Mathematical Methods I
SSCM 1023

Part 2 – Differentiation and integration of trigonometric and hyperbolic functions
Further Applications of Integration
Improper Integrals

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Differentiation of Hyperbolic Functions

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \quad = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \quad = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \quad = \frac{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{4}{\left(e^x + e^{-x}\right)^2} \quad = \left(\frac{2}{\left(e^x + e^{-x}\right)}\right)^2 \quad = \left(\frac{1}{\cosh x}\right)^2 \quad = \operatorname{sech}^2 x$$

Differentiation of Hyperbolic Functions

Standard Derivatives

$y = f(x)$	$\dfrac{dy}{dx} = f'(x)$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\text{sech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
$\coth x$	$-\text{cosech}^2 x$

Differentiation of Hyperbolic Functions

Example: Find the derivatives of the following functions

(a) $y = \cosh(3x)$ (b) $r = \sinh(2t^2 - 1)$

(c) $g(x) = (x-1)^3 \operatorname{sech}^2 x$ (d) $y = \tanh(\ln x)$

Implicit differentiation (introduction)

Find the derivatives of the following expressions

(a) $x = y^2 \sinh 4x + \cosh y$

(b) $y = \tanh(x + y)$

$$\frac{dy}{dx} = \sec h^2(x+y)\frac{d}{dx}(x+y)$$

$$= \sec h^2(x+y) \cdot \left(1 + \frac{dy}{dx}\right)$$

Derivatives of Inverse Trigonometric Functions

Derivatives of $\sin^{-1}x$

Let $\sin y = x$ $\quad \dfrac{d}{dx}(\sin y) = \dfrac{d}{dx}(x)$ $\Rightarrow$ $\cos y \dfrac{dy}{dx} = 1$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ $\quad \cos y \ge 0$ $\quad$ We get $\quad \dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$

Exercise Differentiate each of the following functions

$f(x) = \tan^{-1}\sqrt{x}$ $\qquad g(t) = \sin^{-1}(1-t)$ $\qquad h(x) = \sec^{-1} e^{2x}$

Find the derivative dy/dx

$x\tan^{-1} y = x^2 + y$ $\qquad \sin^{-1}(xy) + \dfrac{\pi}{2} = \cos^{-1} y$

Definition of inverse trigonometric functions

Function	**Domain**	**Range**
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y < \frac{\pi}{2} \cup \frac{\pi}{2} < y < \pi$
$\cot^{-1} x$	$-\infty \le x \le \infty$	$0 < y < \pi$
$\text{cosec}^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} < y < 0 \cup 0 < y < \frac{\pi}{2}$

Derivatives of Inverse Trigonometric Functions

1. $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

2. $$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

3. $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

4. $$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

5. $$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x \rvert \sqrt{x^2-1}}$$

6. $$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{\lvert x \rvert \sqrt{x^2-1}}$$

If *u* is a differentiable function of *x*, then

1. $$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \bullet \frac{du}{dx}$$

2. $$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \bullet \frac{du}{dx}$$

3. $$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \bullet \frac{du}{dx}$$

4. $$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2} \bullet \frac{du}{dx}$$

5. $$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \bullet \frac{du}{dx}$$

6. $$\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{|u|\sqrt{u^2-1}} \bullet \frac{du}{dx}$$

Derivatives of Inverse Trigonometric Functions

Exercise Differentiate each of the following functions

$f(x) = \tan^{-1}\sqrt{x}$ $\qquad g(t) = \sin^{-1}(1-t)$ $\qquad h(x) = \sec^{-1} e^{2x}$

Find the derivative dy/dx (include implicit)

$y = (\tan^{-1} x^2)^4$ $\qquad y = \ln(\sin^{-1} 4x)$

$x\tan^{-1} y = x^2 + y$ $\qquad \sin^{-1}(xy) + \dfrac{\pi}{2} = \cos^{-1} y$

Inverse Hyperbolic Functions

Function	Domain	Range
$y = \sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = \cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$y = \tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$y = \coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \text{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$y = \text{cosech}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Function	Logarithmic form
$y = \sinh^{-1} x$	$\ln\left(x + \sqrt{x^2 + 1}\right)$
$y = \cosh^{-1} x$	$\ln\left(x + \sqrt{x^2 - 1}\right)$
$y = \tanh^{-1} x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); \ \lvert x \rvert < 1$

Derivatives of Inverse Hyperbolic Functions

Derivatives of $y = \sinh^{-1} x$

Given $x = \sinh y$ ⇒ $\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$

$$1 = \cosh y \frac{dy}{dx}$$

⇒ $\therefore \frac{dy}{dx} = \frac{1}{\cosh y}$

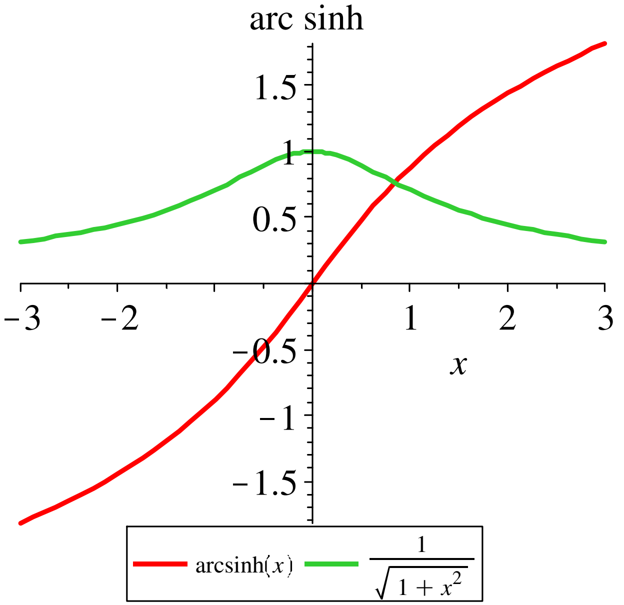
Other ways to obtain the derivatives are

$$y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Since $-\infty < y < \infty$, $\cosh y \geq 0$: so using the identity : $\mathbf{\cosh^2 y - \sinh^2 y = 1}$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$



Derivatives of Inverse Hyperbolic Functions

Function, y	**Derivatives, $\frac{dy}{dx}$**
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}};\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2};\ \lvert x\rvert < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2};\ \lvert x\rvert > 1$
$\text{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}};\ 0 < x < 1$
$\text{cosech}^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{1+x^2}};\ x \neq 0$

Exercise: Find the derivatives of

$$y = \sinh^{-1}(1-3x)$$

$$y = \cosh^{-1}\left(\frac{1}{x}\right)$$

$$y = e^x \text{sech}^{-1} x$$

$$y = \sinh^{-1}(\tan 3x)$$

$$y^3 - \sinh^{-1} xy = 0$$

Techniques of integration

Integration by substitution

$$\int \frac{\sin x}{1-\cos x}\,dx$$ Let $u = 1 - \cos x$

$$\int \sin x \cos^4 x \; dx$$ Let $u = \cos x$

$$\int x \cos x^2 \, e^{\sin x^2} \; dx$$ Let $u = \sin x^2$

Rule of thumb: $\dfrac{simple}{complex}$, $simple(complex)^n$, $simple\sqrt{complex}$

Techniques of integration

Integration by parts

Start with the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$d(uv) = u\ dv + v\ du$$

$$d(uv) - v\ du = u\ dv$$

$$\int u\ dv = \int \left(d(uv) - v\ du \right)$$

$$\int u\ dv = \int \left(d(uv) \right) - \int v\ du$$

$$\int u\ dv = uv - \int v\ du$$

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

Techniques of integration

Integration by parts

$$\int u\,dv = uv - \int v\,du \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Let $u = x$ and $dv = e^x\,dx$. Then $du = 2x\,dx$ and $v = e^x$.

$$\int x^2 e^x dx = x^2 e^x - \int 2x\,e^x dx$$

Exercise

$$\int x\cos x\,dx$$

$$\int x\sin 2x\,dx$$

Continuing:

We can use parts again on this integral

(with $u = 2x$ and $dv = e^x\,dx$) to get:

$$\int x^2 e^x dx = x^2 e^x - 2x\,e^x + \int 2e^x dx$$

$$= x^2 e^x - 2x\,e^x + 2e^x + C$$

Techniques of integration

$$\int u \ dv = uv - \int v \ du$$

u differentiates to zero (usually).

dv is easy to integrate.

Choose u in this order: LIPET

Logs, **I**nverse trig, **P**olynomial, **E**xponential, **T**rig

$$\int x \cdot \cos x \ dx$$

$$u = x \quad dv = \cos x \ dx$$

polynomial factor

$$\Rightarrow \quad x \cdot \sin x - \int \sin x \ dx$$

$$\int \ln x \ dx$$

$$u = \ln x \qquad dv = dx$$

logarithmic factor

$$\Rightarrow \quad \ln x \cdot x - \int x \cdot \frac{1}{x} \ dx$$

Techniques of integration

Tabular Integration

Tabular integration works for integrals of the form:

$$\int f(x)\,g(x)\,dx$$

Differentiates to zero in several steps. ($f(x)$)

Integrates repeatedly. ($g(x)$)

differentiate	Integrate
x^2 +1	$\sin nx$
$2x$ -1	$\left(-\frac{1}{n}\right)\cos nx$
2 +1	$\left(-\frac{1}{n^2}\right)\sin nx$
0	$\left(\frac{1}{n^3}\right)\cos nx$

Zero!

$$\int_a^b x^2\sin(nx)dx = x^2(+1)\left(-\tfrac{1}{n}\right)\cos(nx) + (2x)(-1)\left(-\tfrac{1}{n^2}\right)\sin(nx)\Big|_a^b$$

$$+\int_a^b 2\left(-\tfrac{1}{n^2}\right)\sin(nx)dx$$

$$\Rightarrow \int_a^b x^2\sin(nx)dx = x^2\left(-\tfrac{1}{n}\right)\cos(nx) + (2x)\left(\tfrac{1}{n^2}\right)\sin(nx) + (2)\left(\tfrac{1}{n^3}\right)\cos(nx)\Big|_a^b$$

$$+\int_a^b -0\left(\tfrac{1}{n^3}\right)\cos(nx)dx$$

Techniques of integration

Tabular Integration

Exercise

$$\int \sec h^2 x dx = \tanh x + C$$

$$\int x \sec^2 x \, dx$$

$$\int e^{3x} \cos 2x \, dx$$

Techniques of integration

Integration using partial fractions

$$\int \frac{3x+2}{x^2+3x+2}\,dx \qquad \frac{3x+2}{x^2+3x+2} = -\frac{1}{x+1} + \frac{4}{x+2}$$

$$\int \frac{1}{x^3+2x^2+x}\,dx \qquad \frac{1}{x^3+2x^2+x} = -\frac{1}{x+1} + \frac{1}{x} - \frac{1}{(x+1)^2}$$

Techniques of integration

Partial Fractions

$$\frac{2}{x(x+2)}=\frac{A}{x}+\frac{B}{x+2}$$

Break into 2 smaller fractions

$$\frac{11x-2}{10x^2-3x-1}=\frac{A}{5x+1}+\frac{B}{2x-1}$$

$$\frac{6x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

$$\frac{4x^2-7x-3}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Repeated roots: we must use two terms for partial fractions.

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} \Rightarrow$$

$$\begin{array}{r} 2x \\ x^2-2x-3 \overline{\smash{\big)}\, 2x^3-4x^2-x-3} \\ \underline{2x^3-4x^2-6x} \\ 5x-3 \end{array}$$

$$2x+\frac{5x-3}{x^2-2x-3}$$

$$= 2x+\frac{3}{(x-3)}+\frac{2}{(x+1)}$$

If the degree of the numerator is higher than the degree of the denominator, use **long division** first.

Techniques of integration

Partial Fractions

$$\frac{7x^2-4x}{(x^2+1)(x-2)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-2}$$

A nice shortcut if you have non-repeated linear factors—the **Heaviside Shortcut**

$$\frac{x-2}{(x-5)(x-1)} \Longrightarrow \frac{x-2}{(x-5)(x-1)}=\frac{A}{x-5}+\frac{B}{x-1}$$

Calculate A, multiply $(x–5)$, let $x\to 5$:

$$\frac{x-2}{(x-5)(x-1)}(x-5)=\frac{A}{x-5}(x-5)+\frac{B}{x-1}(x-5)$$

$$\frac{x-2}{(x-1)}=A+\frac{B}{x-1}(x-5)$$

$$\Longrightarrow \frac{5-2}{5-1}=\frac{3}{4}=A$$

Techniques of integration

Partial Fractions

first degree numerator

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

irreducible quadratic factor

repeated root

The factor $x - 1$ is repeated three times in the denominator

$$\frac{x^2+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Since $x^3 + 4x = x(x^2 + 4)$, which can't be factored further, we write

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Integrals of Hyperbolic Functions

1. $\int \sinh x dx = \cosh x + C$

2. $\int \cosh x dx = \sinh x + C$

3. $\int \sec h^2 x dx = \tanh x + C$

4. $\int \cos \mathrm{ech}^2 \mathrm{x} \mathrm{dx} = -\coth \mathrm{x} + \mathrm{C}$

5. $\int \sec hx \tanh x dx = -\sec hx + C$

6. $\int \cos echx \coth x dx = -\cos echx + C$

Integrals of Hyperbolic Functions

$\cosh^2 t - \sinh^2 t = 1$

$1 - \tanh^2 t = \text{sech}^2\, t$

$\coth^2 t - 1 = \text{cosech}^2\, t$

$\cosh 2t = 2\cosh^2 t - 1$

$\qquad = 1 + 2\sinh^2 t$

$\sinh 2t = 2\sinh t \cosh t$

Exercise: Evaluate the following integrals

$$\int \sinh x \cosh x \, dx$$

$$\int \sqrt{\tanh x}\, \text{sec}h^2 x \, dx$$

$$\int x \cosh x \, dx$$

$$\int x^3 \cosh x \, dx$$

Integration of Inverse Trigonometric Functions

Differentiation	Integration
$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$	$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$	$\displaystyle\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$	$\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
$\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$	$\displaystyle\int \frac{-dx}{1+x^2} = \cot^{-1} x + C$
$(\sec^{-1} x) = \dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\displaystyle\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \sec^{-1} x + C$
$(\csc^{-1} x) = \dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\displaystyle\int \frac{-dx}{\lvert x\rvert\sqrt{x^2-1}} = \csc^{-1} x + C$

Integration of Inverse Trigonometric Functions

Exercise: Evaluate the following integrals

a) $\int_0^1 \tan^{-1} x \, dx$

b) $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$

c) $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} \, dx$

Let $x = au$, d$x = a$du

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{du}{\sqrt{1-u^2}}$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

Integration of Inverse Trigonometric Functions

Let $x = au$, $dx = a\,du$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{-dx}{a^2 + x^2} = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{-dx}{\lvert x\rvert\sqrt{x^2 - a^2}} = \frac{1}{a}\csc^{-1}\left(\frac{x}{a}\right) + C$

Exercise: Evaluate the following integrals

1. a) $\int \frac{dx}{\sqrt{16 - x^2}}$

   b) $\int \frac{2\,dx}{3 + x^2}$

2. a) $\int \frac{dx}{\sqrt{1 - 4x^2}}$

   b) $\int \frac{dx}{4 + 3x^2}$

$$\int \frac{dx}{x^2 - 2x + 10}$$

Integration of Inverse Hyperbolic Functions

Differentiation	Integration
$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\int \frac{dx}{1-x^2} = \tanh^{-1} x + C$

Integration of Inverse Hyperbolic Functions

Let $x=au$, $dx=a\,du$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{du}{\sqrt{1+u^2}}$$

$$= \sinh^{-1} u + C$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C, & \lvert x\rvert < a \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C, & \lvert x\rvert > a \end{cases}$
$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\sec h^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\cos ech^{-1}\left(\frac{x}{a}\right) + C$

Integration of Inverse Hyperbolic Functions

Exercise: Evaluate the following integrals

a) $\int \frac{dx}{\sqrt{3x^2+2}}$

2. Show that $\int \frac{x+1}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + \sinh^{-1} x + C$.

b) $\int \frac{dx}{x\sqrt{9-4x^2}}$

c) $\int \frac{dx}{\sqrt{2(x-3)^2+1}}$

$$\int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) + c$$

d) $\int \frac{dx}{\sqrt{x^2+4x+3}}$

Integration with Trigo substitution

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.
$\sin^2 x + \cos^2 x = 1$
$\sec^2 x = 1 + \tan^2 x$

E.g.

$$\int \sin^4 x \cos x \, dx \quad \rightarrow u = \sin x,\ du = \cos x \, dx \quad \rightarrow \int u^4 du = \frac{u^5}{5} = \frac{\sin^5 x}{5}$$

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int \sin^2 x \left(1 - \sin^2 x\right) \cos x \, dx \quad \rightarrow u = \sin x,\ du = \cos x \, dx$$

$$\int \sin^2 x \cos^2 x \, dx = \int \tfrac{1}{2}\left(1 - \cos 2x\right) \tfrac{1}{2}\left(1 + \cos 2x\right) dx$$

$$= \tfrac{1}{4} \int \left(1 - \cos^2 2x\right) dx = \tfrac{1}{8} \int \left(1 - \cos 4x\right) dx = \frac{x}{8} - \frac{\sin 4x}{32}$$

Integration with Trigo substitution

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|\cos x|$$

$\rightarrow u = \cos x$, d$u = -\sin x$ dx

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln|\sin x|$$

$\rightarrow u = \sin x$, d$u = \cos x$ dx

$$\int \sec x dx = \int \frac{\sec x(\tan x + \sec x)}{(\tan x + \sec x)} dx = \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} dx = \int \frac{du}{u} = \ln|\sec x + \tan x|$$

$\rightarrow u = \sec x + \tan x$, d$u = (\sec x \tan x + \sec^2 x)$ dx

$$\int \csc x dx = -\ln|\csc x + \cot x|$$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.
$\sin^2 x + \cos^2 x = 1$
$\sec^2 x = 1 + \tan^2 x$

$$\int \tan^m x \sec^n x \, dx$$

1) If m is odd, $\rightarrow u = \sec x$, d$u = \sec x \tan x$ dx
2) If n is even, $\rightarrow u = \tan x$, d$u = \sec^2 x$ dx

Integration with Trigo substitution

Exercise: Evaluate the following integrals

$$\int \sqrt{1-x^2}\,dx \qquad \to x = \sin t,\ \mathrm{d}x = \cos t\ \mathrm{d}t \qquad \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t$$

$$\int \sqrt{1-x^2}\,dx = \int \cos t \cos t\ dt = \int \cos^2 t\ dt$$

$$= \int \tfrac{1}{2}(1+\cos 2t)\ dt = \frac{t}{2} + \frac{2\sin t \cos t}{4} = \frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2}$$

expression	substitution	identity
$a^2 - u^2$	$u = a \sin t$	$1 - \sin^2 t = \cos^2 t$
$a^2 + u^2$	$u = a \tan t$	$1 + \tan^2 t = \sec^2 t$
$u^2 - a^2$	$u = a \sec t$	$\sec^2 t - 1 = \tan^2 t$

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1} = \tfrac{1}{2}\int \frac{du}{u^2+1} = \tfrac{1}{2}\int \frac{\sec^2 \theta d\theta}{\sec^2 \theta}$$

Dif: $\sec t$, Int: $\sec^2 t$

$\tan^2 t = \sec^2 t - 1$

$$I = \int \sqrt{x^2-1}\,dx \qquad I = \int \tan^2 t \sec t\, dt = \tan t \sec t - \int \sec^3 t\, dt$$

$\to x = \sec t$, $\mathrm{d}x = \sec t \tan t\ \mathrm{d}t$

Integration with hyperbolic substitution

$\cosh^2 t - \sinh^2 t = 1$
$1 - \tanh^2 t = \text{sech}^2\, t$
$\coth^2 t - 1 = \text{cosech}^2\, t$
$\cosh 2t = 2\cosh^2 t - 1$
$\quad = 1 + 2\sinh^2 t$
$\sinh 2t = 2\sinh t \cosh t$

$$\int \frac{dt}{\cosh t} = 2\int \frac{ds}{s^2+1} = 2\tan^{-1}\left(e^t\right)$$

$\to s = e^t$, $ds = e^t\, dt$

$$\int \frac{\tanh(x/2)dx}{\cosh x} = 2\int \frac{\left(e^x - 1\right)e^x dx}{\left(e^x+1\right)\left(e^{2x}+1\right)} = 2\int \frac{(1-t)dt}{(t+1)\left(t^2+1\right)}$$

$\to t = e^{-x}$, $dt = -e^{-x}\, dx$

Solving using partial fraction!

$$I = \int \sqrt{x^2-1}\, dx$$

$\to x = \cosh t$, $dx = \sinh t\, dt$, $\sinh t = \sqrt{(x^2-1)}$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$I = \int \sqrt{\sinh^2 t}\, \sinh t\, dt = \int \frac{\cosh 2t - 1}{2} dt = \frac{\sinh 2t}{4} - \frac{t}{2}$$

$$= \frac{\sinh t \cosh t}{2} - \frac{t}{2} = \frac{x\sqrt{x^2-1}}{2} - \frac{\cosh^{-1} x}{2}$$

Integration with hyperbolic substitution

expression	substitution	identity
$a^2 - u^2$	$u = a \sin t$	$1 - \sin^2 t = \cos^2 t$
$a^2 + u^2$	$u = a \tan t$	$1 + \tan^2 t = \sec^2 t$
$u^2 - a^2$	$u = a \sec t$	$\sec^2 t - 1 = \tan^2 t$

2 and 3 longer Calculation!

We can also use other substitution to eliminate the square root. For example, the hyperbolic substitution

expression	substitution
$a^2 + u^2$	$u = a \sinh t$
$u^2 - a^2$	$u = a \cosh t$

For example, find $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}dx$

$\rightarrow x = a \sinh t$, $dx = a \cosh t\, dt$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{a\cosh t}{a\cosh t}dt = \int dt = t + C$$

$$= ar\sinh\frac{x}{a} + C = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

$\cosh^2 t - \sinh^2 t = 1$

$1 - \tanh^2 t = \text{sech}^2\, t$

$\coth^2 t - 1 = \text{cosech}^2\, t$

$\cosh 2t = 2\cosh^2 t - 1$

$\quad = 1 + 2\sinh^2 t$

$\sinh 2t = 2 \sinh t \cosh t$

Further Applications of Integration

Arc Length in Cartesian Form

The length of the curve *(x(t), y(t)) as t varies from t_0 t to t_1 is given*

Let r=*x*i+*y*j

$$L = \int_C ds = \int_{t_0}^{t_1} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

$$\int_C ds = \int_C \left|\frac{d\mathbf{r}}{dt}\right| dt \qquad \left|\frac{d\mathbf{r}}{dt}\right| = \left|\langle x'(t), y'(t)\rangle\right| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$$

Example: given $x(t)=\cos t$, $y(t) = \sin t$, $0 \le t \le \pi$. Find the length of curve

Solution: length of half circle is π.

For curve in the form: $y = f(x),\ a \le x \le b$.

Let $x(t) = t$, $y(t)=f(x)=f(t)$,
We get $x'(t)=1$, $y'(t) = f'(t)$

$$L = \int_{t_0}^{t_1} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt = \int_{t_0}^{t_1} \sqrt{1 + \left(f'(t)\right)^2}\, dt = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example: Find the arc length of

$$y = \frac{1}{3}\left(x^2 + 2\right)^{3/2},\ 0 \le x \le 3.$$

Answer: **12**

Further Applications of Integration

Arc Length in Polar Coordinate

The length of the curve $L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

the length of a polar curve $r = f(\theta)$, $a \le \theta \le b$,
The parametric equations:

$\boldsymbol{x = r\cos\theta = f(\theta)\cos\theta}$

$\boldsymbol{y = r\sin\theta = f(\theta)\sin\theta}$

Using the Product Rule

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

using $\cos^2\theta + \sin^2\theta = 1$,we have

$$\Rightarrow \quad L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Further Applications of Integration

Arc Length in Polar Coordinate

The length of curve with polar equation $\boldsymbol{r = f(\theta)}$, a≤θ≤b. is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$1 + \sinh^2 t = \cosh^2 t$

$\cosh 2t = 2\cosh^2 t - 1$

$= 1 + 2\sinh^2 t$

E.g. Find the length of curve $\boldsymbol{r = \theta}$, $0 \le \theta \le 1$.

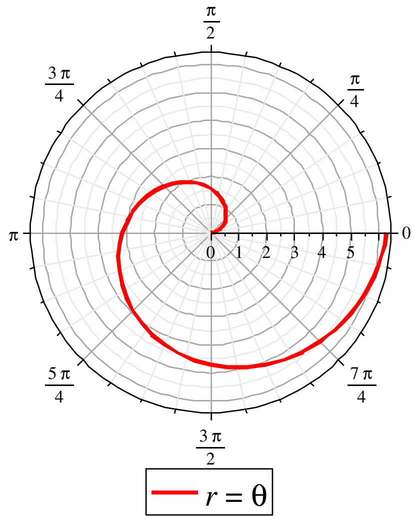
$\theta = \sinh t$

$d\theta/dt = \cosh t$

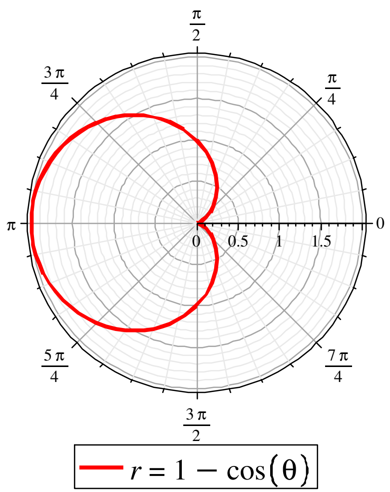
$$L = \int_0^1 \sqrt{\theta^2 + (1)^2}\, d\theta$$

Solution: $\frac{1}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$

$$L = \int_0^1 \sqrt{\sinh^2 t + (1)^2}\, \cosh t\, dt = \int_0^1 \cosh^2 t\, dt$$



E.g. Find the length of cardioid $\boldsymbol{r = 1 - \cos\theta}$, $0 \le \theta \le 2\pi$.



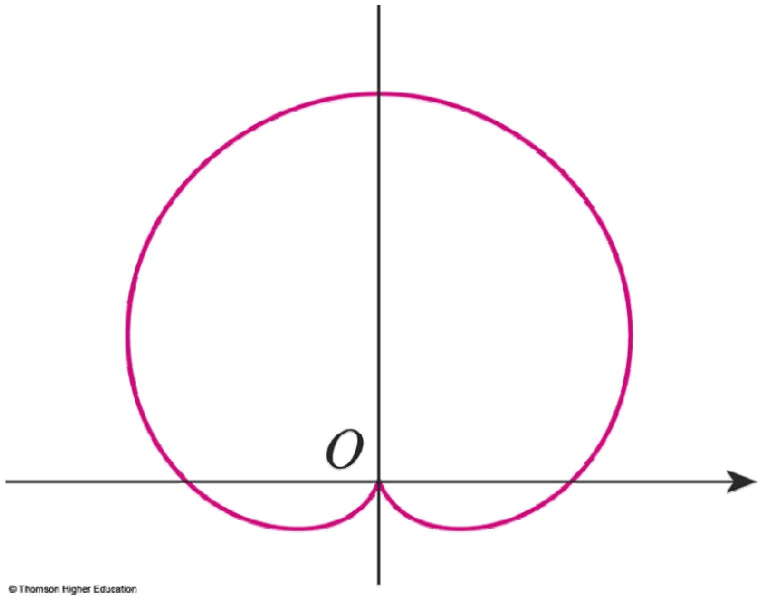
Solution: 8

Further Applications of Integration

Arc Length in Polar Coordinate

E.g: Find the length of the cardioid

$$r = 1 + \sin\theta$$



$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta$$

$$L = \int_0^{2\pi} \frac{\sqrt{2+2\sin\theta}\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\, d\theta = \int_0^{2\pi} \frac{\sqrt{2^2 - 2^2\sin^2\theta}}{\sqrt{2-2\sin\theta}}\, d\theta$$

$$= \int_0^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta$$

$u = 2 - 2\sin\theta$

$du/d\theta = -2\cos\theta$

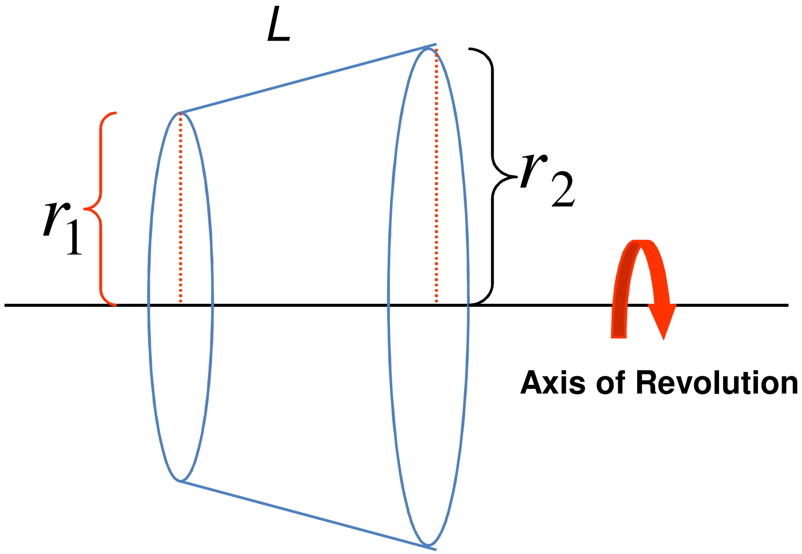
Solution: 8

Further Applications of Integration

area of surface of revolution in Cartesian form.

Definition:

If the graph of a continuous function is revolved about a line, the resulting surface is called a surface of revolution



Surface Area of a Frustum of a Cone:

$$S = 2\pi r L \quad \text{where} \quad L = \text{length of line segment}$$

$$r = \text{average radius} = \frac{1}{2}(r_1 + r_2)$$

Surface Area:

$$S = \int 2\pi y\,ds$$ If the curve is rotated about the x-axis

$$S = \int 2\pi x\,ds$$ If the curve is rotated about the y-axis

Use

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad \text{when} \quad y = f(x)$$

or

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \quad \text{when} \quad x = g(y)$$

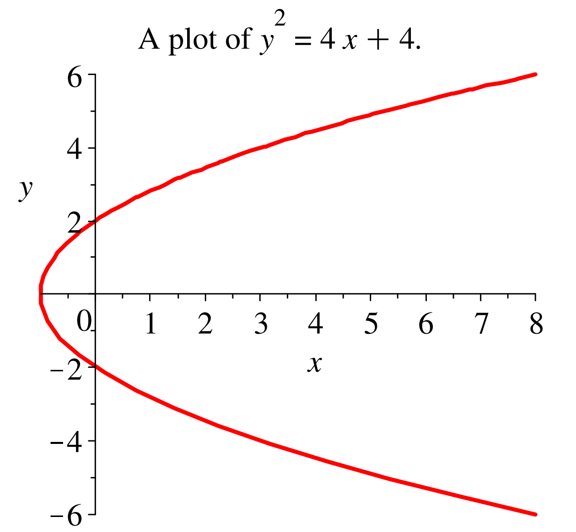
Further Applications of Integration

area of surface of revolution in Cartesian form.

Examples:

Find the area of the surface obtained by rotating the curve about the x-axis:

$$y^2 = 4x + 4,\ 0 \le x \le 8$$



Solutions:

We have $S = \int 2\pi y\, ds$

Now, $y^2 = 4x + 4$

$$\Rightarrow x = \frac{y^2 - 4}{4}$$

Use $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{4}(2y) = \frac{y}{2} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4} = \frac{4 + y^2}{4}$$

Upper Limit : when $x = 8 \Rightarrow y = 6$

Lower Limit : when $x = 0 \Rightarrow y = 2$

$$\therefore S = 2\pi \int_2^6 y\sqrt{\frac{4 + y^2}{4}}\, dy = \pi \int_2^6 y\left(4 + y^2\right)^{1/2} dy = \left[\frac{\pi}{2} \cdot \frac{2}{3}\left(4 + y^2\right)^{3/2}\right]_2^6$$

$$= \frac{\pi}{3}\left(40^{3/2} - 8^{3/2}\right) = \frac{16\pi}{3}\left(5\sqrt{10} - \sqrt{2}\right)$$

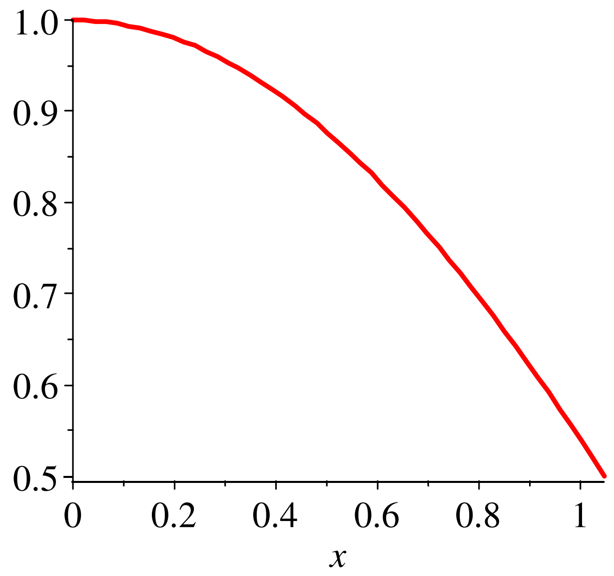
Further Applications of Integration

Area of surface of revolution in Cartesian form.

E.g: $y = \cos x$, $0 \le x \le \pi/3$, rotating about the x-axis.

We have $S = \int 2\pi y\,ds$

Solutions:



Use $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(-\sin x)^2}\,dx = \sqrt{1+\sin^2 x}\,dx$

$$\therefore S = 2\pi\int_0^{\pi/3}\cos x\sqrt{1+\sin^2 x}\,dx$$

let $u = \sin x$ $\qquad$ Up per Limit : $u = \frac{\sqrt{3}}{2}$

$\Rightarrow du = \cos x\,dx$ $\qquad$ Lower Limit : $u = 0$

$$S = 2\pi\int_0^{\sqrt{3}/2}\sqrt{1+u^2}\,du$$

let $v = \tan\theta,\ -\pi/2 < \theta < \pi/2$

$\Rightarrow dv = \sec^2\theta\,d\theta$

$$S = 2\pi\int_0^{Arc\tan(\sqrt{3}/2)}\sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta = 2\pi\int_0^{Arc\tan(\sqrt{3}/2)}\sec^3\theta\,d\theta \text{ Integratio n by Parts}$$

$$= 2\pi\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta+\tan\theta)\right]_0^{Arc\tan(\sqrt{3}/2)} = \pi\left[\frac{\sqrt{7}}{2}\cdot\frac{\sqrt{3}}{2} + \ln\left(\frac{\sqrt{7}}{2}+\frac{\sqrt{3}}{2}\right)\right]$$

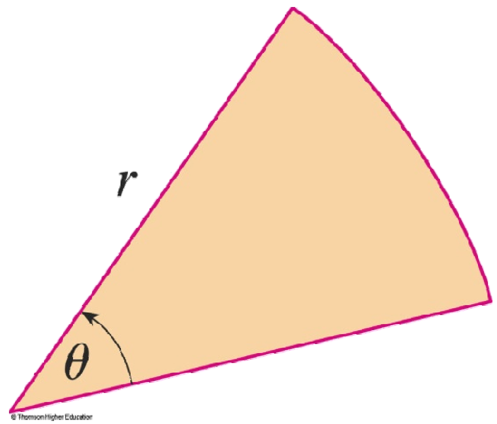
$$= \pi\left[\frac{\sqrt{21}}{4} + \ln\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)\right]$$

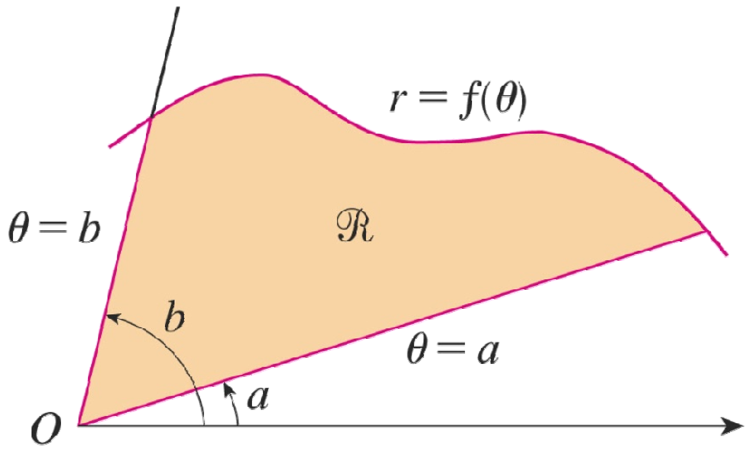
Not in syllabus!

Further Applications of Integration
Area in Polar form.

We need to use the formula for the area of a sector of a circle

$$A = \tfrac{1}{2} r^2 \theta$$

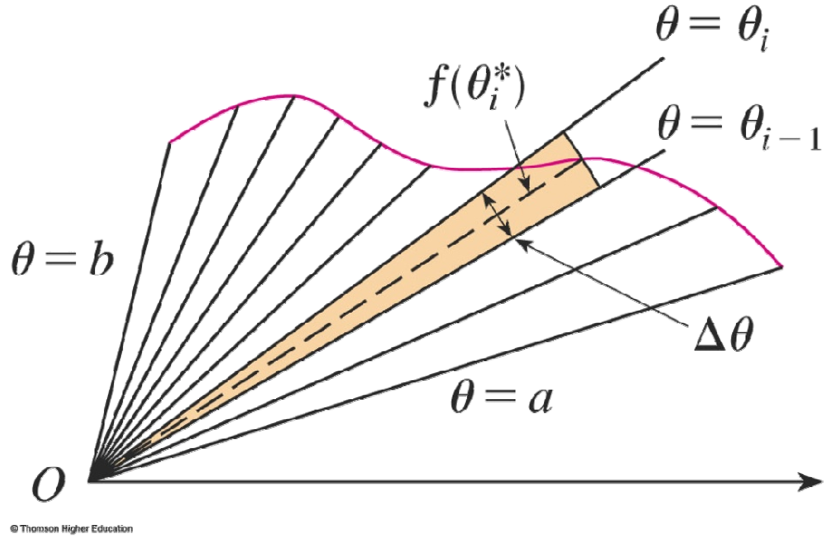




Let R be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where: f is a positive continuous function.

We divide the interval [a, b] into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$, and equal width $\Delta\theta$.

Then, the rays $\theta = \theta_i$ divide R into smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$.



So, an approximation to the total area A of R is

$$A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$$

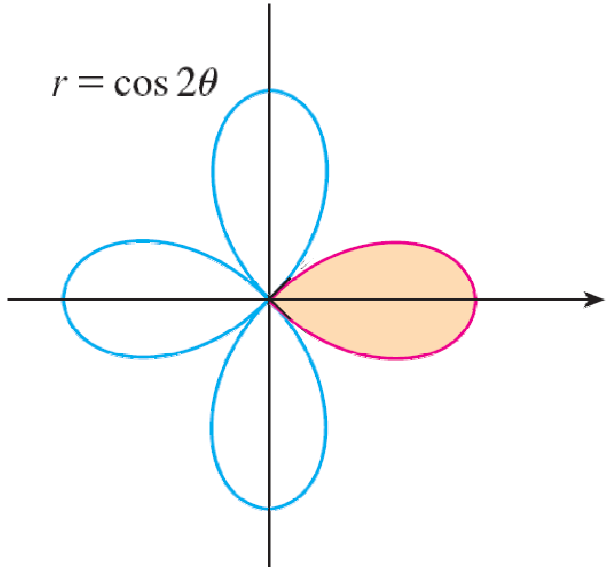
$$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

Not in syllabus!

Further Applications of Integration

Area in Polar form. curve $r = f(\theta)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}[f(\theta_i^*)]^2\Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$



Eg.: Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$$A = \int_{-\pi/4}^{\pi/4}\tfrac{1}{2}r^2\,d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta$$

$$= \int_0^{\pi/4}\cos^2 2\theta\,d\theta$$

$$= \int_0^{\pi/4}\tfrac{1}{2}(1+\cos 4\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}$$

Further Applications of Integration

Area of surface of revolution in parametric equations.

If a smooth curve C given by $\boldsymbol{x=f(t), y=g(t)}$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by:

$$\int_C ds = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$S = \int 2\pi y\, ds \rightarrow \quad S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

⇒ Revolution about x-axes, $\boldsymbol{g(t) \ge 0}$

$$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

⇒ Revolution about y-axes, $\boldsymbol{f(t) \ge 0}$

E.g: $x = a \cos t$, $y = a \sin t$, $0 \le t \le \pi$, rotating about the x-axis.

$$S = 2\pi \int_0^{\pi} a \sin(t) \sqrt{(-a \sin t)^2 + (a \cos t)^2}\, dt$$

$$= 2\pi a^2 \int_0^{\pi} \sin t\, dt = 2\pi a^2 [-\cos t]_0^{\pi} = 4\pi a^2$$

Surface of revolution is equal to sphere surface.

Further Applications of Integration

Area of surface of revolution in polar form.

If f be a function whose derivative is continuous on an interval $a \le \theta \le b$, then the area S of the surface of revolution formed by revolving $\boldsymbol{r = f(\theta)}$ from $\theta = a$ to $\theta = b$ is given by:

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$S = \int 2\pi y\,ds \qquad S = 2\pi\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$S = 2\pi\int_a^b f(\theta)\sin\theta\sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\,d\theta$$ ⟹ Revolution about polar axis

$$S = 2\pi\int_a^b f(\theta)\cos\theta\sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\,d\theta$$ ⟹ Revolution about the line $\theta = \pi/2$

E.g: $r = a = f$, rotating about the polar axis.

$$S = 2\pi\int_0^\pi a\sin\theta\sqrt{(a)^2 + (0)^2}\,d\theta = 2\pi a^2\int_0^\pi \sin\theta\,d\theta$$
$$= 4\pi a^2$$

Improper Integrals

If both the numerator and the denominator are finite at a and , then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}.$$

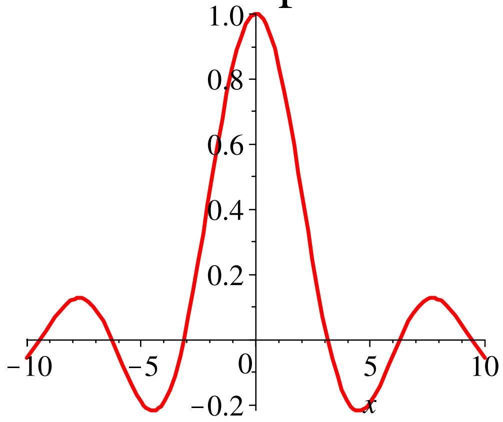
Example 1

$$\lim_{x\to 3}\frac{x^2+1}{x+2}=\frac{10}{5}=2.$$

L'Hopital Rule

If you are doing any limit and you get something in the **form 0/0 or ∞/∞**, then you should probably try to use L'Hopital rule. The basic idea of L'Hospital rule is simple. Consider the limit

If $\lim\frac{f'(x)}{g'(x)}=L$ Then $\rightarrow$ $\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}=L.$



A plot of $\frac{\sin(x)}{x}$.

L'Hopital Rule for 0/0

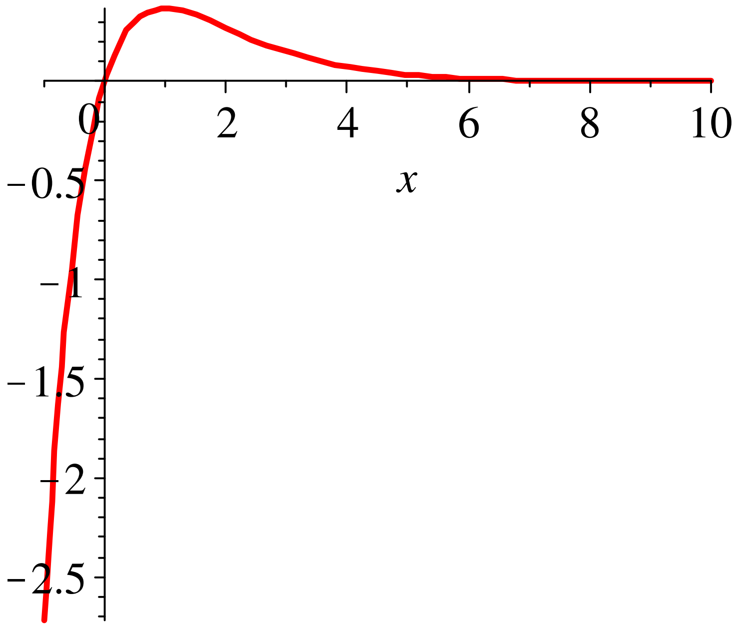
Example: find $\lim_{x\to 0}\frac{\sin x}{x}$ $\lim_{x\to 1}\frac{2\ln x}{x-1}$ $\lim_{x\to 0}\frac{e^x-1}{x^2}$

Improper Integrals

L'Hopital Rule

L'Hopital Rule for ∞/∞

Example: find $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$ $\lim\limits_{x\to\infty}\dfrac{\ln\left(\ln x^{1000}\right)}{\ln x}$



A plot of $\dfrac{x}{e^x}$.

$$\lim_{x\to\infty} xe^{-x} \neq \infty\cdot 0$$

$$= \lim_{x\to\infty}\frac{x}{e^x} = \frac{\infty}{\infty}$$

Improper Integrals

The definite integral $$\int_a^b f(x)\,dx$$

is known as improper integral if either

1) one or both limits are infinite, or
2) $f(x)$ is undefined at certain points on/in the interval.

Note: We called case: 1) as Type I, 2) as Type II

Improper Integral Type 1

(1) If f(x) is continuous in the interval $[a,\infty)$

$$\Rightarrow \quad \int_a^\infty f(x)\ dx = \lim_{T\to\infty} \int_a^T f(x)\ dx.$$

Improper Integrals

Improper Integral Type 1

(2) If $f(x)$ is continuous in the interval (-∞, b]

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{T\to-\infty} \int_{T}^{b} f(x)\,dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist

(3) If $f(x)$ is continuous in the interval (-∞,∞),

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$ with any real number c

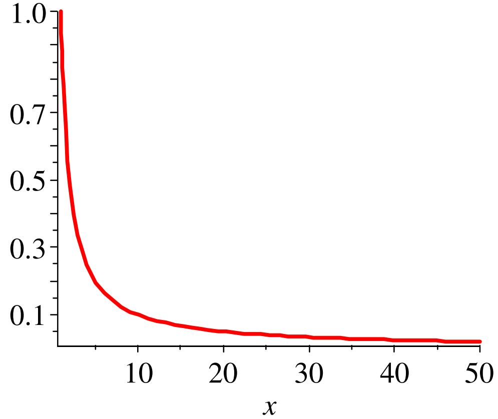
Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges

Improper Integrals

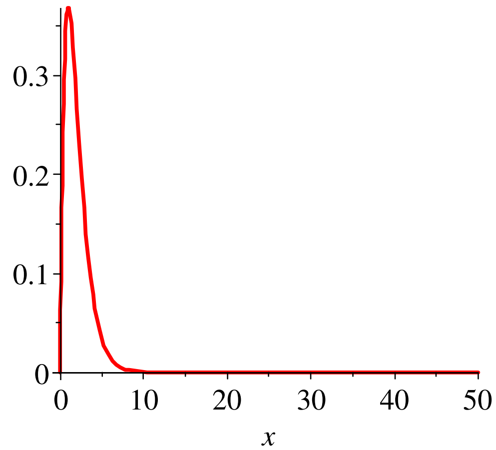
Improper Integral Type 1

E.g.: Determine whether the following integral are convergent or divergent

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{T\to\infty}\int_1^T \frac{1}{x}\,dx$$



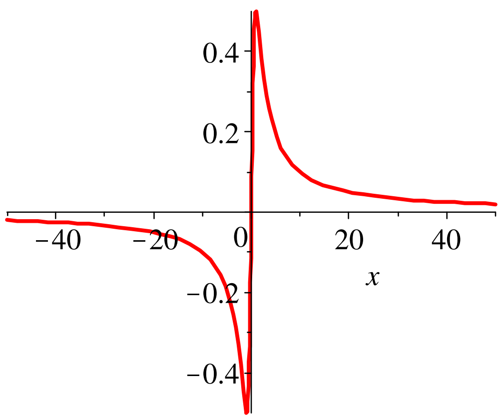
A plot of $\frac{1}{x}$.



A plot of $x\,e^{-x}$.

$$\int_0^\infty xe^{-x}\,dx$$

Int = – (1+x) e^{-x}



A plot of $\frac{x}{1+x^2}$.

$$\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx$$

Int = ½ ln(1+x^2)

Improper Integrals

Improper Integral Type 1

Example : For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Example : Determine whether the below integral is convergent or divergent?

$$\int_0^\infty e^{-2x}\,dx \qquad \int_{-\infty}^2 \frac{1}{5-2x}\,dx$$

Improper Integrals

Improper Integral Type 2

(1) If $f(x)$ is continuous on $[a,b)$, and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{T\to b^-} \int_a^T f(x)\,dx.$$

(2) If $f(x)$ is continuous on $(a,b]$, and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{T\to a^+} \int_T^b f(x)\,dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist

Improper Integrals

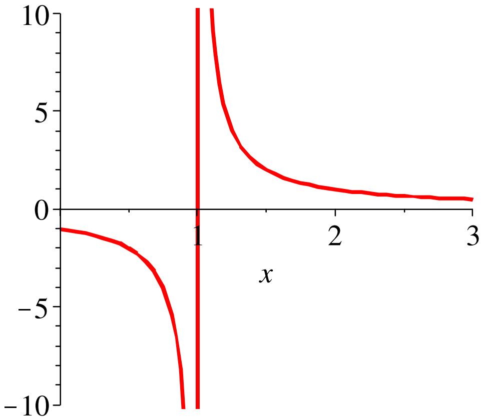
Improper Integral Type 2

(3) If *f(x)* has discontinuity at *c*, where $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note: the improper integrals in 3) is said to *converge* if **both terms converge** and *diverge* if **either term diverges**

E.g.: Determine whether it converge or diverge



A plot of $\frac{1}{x-1}$.

$$\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{T\to 9^-} \int_1^T \frac{1}{\sqrt[3]{x-9}}\,dx$$

$$\text{int} = \frac{3}{2}(x-9)^{2/3}$$

E.g.: Find this improper integral if possible

$$\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

$$= \lim_{a\to 1^{-1}} \int_0^a \frac{1}{x-1}\,dx + \lim_{b\to 1^{+1}} \int_b^3 \frac{1}{x-1}\,dx$$

$-\infty + \infty$ = undefined

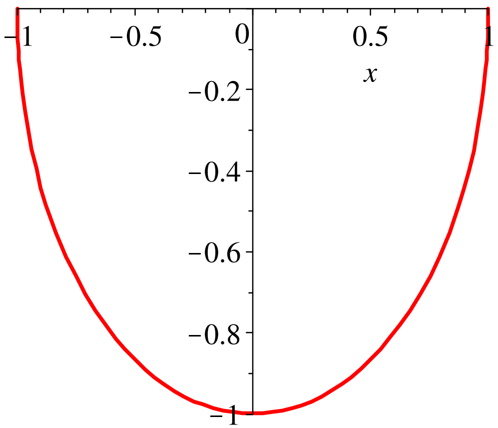
= diverges

Improper Integrals

Improper Integral Type 2

E.g.: Determine whether it converge or diverge

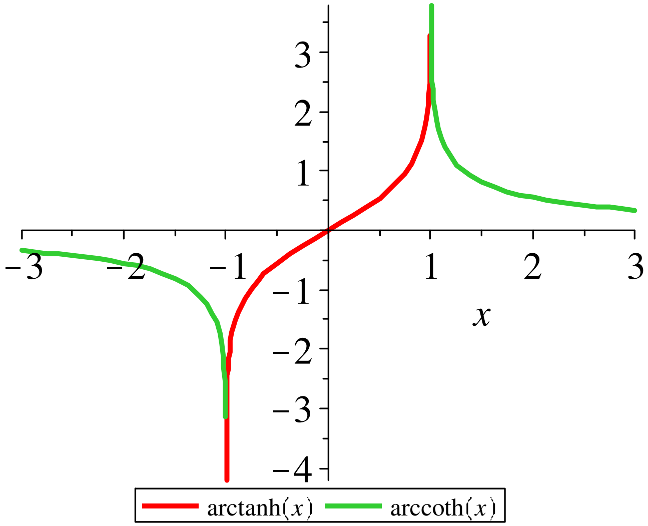
$$I = \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx \qquad Int = -\sqrt{1-x^2}$$



A plot of $-\sqrt{1-x^2}$.

$$I = \int_1^2 \frac{1}{1-x^2}\,dx \qquad Int = \begin{cases} \tanh^{-1}(x), & |x| < 1 \\ \coth^{-1}(x), & |x| > 1 \end{cases}$$

A plot of integral of $\frac{1}{1-x^2}$.



$$I = \int_{-1}^1 \ln x\,dx$$

$$\text{Int} = x\ln(x) - x$$