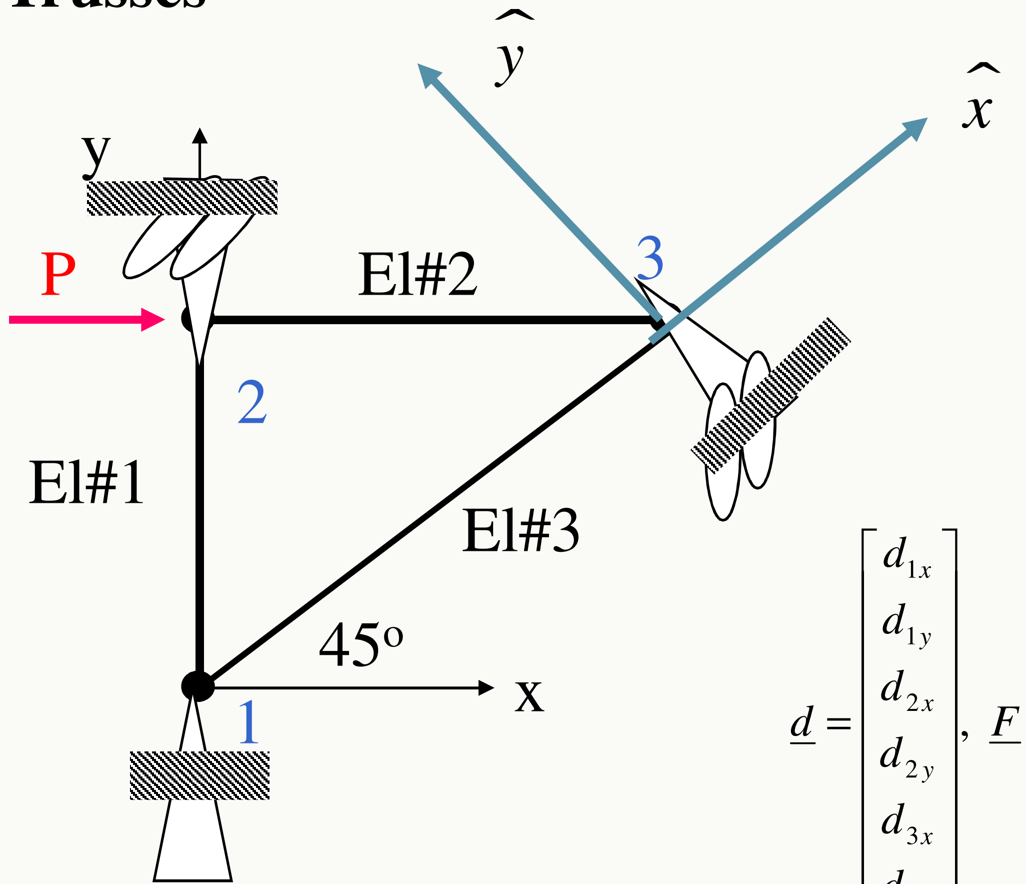
$$\underline{d} = \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Also, $\boxed{\hat{d}_{3y} = 0}$ in the local coordinate system of element 3

How do I convert this to a boundary condition in the global (x,y) coordinates?

Trusses

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$



$$\underline{d} = \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Also, $\hat{F}_{3x} = 0$ in the local coordinate system of element 3

How do I convert this to a boundary condition in the global (x,y) coordinates?

Trusses

Using coordinate transformations

$$\begin{Bmatrix} \hat{d}_{3x} \\ \hat{d}_{3y} \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix} \quad l = m = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{Bmatrix} \hat{d}_{3x} \\ \hat{d}_{3y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{2}}\left(d_{3x} + d_{3y}\right) \\ \frac{1}{\sqrt{2}}\left(d_{3y} - d_{3x}\right) \end{Bmatrix}$$

$\boxed{\hat{d}_{3y} = 0}$ (Multi-point constraint)

$$\Rightarrow \hat{d}_{3y} = \frac{1}{\sqrt{2}}\left(d_{3y} - d_{3x}\right) = 0$$

$$\Rightarrow d_{3y} - d_{3x} = 0 \qquad \text{Eq (2)}$$

Similarly for the forces at node 3

$$\begin{Bmatrix} \hat{F}_{3x} \\ \hat{F}_{3y} \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix}, \quad l = m = \frac{1}{\sqrt{2}}$$

$$\begin{Bmatrix} \hat{F}_{3x} \\ \hat{F}_{3y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{2}}\left(F_{3x} + F_{3y}\right) \\ \frac{1}{\sqrt{2}}\left(-F_{3x} + F_{3y}\right) \end{Bmatrix}$$

$$\boxed{\hat{F}_{3x} = 0}$$

$$\hat{F}_{3x} = \frac{1}{\sqrt{2}}\left(F_{3y} + F_{3x}\right) = 0$$

Eq (3)

$$F_{3y} + F_{3x} = 0$$

Trusses

Therefore we need to solve the following equations simultaneously

$$\underline{K}\,\underline{d} = \underline{F} \qquad \text{Eq(1)}$$

$$d_{3y} - d_{3x} = 0 \qquad \text{Eq(2)}$$

$$F_{3y} + F_{3x} = 0 \qquad \text{Eq(3)}$$

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Incorporate boundary conditions and reduce Eq(1) to

$$1260 \times 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ 0 \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

$$\underline{\mathrm{K}} = 1260 \times 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Write these equations out explicitly

$$1260\times10^5(d_{2x}-d_{3x}) = P \qquad \text{Eq(4)}$$

$$1260\times10^5(-d_{2x}+1.5d_{3x}+0.5d_{3y}) = F_{3x} \qquad \text{Eq(5)}$$

$$1260\times10^5(0.5d_{3x}+0.5d_{3y}) = F_{3y} \qquad \text{Eq(6)}$$

Add Eq (5) and (6)

$$1260\times10^5(-d_{2x}+2d_{3x}+d_{3y}) = F_{3x}+F_{3y} = 0 \text{ using Eq(3)}$$

$$\Rightarrow 1260\times10^5(-d_{2x}+3d_{3x}) = 0 \quad \text{using Eq(2)}$$

$$\Rightarrow d_{2x} = 3d_{3x} \quad \text{Eq(7)}$$

Plug this into Eq(4)

$$\Rightarrow 1260\times10^5(3d_{3x}-d_{3x}) = P$$

$$\Rightarrow 2520\times10^5 d_{3x} = 10^6$$

P=1000 kN

Trusses

$$\Rightarrow \quad d_{3x} = 0.003968\,m$$

$$d_{2x} = 3d_{3x} = 0.0119\,m$$

Compute the reaction forces

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 1260 \times 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

$$= \begin{Bmatrix} -500 \\ -500 \\ 0 \\ -500 \\ 500 \end{Bmatrix} kN$$

Physical significance of the stiffness matrix

In general, we will have a stiffness matrix of the form

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

And the finite element force-displacement relation

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Trusses

Physical significance of the stiffness matrix

The first equation is

$$k_{11}d_1 + k_{12}d_2 + k_{13}d_3 = F_1$$

Force equilibrium equation at node 1

Columns of the global stiffness matrix

What if $d_1=1$, $d_2=0$, $d_3=0$?

While **d.o.f** 2 and 3 are held fixed

$F_1 = k_{11}$ Force along **d.o.f** 1 due to unit displacement at **d.o.f** 1

$F_2 = k_{21}$ Force along **d.o.f** 2 due to unit displacement at **d.o.f** 1

$F_3 = k_{31}$ Force along **d.o.f** 3 due to unit displacement at **d.o.f** 1

Similarly we obtain the physical significance of the other entries of the global stiffness matrix

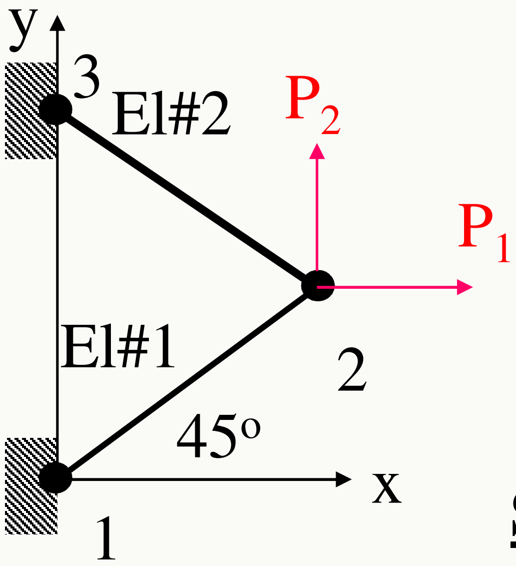
Trusses

In general

k_{ij} = Force at d.o.f 'i' due to **unit displacement** at d.o.f 'j' keeping **all the other d.o.fs fixed**

Trusses

Example



> ***The length of bars 12 and 23 are equal (L)***
> ***E: Young's modulus***
> ***A: Cross sectional area of each bar***
> ***Solve for d_{2x} and d_{2y} using the "physical interpretation" approach***

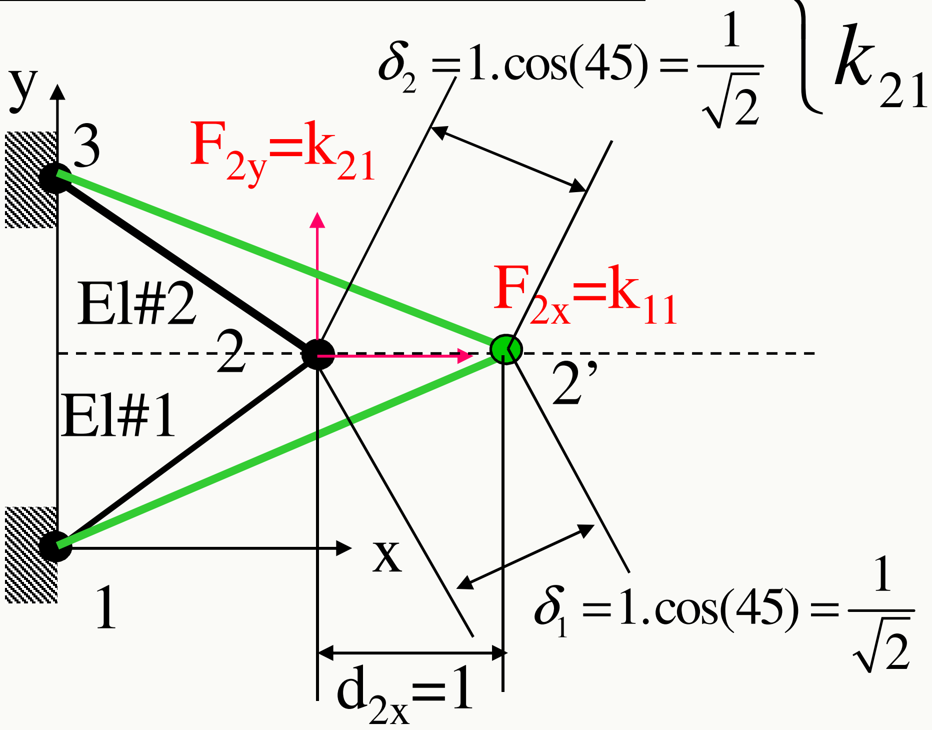
<u>Solution</u>

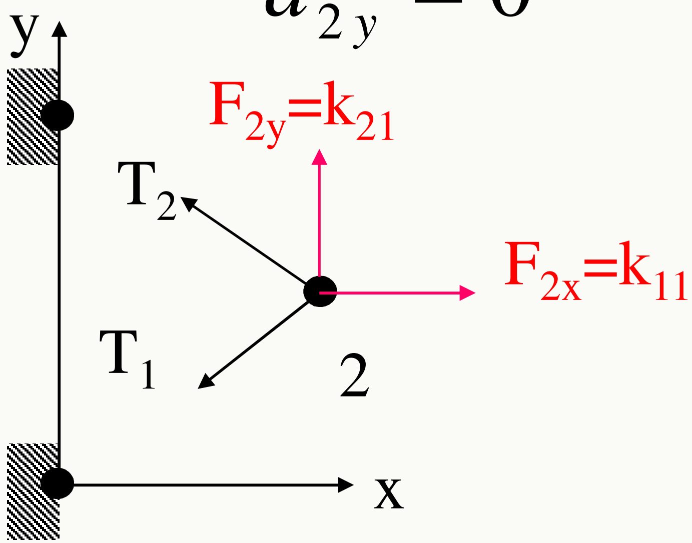
Notice that the final set of equations will be of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Where k_{11}, k_{12}, k_{21} and k_{22} will be determined using the "physical interpretation" approach

To obtain the first column $\begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix}$ apply $d_{2x} = 1$, $d_{2y} = 0$





Force equilibrium

$$\sum F_x = k_{11} - T_1\cos(45) - T_2\cos(45) = 0$$

$$\sum F_y = k_{21} - T_1\sin(45) + T_2\sin(45) = 0$$

Force-deformation relations

$$T_1 = \frac{EA}{L}\delta_1$$

$$T_2 = \frac{EA}{L}\delta_2$$

Combining force equilibrium and force-deformation relations

$$k_{11} = \frac{(T_1 + T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2)$$

$$k_{21} = \frac{(T_1 - T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2)$$

Now use the geometric (compatibility) conditions (see figure)

$$\delta_1 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

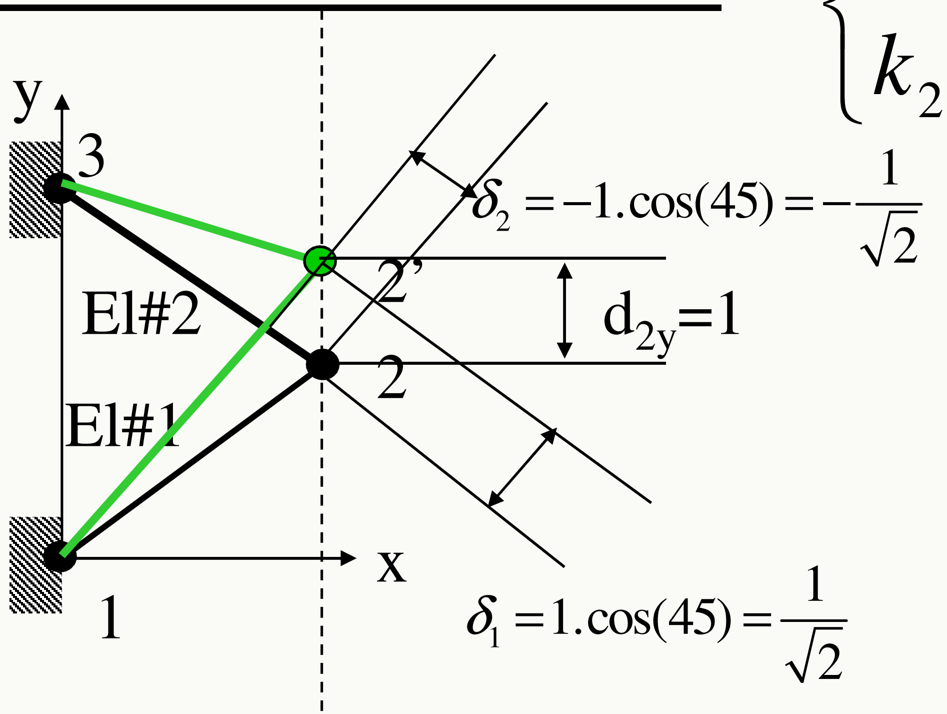
$$\delta_2 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

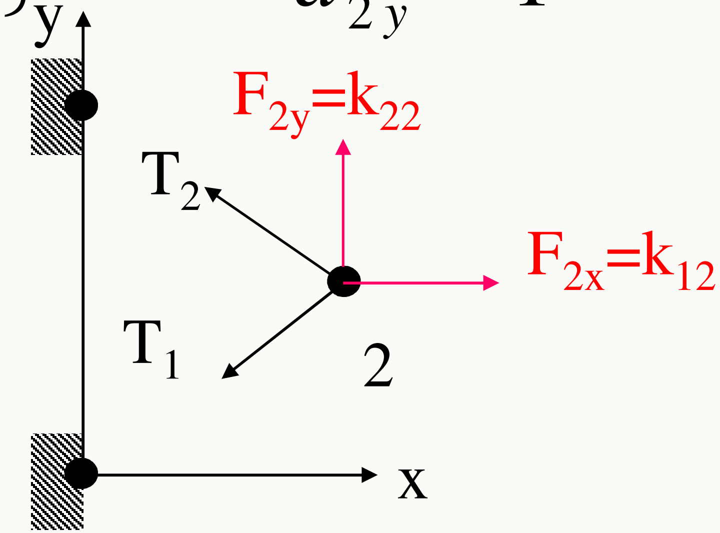
Finally

$$k_{11} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2) = \frac{EA}{\sqrt{2}L}(\frac{2}{\sqrt{2}}) = \frac{EA}{L}$$

$$k_{21} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2) = 0$$

To obtain the second column $\begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix}$ apply $d_{2x} = 0$, $d_{2y} = 1$





Force equilibrium

$$\sum F_x = k_{12} - T_1\cos(45) - T_2\cos(45) = 0$$

$$\sum F_y = k_{22} - T_1\sin(45) + T_2\sin(45) = 0$$

Force-deformation relations

$$T_1 = \frac{EA}{L}\delta_1$$

$$T_2 = \frac{EA}{L}\delta_2$$

Combining force equilibrium and force-deformation relations

$$k_{12} = \frac{(T_1 + T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2)$$

$$k_{22} = \frac{(T_1 - T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2)$$

Now use the geometric (compatibility) conditions (see figure)

$$\delta_1 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

$$\delta_2 = -1.\cos(45) = -\frac{1}{\sqrt{2}}$$

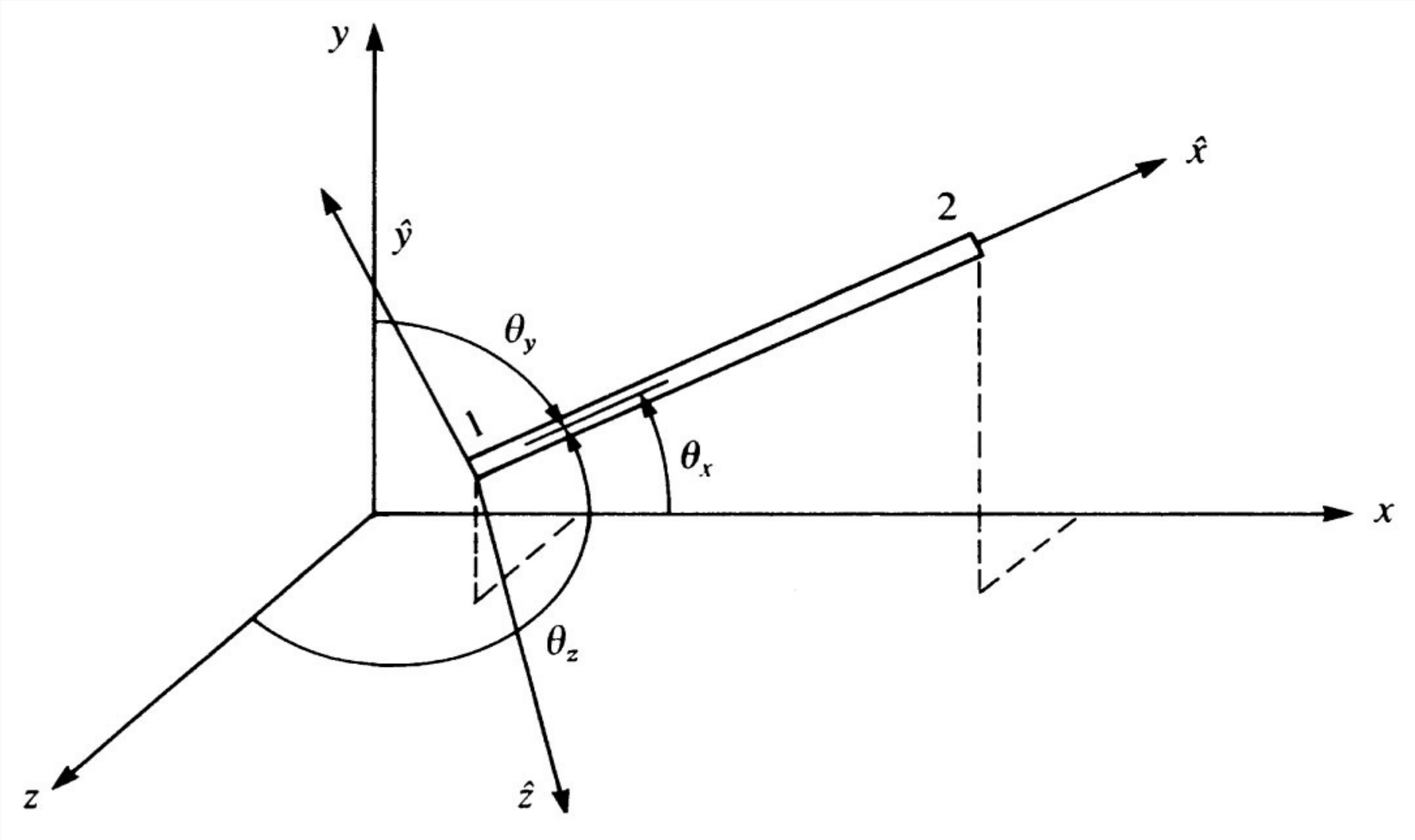
This negative is due to **compression**

Finally

$$k_{12} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2) = 0$$

$$k_{22} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2) = \frac{EA}{\sqrt{2}L}\left(\frac{2}{\sqrt{2}}\right) = \frac{EA}{L}$$

3D Truss (space truss)



In local coordinate system $\boxed{\underline{\hat{f}} = \underline{\hat{k}}\ \underline{\hat{d}}}$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{1z} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{f}_{2z} \end{Bmatrix} = \begin{bmatrix} k & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{1z} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{d}_{2z} \end{Bmatrix}$$

The transformation matrix for a **single vector** in 3D

$$\boxed{\underline{\hat{d}} = \underline{T}^{*}\,\underline{d}}$$

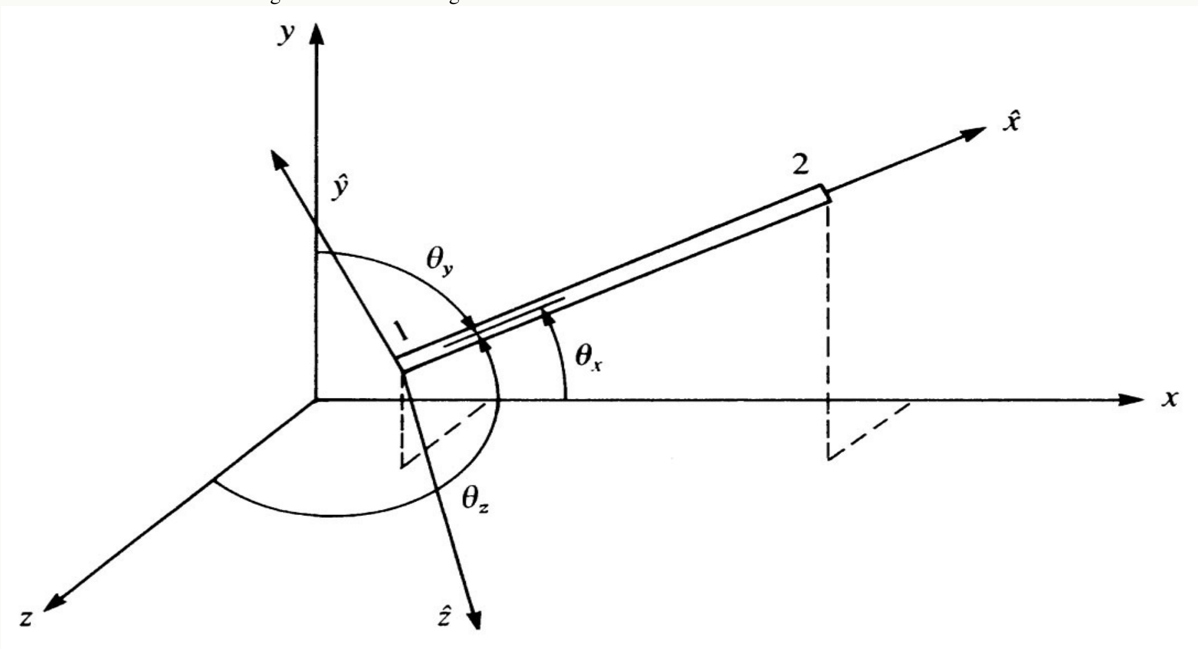
$$\underline{T}^{*} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

l_1, m_1 *and* n_1 are the direction cosines of $\hat{x}$

$$l_1 = \cos\,\theta_x$$

$$m_1 = \cos\,\theta_y$$

$$n_1 = \cos\,\theta_z$$



Transformation matrix **T** relating the local and global displacement and load vectors of the truss element

$$\boxed{\hat{\underline{d}} = \underline{T}\,\underline{d}}$$

$$\boxed{\hat{\underline{f}} = \underline{T}\,\underline{f}}$$

$$\underset{6\times 6}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

Element stiffness matrix in global coordinates

$$\boxed{\underset{6\times 6}{\underline{k}} = \underset{6\times 6}{\underline{T}^T}\;\underset{6\times 6}{\hat{\underline{k}}}\;\underset{6\times 6}{\underline{T}}}$$

$$\underline{\mathrm{k}} = \underline{\mathrm{T}}^T \,\underline{\hat{\mathrm{k}}}\underline{\mathrm{T}} = \frac{\mathrm{EA}}{\mathrm{L}}\begin{bmatrix} l_1^{\ 2} & l_1 m_1 & l_1 n_1 & -l_1^{\ 2} & -l_1 m_1 & -l_1 n_1 \\ l_1 m_1 & m_1^{\ 2} & m_1 n_1 & -l_1 m_1 & -m_1^{\ 2} & -m_1 n_1 \\ l_1 n_1 & m_1 n_1 & n_1^{\ 2} & l_1 n_1 & m_1 n_1 & -n_1^{\ 2} \\ -l_1^{\ 2} & -l_1 m_1 & -l_1 n_1 & l_1^{\ 2} & l_1 m_1 & l_1 n_1 \\ -l_1 m_1 & -m_1^{\ 2} & -m_1 n_1 & l_1 m_1 & m_1^{\ 2} & m_1 n_1 \\ -l_1 n_1 & -m_1 n_1 & -n_1^{\ 2} & l_1 n_1 & m_1 n_1 & n_1^{\ 2} \end{bmatrix}$$

Notice that the direction cosines of **only** the local $\hat{\mathrm{x}}$ axis enter the $\underline{\mathrm{k}}$ matrix