Mathematical Methods III
SSCM 2043

Part 1 – Fourier Series
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Periodic Functions

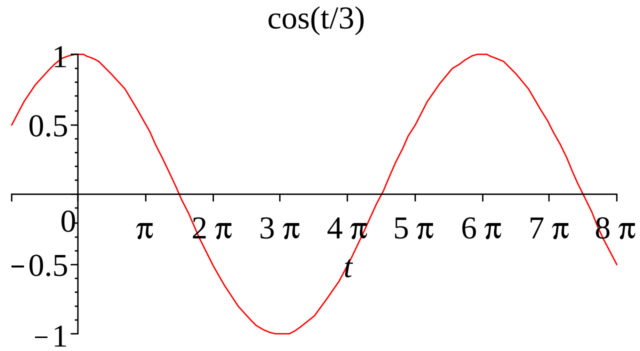
Any function that satisfies

$$f(t) = f(t+T)$$

where *T* is a constant and is called the *period* of the function.

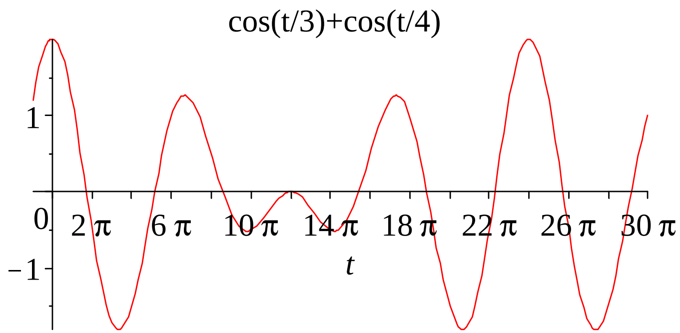
$$f(t) = \cos\frac{t}{3} + \cos\frac{t}{4}$$

Example: Find its period



$$f(t) = f(t+T)$$

$$\Rightarrow \quad \cos\frac{t}{3} + \cos\frac{t}{4} = \cos\frac{1}{3}(t+T) + \cos\frac{1}{4}(t+T)$$



Fact: $\cos\theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi \quad \Rightarrow \quad T = 6m\pi$$

$$\frac{T}{4} = 2n\pi \quad \Rightarrow \quad T = 8n\pi$$

$$\Rightarrow \quad T = 24\pi$$ **smallest *T*** ***(for both)***

Periodic Functions

Example:

$$f(t) = \cos \omega_1 t + \cos \omega_2 t$$

Find its period.

$$f(t) = f(t+T) \Rightarrow \cos \omega_1 t + \cos \omega_2 t = \cos \omega_1 (t+T) + \cos \omega_2 (t+T)$$

$$\omega_1 T = 2m\pi$$

$$\omega_2 T = 2n\pi$$

$$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{m}{n} \Rightarrow \frac{\omega_1}{\omega_2}$$

must be a rational number

$$f(t) = \cos 10t + \cos(10+\pi)t$$

Is this function a periodic function? ✗

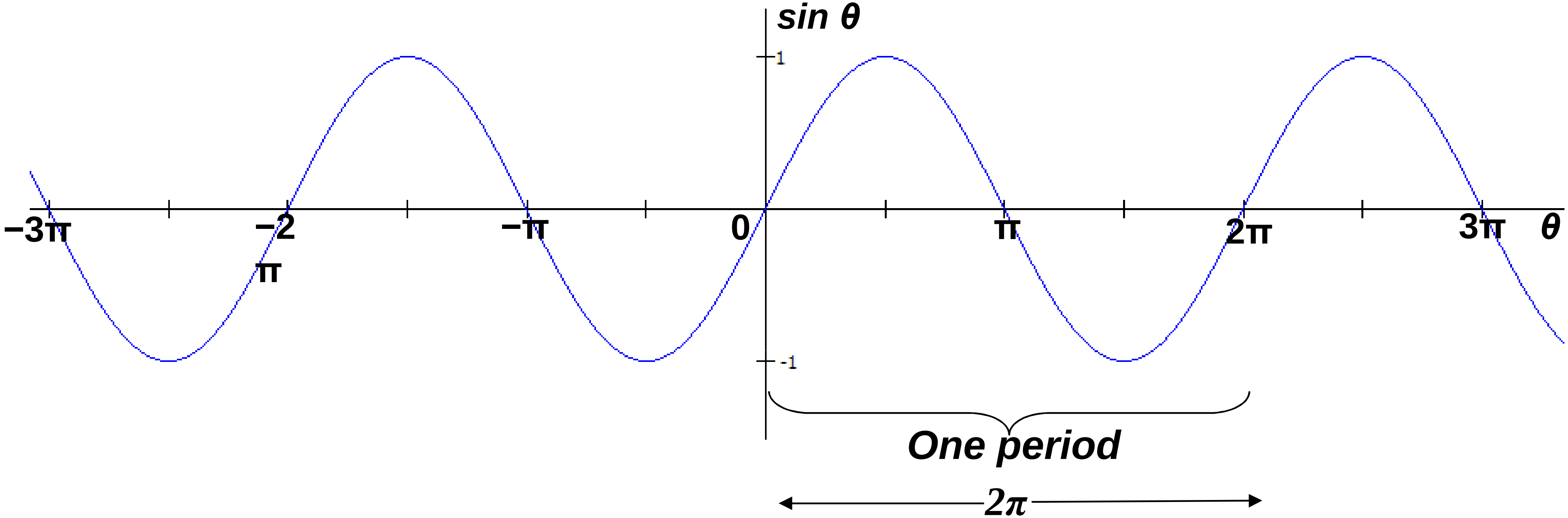
$$\frac{\omega_1}{\omega_2} = \frac{10}{10+\pi}$$

not a rational number

$$\cos \omega(t+T) = \cos \omega t \cos \omega T - \sin \omega t \sin \omega T$$

cos(A+B)=cosAcosB–sinAsinB

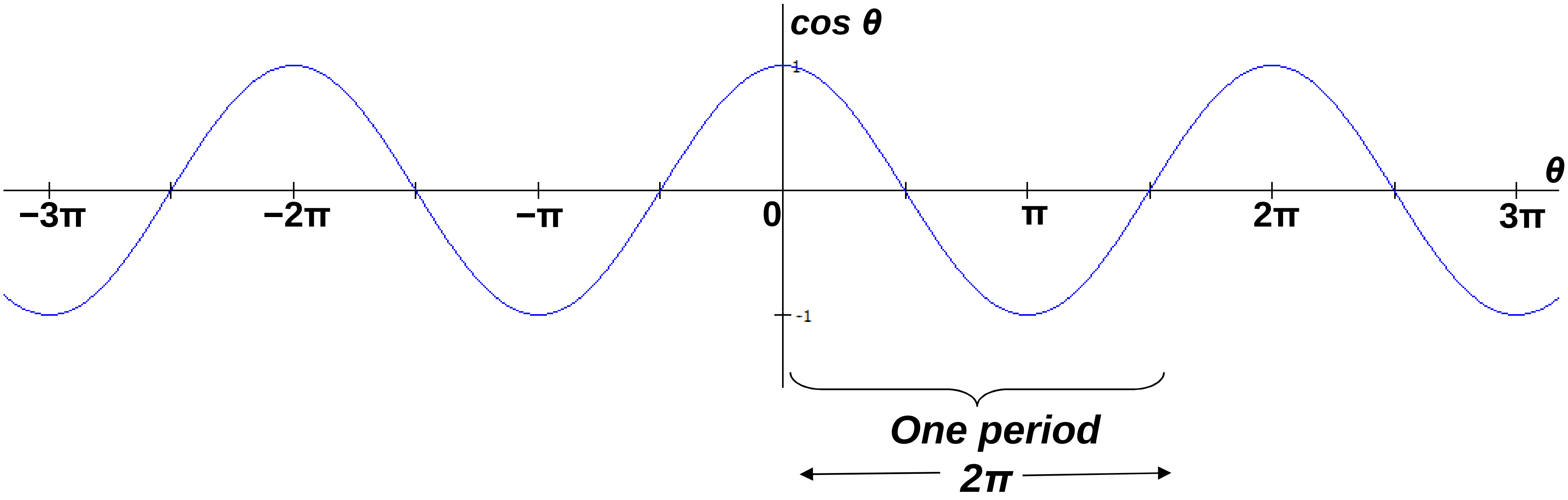
Odd Functions



sin θ* is an odd function*; it is symmetric wrt the origin.** ***f*(-*t*)= - *f*(*t*)**

$$sin(-\theta) = -sin(\theta)$$

Even Functions



$\cos\theta$ **is an even function; it is symmetric wrt the y-axis. : ->** $f(-t) = f(t)$

$$\cos(-\theta) = \cos(\theta)$$

Fourier Series

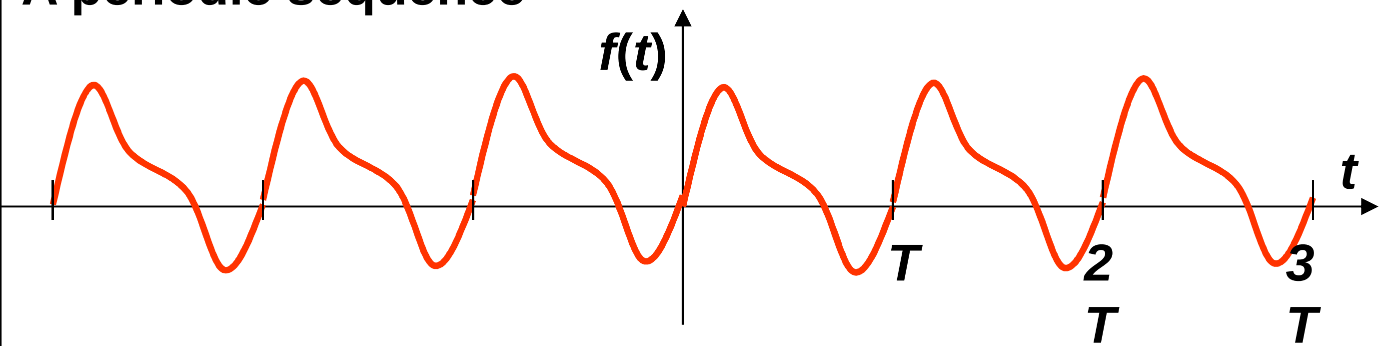
$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}}_{\text{Odd Part}}$$

***T* is a period of all the above signals**

Let $\omega_0 = 2\pi/T$; $T = 2l$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

A periodic sequence



Orthogonal Functions

- Call a set of functions $\{\phi_k\}$ *orthogonal* on an interval $a < t < b$ if it satisfies

$$\int_a^b \phi_m(t)\phi_n(t)dt = \begin{cases} 0 & m \neq n \\ r_n & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi / T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Prove this!

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Proof

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt$$

m ≠ n

$\omega_0=2\pi/T;\;\; T=2l$

$$=\frac{1}{2}\int_{-T/2}^{T/2}\cos[(m+n)\omega_0 t]dt+\frac{1}{2}\int_{-T/2}^{T/2}\cos[(m-n)\omega_0 t]dt$$

$$=\frac{1}{2}\frac{1}{(m+n)\omega_0}\sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2}+\frac{1}{2}\frac{1}{(m-n)\omega_0}\sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}$$

$$=\frac{1}{2}\frac{1}{(m+n)\omega_0}2\underbrace{\sin[(m+n)\pi]}_{0}+\frac{1}{2}\frac{1}{(m+n)\omega_0}2\underbrace{\sin[(m-n)\pi]}_{0}=0$$

$$\begin{Bmatrix} 1, \\ \cos\omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \cdots \\ \sin\omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \cdots \end{Bmatrix}$$

is orthogonal set.

Decomposition

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega_0 t)+\sum_{n=1}^{\infty}b_n\sin(n\omega_0 t)$$

$\omega_0 = 2\pi/T.$

$T = 2l$

$$a_0=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)dt=\frac{1}{l}\int_{-l}^{l}f(t)dt$$

$$a_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\cos n\omega_0 tdt=\frac{1}{l}\int_{-l}^{l}f(t)\cos n\omega_0 tdt,\quad n=1,2,\cdots$$

$$b_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\sin n\omega_0 tdt=\frac{1}{l}\int_{-l}^{l}f(t)\sin n\omega_0 tdt,\qquad n=1,2,\cdots$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi t}{l}\right)$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)dt=0,\qquad m\neq 0$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt=\begin{cases}0 & m\neq n\\ T/2 & m=n\end{cases}$$

$$\int_{-T/2}^{T/2}\sin(m\omega_0 t)dt=0,\qquad m\neq 0$$

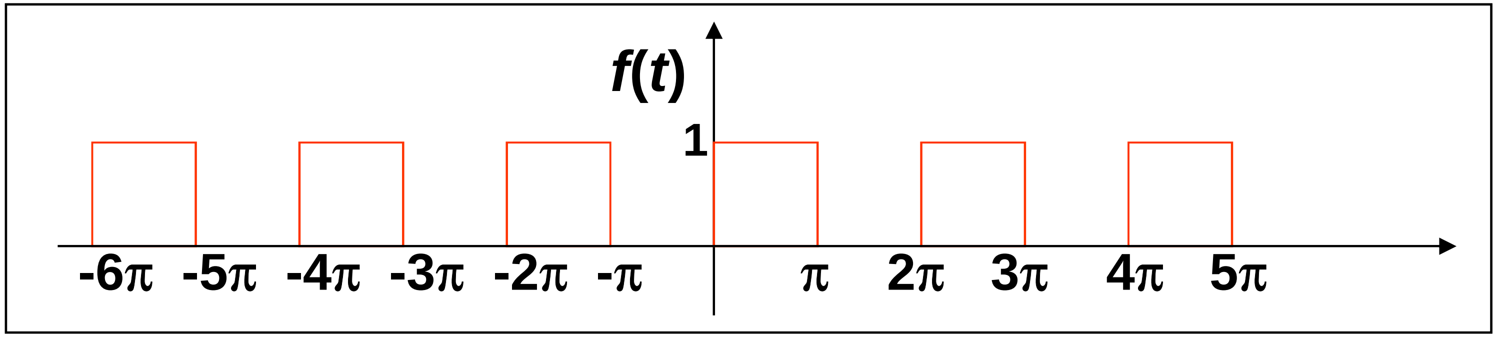
$$\int_{-T/2}^{T/2}\sin(m\omega_0 t)\cos(n\omega_0 t)dt=0,\quad \text{for all } m \text{ and } n$$

$$\int_{-T/2}^{T/2}\sin(m\omega_0 t)\sin(n\omega_0 t)dt=\begin{cases}0 & m\neq n\\ T/2 & m=n\end{cases}$$

Example (Square Wave)

$$f(t)=\begin{cases}1, & 0<t<\pi \\ 0, & \pi<t<2\pi\end{cases}$$

$$f(t)=f(t+2\pi)$$

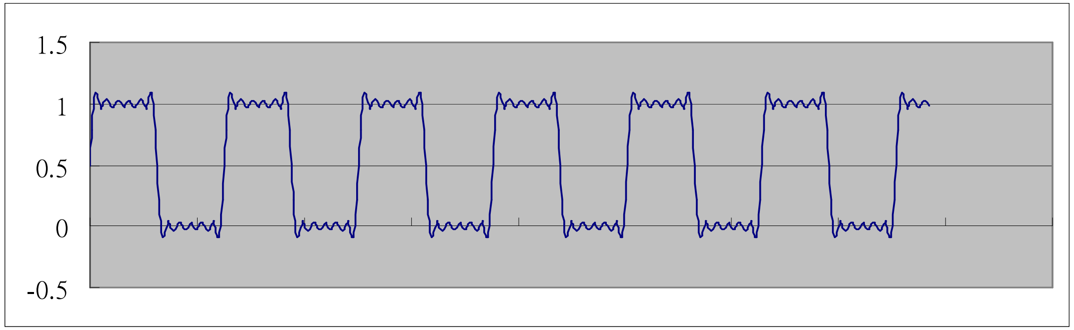


$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega_0 t)+\sum_{n=1}^{\infty}b_n\sin(n\omega_0 t)$$

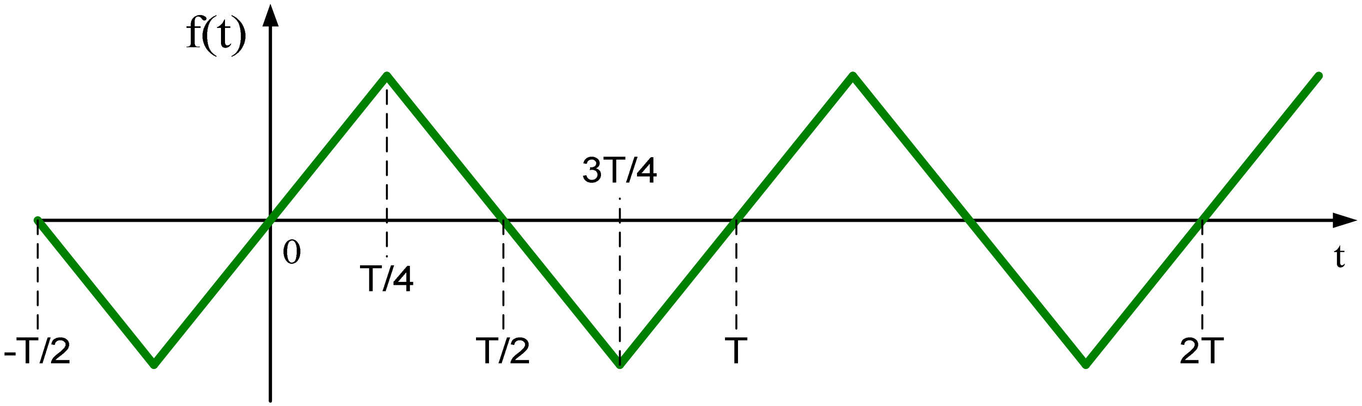
$$a_0=\frac{2}{2\pi}\int_0^{\pi}1dt=1 \qquad a_n=\frac{2}{2\pi}\int_0^{\pi}\cos nt dt=\frac{1}{n\pi}\sin nt\Big|_0^{\pi}=0 \quad n=1,2,\cdots$$

$$b_n=\frac{2}{2\pi}\int_0^{\pi}\sin nt dt=-\frac{1}{n\pi}\cos nt\Big|_0^{\pi}=-\frac{1}{n\pi}(\cos n\pi-1)=\begin{cases}2/n\pi & n=1,3,5,\cdots \\ 0 & n=2,4,6,\cdots\end{cases}$$

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\left(\sin t+\frac{1}{3}\sin 3t+\frac{1}{5}\sin 5t+\cdots\right)$$



Example Find the Fourier series of the following periodic function.



$$f(t)=\begin{cases} t, & -\frac{T}{4}\le t\le\frac{T}{4} \\ -t+\frac{T}{2}, & \frac{T}{4}\le t\le\frac{3T}{4}\end{cases}$$

$$f(t+T)=f(t)$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

This is an odd function. Therefore, a_n=0

$\omega_0=2\pi/T.$

$T=2l$

$$a_0=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)dt=\frac{1}{l}\int_{-l}^{l}f(t)dt$$

$$a_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\cos n\omega_0 t dt$$

$$b_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\sin n\omega_0 t dt$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$=\frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{4}} t\sin\left(\frac{2\pi n}{T}t\right)dt + \frac{4}{T}\int_{\frac{T}{4}}^{\frac{T}{2}}\left(-t+\frac{T}{2}\right)\sin\left(\frac{2\pi n}{T}t\right)dt$$

Use integration by parts.

$$\boldsymbol{b_n = \frac{4}{T}\left[2.\left(\frac{T}{2\pi n}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right] = \frac{2T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)}$$

b_n=0, when n is even.

Therefore, the Fourier series is

$$\frac{2T}{\pi^2}\left[\sin\left(\frac{2\pi}{T}t\right) - \frac{1}{3^2}\sin\left(\frac{6\pi}{T}t\right) + \frac{1}{5^2}\sin\left(\frac{10\pi}{T}t\right) - \cdots\right]$$

Example of integration by parts.

$$\int_{x=a}^{x=b} x^2 \sin(nx)dx$$

differentiate	Integrate
x^2 +1	$\sin nx$
$2x$ -1	$\left(-\frac{1}{n}\right)\cos nx$
2 +1	$\left(-\frac{1}{n^2}\right)\sin nx$
0	$\left(\frac{1}{n^3}\right)\cos nx$

$$\int_{x=a}^{x=b} x^2 \sin(nx)dx = \left[x^2\left(-\frac{1}{n}\right)\cos nx + 2x(-1)\left(-\frac{1}{n^2}\right)\sin nx + 2\left(\frac{1}{n^3}\right)\cos nx\right]_{x=a}^{x=b}$$

Harmonics

Define $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$ **, called the *fundamental angular frequency*.**

Define $\omega_n = n\omega_0$ **, called the *n-th harmonic* of the periodic function.**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

Complex Form of the Fourier Series

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j\sin n\omega_0 t \qquad e^{-jn\omega_0 t} = \cos n\omega_0 t - j\sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) \quad \sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

$\omega_0 = 2\pi/T$.

$T = 2l$

$$e^{-i\pi} + 1 = 0$$

Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$\omega_0 = 2\pi/T$.

$T = 2l$

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt - j\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$c_0=\frac{a_0}{2}$$

$$c_n=\frac{1}{2}(a_n-jb_n)$$

$$c_{-n}=\frac{1}{2}(a_n+jb_n)$$

If *f*(*t*) is real,

$$\Rightarrow \quad c_{-n}=c_n^*$$

$$c_n=|c_n|e^{j\phi_n}, \quad c_{-n}=c_n^*=|c_n|e^{-j\phi_n}$$

$$|c_n|=|c_{-n}|=\frac{1}{2}\sqrt{a_n^2+b_n^2}$$

$$n=\pm1,\pm2,\pm3,\cdots$$

$$\phi_n=\tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

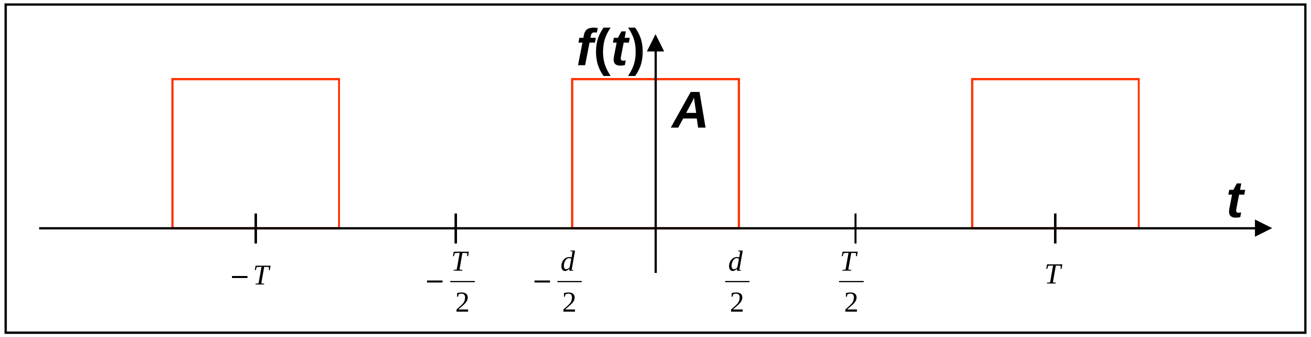
$$c_0=\frac{1}{2}a_0$$

Complex Form of the Fourier Series

Example

$$f(t)=\begin{cases}0, & -T/2<t<-d/2\\ A, & -d/2<t<d/2\\ 0, & d/2<t<T/2\end{cases}$$

$$f(t)=f(t+T)$$



$$c_n=\frac{A}{T}\int_{-d/2}^{d/2}e^{-jn\omega_0 t}dt$$

$$=\frac{A}{T}\frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{-d/2}^{d/2}$$

$$=\frac{A}{T}\left(\frac{1}{-jn\omega_0}e^{-jn\omega_0 d/2}-\frac{1}{-jn\omega_0}e^{jn\omega_0 d/2}\right)$$

$$=\frac{A}{T}\frac{1}{-jn\omega_0}(-2j\sin n\omega_0 d/2)$$

$$=\frac{A}{T}\frac{1}{\frac{1}{2}n\omega_0}\sin n\omega_0 d/2$$

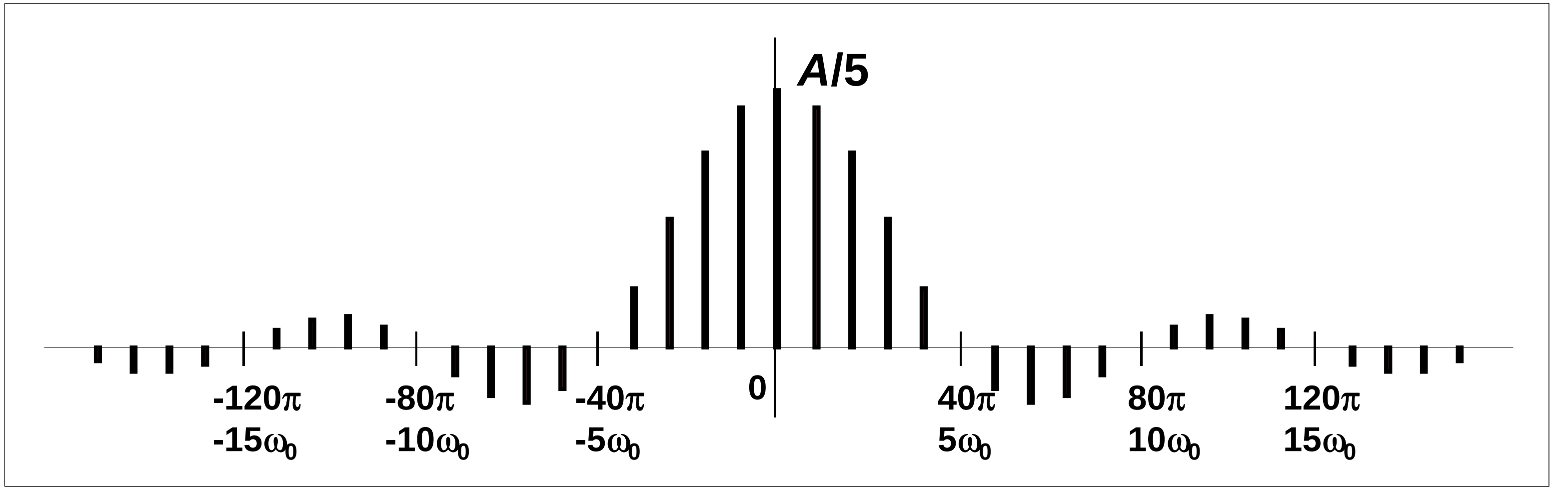
$$=\frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-jn\omega_0 t}dt$$

$$\cos n\omega_0 t=\frac{1}{2}\left(e^{jn\omega_0 t}+e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t=\frac{1}{2j}\left(e^{jn\omega_0 t}-e^{-jn\omega_0 t}\right)=-\frac{j}{2}\left(e^{jn\omega_0 t}-e^{-jn\omega_0 t}\right)$$

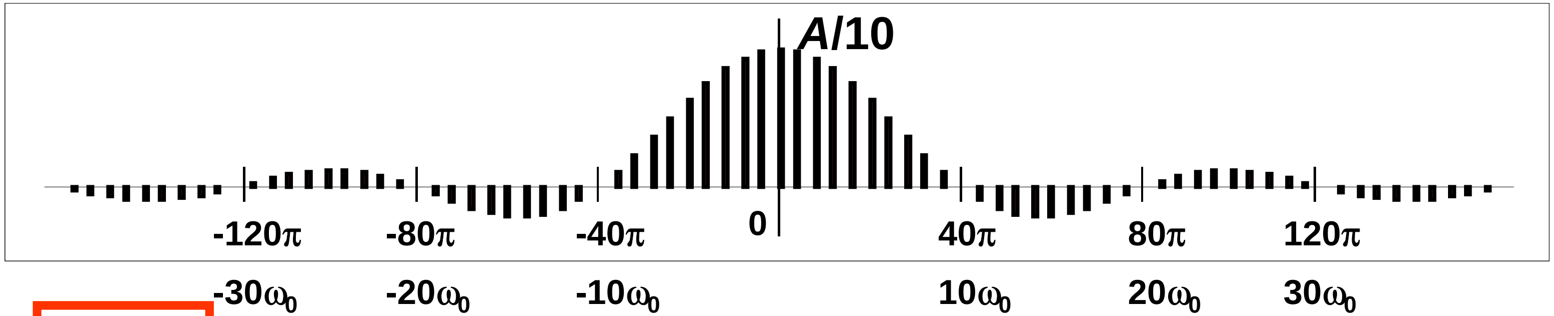
Complex Form of the Fourier Series



$$c_n = \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \quad T = \frac{1}{4}, \quad \frac{d}{T} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$



$$c_n = \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \quad T = \frac{1}{2}, \quad \frac{d}{T} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 4\pi$$

SOLVED PROBLEM

1.Find the complex form of the Fourier series of the periodic function

$$f(x)=\begin{cases}0\ ,\ 0<x<l\\ \alpha\ ,\ l<x<2l\end{cases}$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

$$=\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_0 t}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-jn\omega_0 t}dt$$

$\omega_0=2\pi/T$.

$T=2l$

SOLUTION

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e\frac{in\pi x}{l}$$

$$c_n=\frac{1}{2l}\int_0^{2l}f(x)e^{-\frac{in\pi x}{l}}dx$$

$$c_n=\frac{1}{2l}\left[\int_0^{l}0\left(e^{-\frac{in\pi x}{l}}\right)dx+\int_l^{2l}\alpha e^{-\frac{in\pi x}{l}}dx\right]$$

$$=\frac{\alpha}{2l}\left(-\frac{l}{in\pi}e^{-\frac{in\pi x}{l}}\right)_l^{2l}$$

$$=-\frac{\alpha}{2n\pi i}\left[e^{-2in\pi}-e^{-ni\pi}\right]$$

$$c_n = -\frac{\alpha}{2n\pi i}\left[1-(-1)^n\right]$$

$$= \frac{\alpha i}{2n\pi}\left[1-(-1)^n\right]$$

If $n \neq 0$

If $n=0$

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{2l}\int_{l}^{2l}\alpha e^{-0}dt = \frac{\alpha}{2}$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

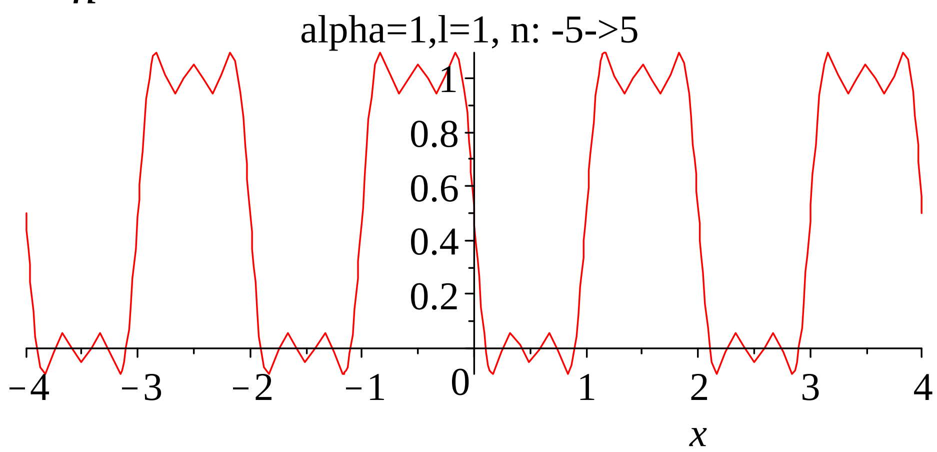
$$f(x) = c_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{\alpha i}{2n\pi}(1-(-1)^n)e^{\frac{in\pi x}{l}}$$

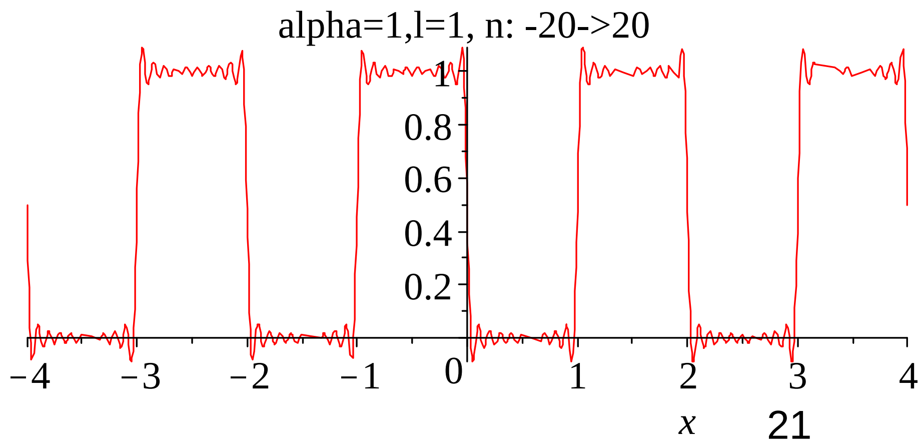
$$f(x) = c_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{\alpha i}{2n\pi}(1-(-1)^n)\left\{\cos\left(n\pi x/l\right) + i\sin\left(n\pi x/l\right)\right\}$$

$$= \frac{\alpha}{2} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i\alpha}{2n\pi}(1-(-1)^n)\cos\left(n\pi x/l\right) + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{-\alpha}{2n\pi}(1-(-1)^n)\sin\left(n\pi x/l\right)$$

$$= \frac{\alpha}{2} + \sum_{k=1}^{\infty} \frac{\alpha}{2k\pi}(1-(-1)^{-k})\sin\left(-k\pi x/l\right) + \sum_{n=1}^{\infty} \frac{-\alpha}{2n\pi}(1-(-1)^n)\sin\left(n\pi x/l\right) = \frac{\alpha}{2} + \sum_{n=1}^{\infty} \frac{-\alpha}{n\pi}(1-(-1)^n)\sin\left(n\pi x/l\right)$$

If $k = -n$

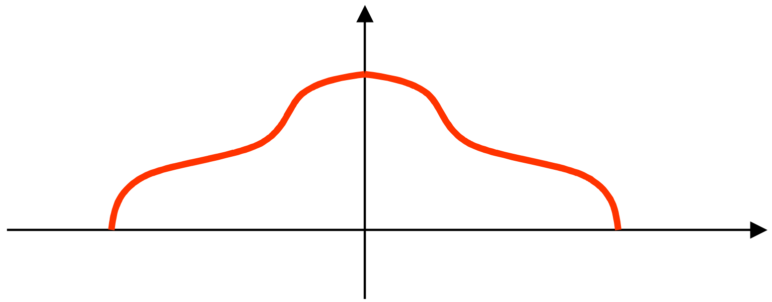




Waveform Symmetry

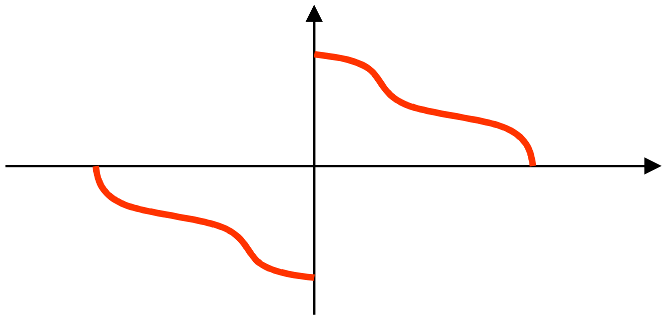
- Even Functions

$$f(t) = f(-t)$$



- Odd Functions

$$f(t) = -f(-t)$$



- Any function $f(t)$ can be expressed as the sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \tfrac{1}{2}[f(t) + f(-t)]$$ **Even Part**

$$f_o(t) = \tfrac{1}{2}[f(t) - f(-t)]$$ **Odd Part**

Waveform Symmetry

$\omega_0 = 2\pi / T.$

Fourier Coefficients of Even Functions

$T = 2l$

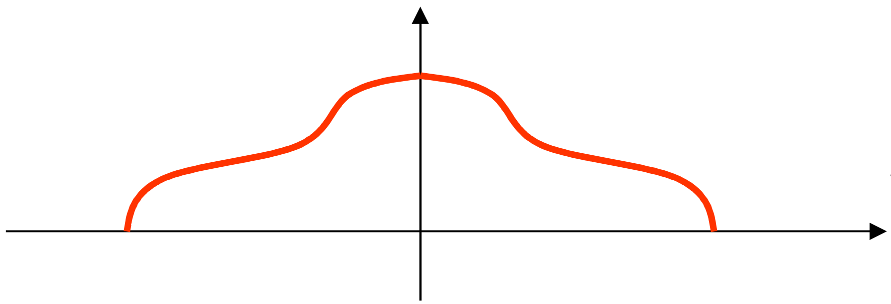
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt = \frac{1}{l}\int_{-l}^{l} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt$$

$$f(t) = f(-t)$$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$\Rightarrow \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt = \frac{2}{l}\int_0^{l} f(t)\cos(n\omega_0 t)dt$$

Waveform Symmetry

Fourier Coefficients of Odd Functions

$\omega_0 = 2\pi/T.$

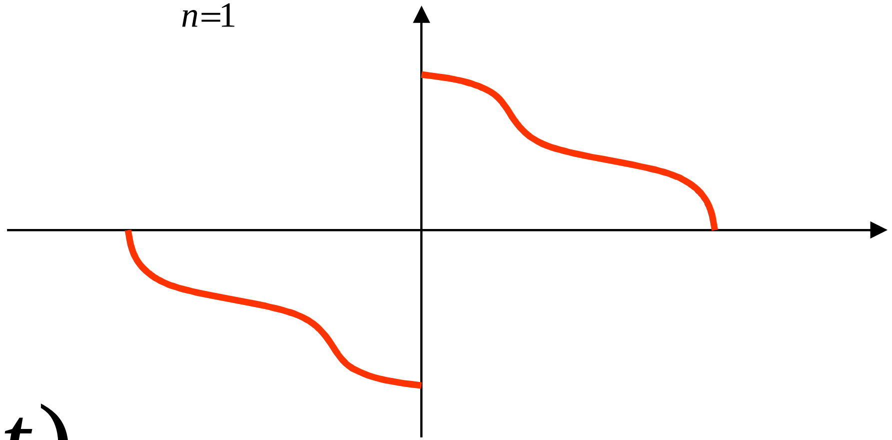
$T = 2l$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt = \frac{1}{l}\int_{-l}^{l} f(t)dt$$

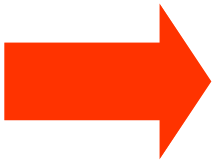
$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt$$



$$f(t) = -f(-t)$$

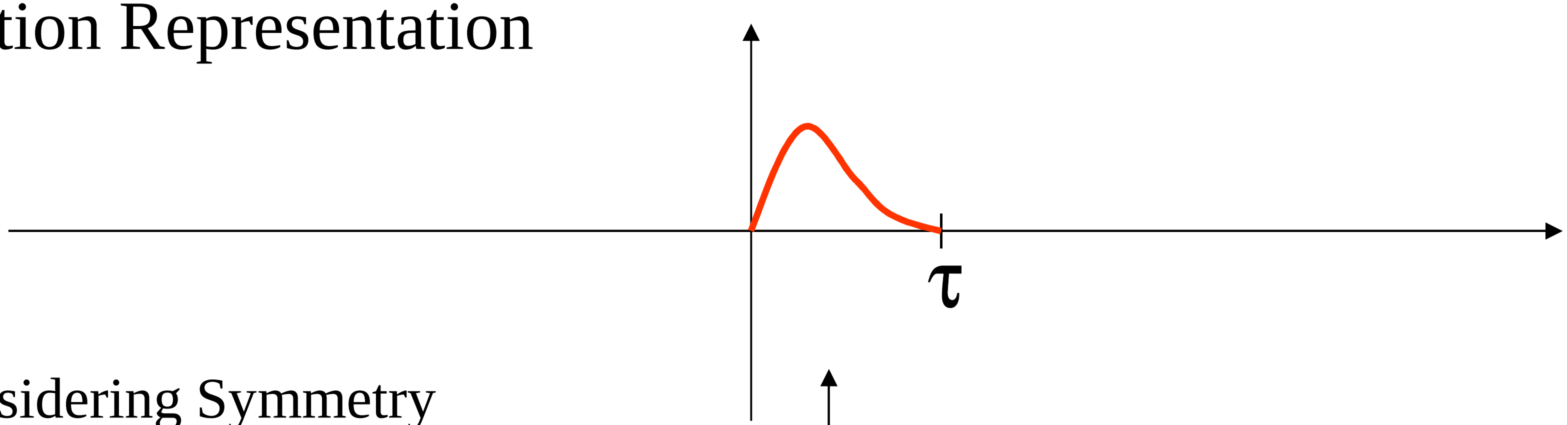
$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$



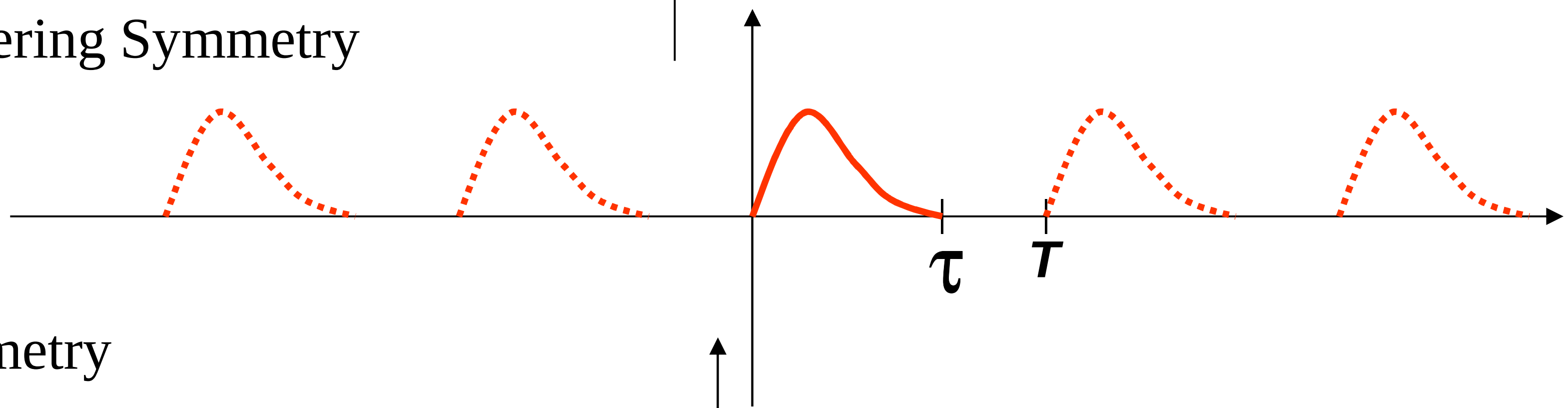
$$b_n = \frac{4}{T}\int_{0}^{T/2} f(t)\sin(n\omega_0 t)dt = \frac{2}{l}\int_{0}^{l} f(t)\sin(n\omega_0 t)dt$$

Fourier Series - Half-Range Expansions

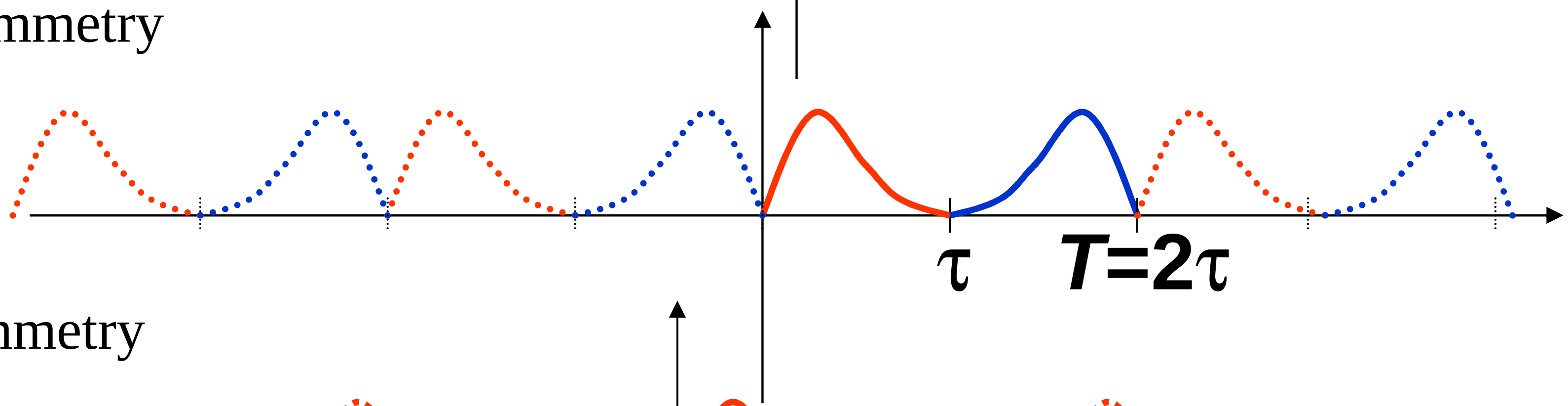
Non-Periodic Function Representation



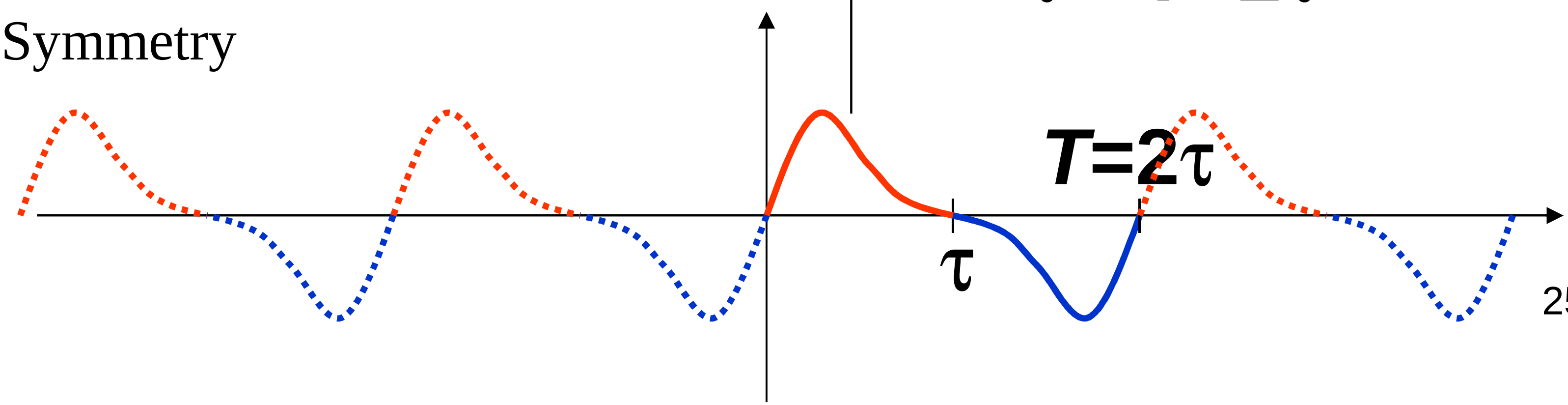
Extension Without Considering Symmetry



Expansion Into Even Symmetry



Expansion Into Odd Symmetry



DIRICHLET CONDITIONS

Suppose that

1. $f(x)$ is defined and single valued except possibly at finite number of points in $(-l,+l)$
2. $f(x)$ is periodic outside $(-l,+l)$ with period $2l$
3. $f(x)$ and $f'(x)$ are piecewise continuous in $(-l,+l)$

Then the Fourier series of $f(x)$ converges to

1. $f(x)$ if x is a point of continuity
2. $[f(x+0)+f(x-0)]/2$ if x is a point of discontinuity

E.g. For a function defined by $f(x)=|x|, -\pi < x < \pi$ obtain a Fourier series. Deduce that

$\omega_0 = 2\pi/T$.
$T = 2l$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$$

$$a_0=\frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt=\frac{1}{l}\int_{-l}^{l} f(t)dt$$

$$a_n=\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt$$

$$b_n=\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt$$

Solution **$f(x) = |x|$ is an even function**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}|x|dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx$$

$$=\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n-1\right]}{n^2}\cos nx$$

$x=0$ is a point of continuity, by Dirichlet condition the Fourier series converges to f(0) and f(0)=0

$$0=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$0=\frac{\pi}{2}-\frac{2}{\pi}\left(\frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\ldots\right)$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$$

HALF RANGE SERIES - COSINE SERIES

A function $f(x)$ defined in $(0,l)$ can be expanded as a Fourier series of period containing only cosine terms by extending $f(x)$ suitably in $(-l,0)$. (As an even function)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$\omega_0 = 2\pi/T.$

$T = 2l$

where

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx, \; n \geq 0$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)dt = \frac{1}{l} \int_{-l}^{l} f(t)dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t dt$$

HALF RANGE SERIES - SINE SERIES

A function $f(x)$ defined in $(0,l)$ can be expanded as a Fourier series of period containing only sine terms by extending $f(x)$ suitably in $(-l,0)$. (As an odd function)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$f(x) = \sum_{u=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$\omega_0 = 2\pi/T$.

$T = 2l$

where

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt = \frac{1}{l}\int_{-l}^{l} f(t)dt$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \quad, n \geq 1$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt$$

E.g. Obtain the Fourier expansion of (x sin$x$)as a cosine series in $(0,\pi)$. Hence find the value of

$$1+\frac{2}{1\,.\,3}-\frac{2}{3\,.\,5}+\frac{2}{5\,.\,7}+\cdots$$

SOLUTION

Given function represents an even function in $(-\pi,\pi)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$$

$\omega_0=2\pi/T.$

$T=2l$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$a_0=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)dt=\frac{1}{l}\int_{-l}^{l}f(t)dt$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x\sin xdx=\frac{2}{\pi}\left[x(-\cos x)-1(-\sin x)\right]_0^{\pi}=2$$

$$a_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\cos n\omega_0 tdt$$

$$b_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\sin n\omega_0 tdt$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nxdx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\left[\frac{1}{2}\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left[\begin{array}{l} \left\{x\left(\frac{-\cos(1+n)x}{1+n}\right) - 1.\left(\frac{-\sin(1+n)x}{(1+n)^2}\right)\right\}_0^{\pi} \\ + \left\{x\left(\frac{-\cos(1-n)x}{1-n}\right) - 1.\left(\frac{-\sin(1-n)x}{(1-n)^2}\right)\right\}_0^{\pi} \end{array}\right]$$

$$= \frac{-1(-1)^{n+1}}{n+1} + \frac{1}{n-1}(-1)^{1-n}$$

$$a_n = (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{n^2-1} \qquad \text{If } n \neq 1$$

If $n = 1$ $$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right] = \frac{-1}{2}$$

$$x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx \qquad \text{in } (0,\pi)$$

At $x = \pi/2$, the above series reduces to

$$1 + 2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos\left(\frac{n\pi}{2}\right)$$

$x = \pi/2$ is a point of continuity

$\therefore$ The given series converges to $f(\pi/2) = \pi/2$

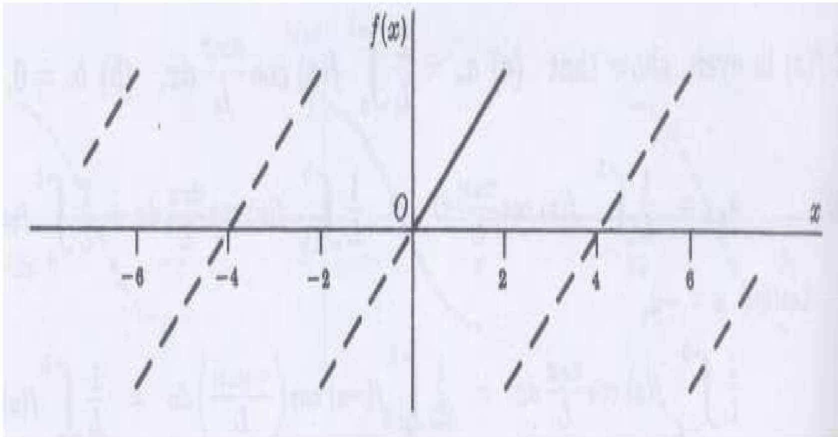
$$x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx \qquad \text{in } (0,\pi)$$

$$\frac{\pi}{2} = 1 + 2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\infty = \frac{\pi-2}{4}$$

2) Expand $f(x)=x, \ 0<x<2$ in half range
(a) sine Series (b) Cosine series.

(a) Extend the definition of given function to that of an odd function of period 4



$$f(x)=\begin{cases} x; & -2<x<0 \\ x; & 0<x<2 \end{cases}$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

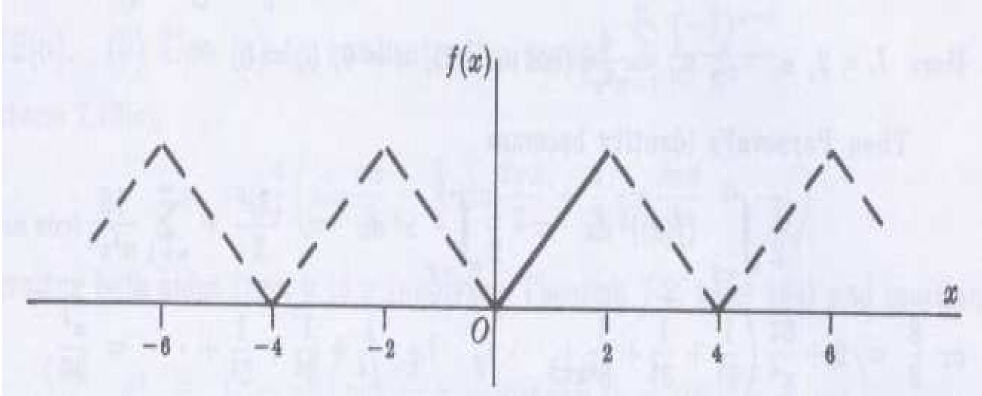
$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right) \right]_0^2 = \frac{-4(-1)^n}{n\pi}$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{-(-1)^n}{n}\sin\frac{n\pi x}{2}$$

(b) Extend the definition of given function to that of an even function of period 4

$$f(x)=\begin{cases}-x; -2<x<0\\ x; 0<x<2\end{cases}$$



$$b_n = 0$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx$$

$$=\left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)-1\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_0^2 = \frac{4\left[(-1)^n-1\right]}{n^2\pi^2};$$

$n \neq 0$

$$a_0 = \int_0^2 x\,dx = 2$$

$$f(x) = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2} \cos \frac{n\pi x}{2}$$

PARSEVAL'S IDENTITY

$$\int_{-l}^{l}[f(x)]^2 dx = l\left[\frac{1}{2}a_0{}^2 + \sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2)\right]$$

Provided the Fourier series for *f*(*x*) converges uniformly in (-*l*, *l*).
The Fourier Series for *f*(*x*) in (-*l*,*l*) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{.......... ......(1)}$$

Multiplying both sides of (1) by *f*(*x*)and integrating term from –*l* to *l*
(which is justified because *f* (*x*) is uniformly convergent)

$$\int_{-l}^{l} f(x)^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)dx + \sum_{n=1}^{\infty} a_n\left\{\int_{n=1}^{\infty} f(x)\cos\frac{n\pi x}{l}dx\right\} + \sum_{n=1}^{\infty} b_n\left\{\int_{n=1}^{\infty} f(x)\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{a_0}{2}\times la_0 + \sum_{n=1}^{\infty} a_n(la_n) + \sum_{n=1}^{\infty} b_n(lb_n)$$

$$\therefore \int_{-l}^{l} f(x)^2 dx = l\left[\frac{a_0{}^2}{2} + \sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2)\right] \text{..................(2)}$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt = \frac{1}{l}\int_{-l}^{l} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt$$

CASE-I If $f(x)$ is defined in $(0,2l)$ then Parseval's Identity is given by

$$\int_0^{2l}\left[f(x)\right]^2 dx = l\left[\frac{1}{2}a_0^{\,2} + \sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})\right]..............(3)$$

CASE-II If half range cosine series in $(0,l)$ for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then Parseval's Identity is given by

$$\int_0^{l}\left[f(x)\right]^2 dx = \frac{l}{2}\left[\frac{1}{2}a_0^{\,2} + \sum_{n=1}^{\infty} a_n^{\,2}\right]......... \;....(4)$$

CASE-III If the half range Sine series in $(0,l)$ for $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{1}{2}a_0^{\,2} + \sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})\right]$$

Then Parseval,s Identity is given by

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^{\,2}\right] \text{.......... ....(5)}$$

E.G. Find the Fourier series of periodic function

$$f(x) = x^2 - x \text{ in } (-\pi, \pi)$$

Hence deduce the sum of series

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{.......}\infty$$

Assuming that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

SOLUTION $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$ in $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 - x)\cos nx dx = \frac{1}{\pi}\left[x^2 \left(\frac{-\sin nx}{n}\right) - 2x\frac{-\cos nx}{n^2} \right]_{-\pi}^{\pi}$$

$$= \frac{4}{n^2}(-1)^n$$ if $n \neq 0$.

$\therefore x \cos nx$ is odd function

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 - x)\sin nx dx = 0 - \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx$$

$$= -\frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi} = \frac{2}{n}(-1)^n$$

$\therefore x^2 \sin nx$ is odd function

$$f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx. \qquad \text{in } (-\pi, \pi)$$

Using the Parseval's Identity $\quad \bar{y}^2=\left[\frac{1}{4}a_0^{\,2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2}+b_n^{\,2})\right].$

$$\overline{y}^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2-x)^2\,dx=\frac{\pi^4}{5}+\frac{\pi^2}{3}$$

$$a_0=\frac{2\pi^2}{3};\,a_n=\frac{4}{n^2}(-1)^n;\,b_n=\frac{2}{n}(-1)^n$$

$$\therefore\ \frac{\pi^4}{5}+\frac{\pi^2}{3}=\frac{1}{4}\frac{4\pi^4}{9}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\rightarrow\quad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90} \qquad\qquad \because\ \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

E.g. 2 By using sine series for $f(x)=1$ in $0<x<\pi$.

Show that $$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots\infty$$

SOLUTION $$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}=\sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi}\sin nx dx=\frac{2}{\pi}\left\{\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi}$$

$$=\frac{2}{n\pi}\left[1-(-1)^n\right]$$ for $n\neq 0$.

$$f(x)=1=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(nx)$$

By Parseval's Identity $$\bar{y}^2=\left[\frac{1}{2}\sum_{n=1}^{\infty}b_n{}^2\right].$$

$$\bar{y}^2=\frac{1}{\pi}\int_0^{\pi}1^2 dx=1=\frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]^2$$

$$\Rightarrow\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots\infty$$