Mathematical Methods III
SSCM 2043

Part 2 – Partial Differential Equations
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First order Partial Differential Equations

preliminary notation and concepts

a <u>partial differential equation</u> is an equation that contains at least one partial derivative.

Example:

Dependent variables ?? Independent variables

$$\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xuy^2$$

$$u_{xx} + u_{yy} + u_{zz} = f(x, y, z)$$

$$\frac{\partial^4 w}{\partial x^4} - 3\frac{\partial^2 w}{\partial z \partial u} + \frac{\partial^3 w}{\partial u^3} + \cos(xu)\frac{\partial^3 w}{\partial x \partial y \partial z} - xyzu\frac{\partial w}{\partial x} = 0$$

First order Partial Differential Equations

preliminary notation and concepts

order of PDE is the order of the highest partial derivative occurring in the equation.

Example:

Find the order of PDE???

$$\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xuy^2$$

$$u_{xx} + u_{yy} + u_{zz} = f(x, y, z)$$

$$\frac{\partial^4 w}{\partial x^4} - 3\frac{\partial^2 w}{\partial z \partial u} + \frac{\partial^3 w}{\partial u^3} + \cos(xu)\frac{\partial^3 w}{\partial x \partial y \partial z} - xyzu\frac{\partial w}{\partial x} = 0$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)^4 - x\left(\frac{\partial^2 u}{\partial y^2}\right)^5 = xy$$

First order Partial Differential Equations

preliminary notation and concepts

Why We Study PDE:

PDE appears in modeling phenomena in the sciences, engineerig, economics, ecology and other areas.

Heat equation $$\frac{\partial u}{\partial t} - \Delta u = 0$$

Wave equation $$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

Schrodinger equation $$i\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\Delta u + V(x)u(t,x)$$

First order Partial Differential Equations

preliminary notation and concepts

a solution is of a PDE is any function that satisfies the equation.

Example

$$u_{xx} + u_{yy} = 2(x^2 + y^2)$$

One solution of this differential equation is

$$u(x, y) = x^2 y^2$$

First order Partial Differential Equations

preliminary notation and concepts

Example **Verify???**

$$4u_x + 3u_y + u = 0$$

One solution of this differential equation is

$$u(x, y) = e^{-x/4} f(3x - 4y)$$

Where f can be any differential function of a single variable.

Example **Verify???**

$$u_{xx} - 9u_{yy} = 0$$

$$u(x, y) = f(3x + y) + g(3x - y)$$

Is a solution for any twice differntiable function f and g of a single variable.

For example:

$$f(t) = \sin(t)$$

$$g(t) = e^{-t} + t$$

First order Partial Differential Equations

preliminary notation and concepts

A PDE is linear if it is linear in the unknown function and in its partial derivatives. An equation that is not linear is nonlinear

A PDE is linear if the PDE is linear in the unknown function and all its derivatives with coefficients depending on the independent variables alone.

$$a(x,y)u_x + b(x,y)u_y = c_0(x,y)u + c_1(x,y)$$

A PDE in u is classified as linear if all of the terms involving u and any of its derivatives can be expressed as a linear combination in which the coefficients of the u-terms are independent of u. In a linear PDE, the coefficients can depend at most on the independent variables

Example:

nonlinear

$$x^2 u_{xx} - y u_{xy} = 2(x^2 + y^2)u^2$$

$$x^2 u_{xx} - y u_{xy} = 2(x^2 + y^2)u$$

nonlinear

$$x^2 (u_{xx})^{\frac{1}{2}} - y u_{xy} = xu$$

nonlinear

$$x^2 u_{xx} - y u_{xy} = u_x u$$

Many linear PDE problems can be solved exactly using techniques such as separation of variables, superposition, Fourier series, Laplace transform and Fourier transform. Exact solutions are valuable in the exercise of code validation.

First order Partial Differential Equations

preliminary notation and concepts

A PDE is quasi-inear if it is linear in its highest order derivative terms with coef depending….

A PDE of **order *m*** is called **Quasi-linear** if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function or order **strictly less than *m***.

Example:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

$$u_{xx} + 4yu_{yy} - (u_x)^3 + u_x u_y = \cos(u)$$

$$u_t + uu_x = 0$$

$$a(x,y,u,u_x,u_y)u_{xx} + b(x,y,u,u_x,u_y)u_{yy} = c(x,y,u)$$

First order Partial Differential Equations

preliminary notation and concepts

A Quasi-linear PDE where the coefficients of derivatives of order *m* are functions of the independent variables alone is called a **Semi-linear PDE**.

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

$$u_t + u_{xxx} + uu_x = 0$$

Basic of partial integration

$$\frac{\partial u(x,y)}{\partial x} = p(x) \rightarrow u(x,y) = \int p(x)dx + f(y)$$

$$\frac{\partial u(x,y,z)}{\partial x} = p(y) \rightarrow u(x,y) = p(y)\int 1dx + f(y,z) = xp(y) + f(y,z)$$

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} = p(y) \rightarrow u_x(x,y,z) = xp(y) + f(y,z) \rightarrow u(x,y,z) = x^2 p(y) + \int f(y,z)dx + g(y,z)$$

First order Partial Differential Equations

First order PDE derivation

Example Derive a first order PDE such that $u = f(x^2 - xy)$ is it general solution where f is an arbitrary function.

Solution:

$$u = f(x^2 - xy)$$

$$\rightarrow u_x = (2x - y)f', \; u_y = -xf',$$

Then, $u_x = -2u_y - y\dfrac{u_y}{-x}$ ⇨ $-xu_x = 2xu_y - yu_y \;\rightarrow\; (2x - y)u_y + xu_x = 0$

Example Derive a first order PDE such that $u = f(2x - 4y)$ is it general solution where f is an arbitrary function.

Solution:

$$u = f(2x - 4y)$$

$$\rightarrow u_x = 2f', \; u_y = -4f',$$

Then, $2u_x + u_y = 0$

First order Partial Differential Equations

Simple form

b.c. $u(0,y)=y^2$

$$\frac{\partial u(x,y)}{\partial x}+u(x,y)=x \quad \leftarrow \text{nonhomogeneous eq}$$

$Lu=G$ (nonhomogeneous eq) $\to u_p$
$Lu=0$ (homogeneous eq) $\to u_h$
General solution $\to u=u_h+u_p$.

homogeneous eq $\to$ $$\frac{\partial u(x,y)}{\partial x}+u(x,y)=0$$

$u_x+bu=f$: **Integrating factor** $\rho = e^{\int bdx}$

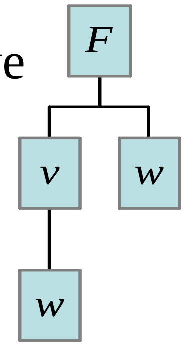
$$e^x\frac{\partial u}{\partial x}+ue^x=0 \to \frac{\partial}{\partial x}\left(e^x u\right)=0 \to e^x u = f(y) \to u_h = e^{-x}f(y)$$

f is arbitrary function

Note that the homogeneous solution can be written: $\boldsymbol{F(e^x u,y)=0}$.

where F is an arbitrary differentiable function $F(v,w)$ such that $\boldsymbol{F_v \neq 0}$. Then it is possible to solve $F(v,w)=0$ and obtain $v=f(w)$ for some function f.

$$\frac{dF}{dw}=\frac{\partial F}{\partial w}+\frac{\partial F}{\partial v}\frac{dv}{dw}$$



For nonhomogeneous eq, assume, $\boldsymbol{u_p=a+bx}$, then we get $\boldsymbol{u_p=x-1}$.

General solution $\to u=u_h+u_p=e^{-x}f(y)+x-1$.

b.c. $u(0,y)=y^2 \to y^2=e^0 f(y)+0-1 \to y^2+1=f(y)$
specified solution : $\boldsymbol{u=e^{-x}(y^2+1)+x-1}$

Or we can **solve directly**: $\rho=e^x$:

$$\frac{\partial}{\partial x}\left(e^x u\right)=e^x x \to e^x u = \int xe^x dx + f(y)$$

$$\to e^x u = (x-1)e^x + f(y) \to u = x-1+e^{-x}f(y)$$

First order Partial Differential Equations

The Linear Equation : Coordinate Transformation (constant coefficients)

e.g. $au_x+bu_y+cu=d(x,y)$. (a)
The coefficients a, b, c are constants.

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + Au_\eta$$

$$u_y = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = Bu_\eta$$

Introduce **coordinate transformation** is: $\boldsymbol{\xi=x,\ \eta=Ax+By}$.

Eq (a) become: $au_\xi+(aA+bB)u_\eta+cu=d$ (b)
Now choose $\boldsymbol{(aA+bB)=0}$, eq. (b) become : $\boldsymbol{au_\xi+cu=d}$ (c)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Integrating factor:

$$\rho = \exp\left(\int \frac{c}{a} d\xi\right)$$

$$\frac{\partial}{\partial \xi}(\rho u) = \frac{\rho d}{a} \rightarrow \rho u = \int \frac{\rho d}{a} d\xi + f(\eta)$$

First order Partial Differential Equations

The Linear Equation : Coordinate Transformation (constant coefficients)

e.g. $u_x - u_y = 2y$. Condition: $u(x,2x+1)=e^x$.

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + Au_\eta$$

$$u_y = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = Bu_\eta$$

Introduce **coordinate transformation** is: $\boldsymbol{\xi = x,\ \eta = Ax + By}$.

$au_x + bu_y + cu = d(x,y)$

We get: $au_\xi + (aA+bB)u_\eta + cu = d \quad \rightarrow u_\xi + (A-B)u_\eta = 2(\eta - A\xi)/B$.

Now choose $\mathbf{(A-B)=0}$, or set $A=1$, $B=1$.

$\rightarrow u_\xi = 2(\eta - \xi)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The general solution

By direct integration: $$u(\xi,\eta) = 2\xi\eta - \xi^2 + F(\eta)$$

The general solution: $u(x,y) = 2x(x+y) - x^2 + F(x+y)$.

Apply conditions: $u(x,2x+1)=e^x$.

$u(x,2x+1) = 2x(3x+1) - x^2 + F(3x+1) = e^x$.

$\rightarrow F(3x+1) = e^x - 5x^2 - 2x$

$$\rightarrow \theta = 3x+1 \rightarrow x = \frac{\theta - 1}{3}$$

$$\rightarrow F(\theta) = \exp\left(\frac{\theta-1}{3}\right) - 5\left(\frac{\theta-1}{3}\right)^2 - 2\left(\frac{\theta-1}{3}\right)$$

$$\rightarrow F(x+y) = \exp\left(\frac{x+y-1}{3}\right) - 5\left(\frac{x+y-1}{3}\right)^2 - 2\left(\frac{x+y-1}{3}\right)$$

The general solution: $$u(x,y) = 2x(x+y) - x^2 + \exp\left(\frac{x+y-1}{3}\right) - 5\left(\frac{x+y-1}{3}\right)^2 - 2\left(\frac{x+y-1}{3}\right).$$

First order Partial Differential Equations

The Linear Equation : Coordinate Transformation (constant coefficients)

Exercise

e.g. $u_x - 4u_y + u = x$.

Introduce **coordinate transformation** is: $\boldsymbol{\xi = x,\ \eta = Ax + By}$.

$au_x + bu_y + cu = d(x,y)$

We get: $au_\xi + (aA + bB)u_\eta + cu = d$

$\rightarrow u_\xi + u = \xi$

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + Au_\eta$$

$$u_y = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = Bu_\eta$$

$$u(\xi,\eta) = \xi - 1 + e^{-\xi}F(\eta)$$

First order Partial Differential Equations

The Linear Equation : Coordinate Transformation

Consider the general solution of linear first order homogeneous eq. $$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=0 \qquad \text{(a0)}$$

Let ξ=ξ(x,y), η=η(x,y) be a transformation on domain with Jacobian $\partial(\xi,\eta)/\partial(x,y)\neq 0$,

$$\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}$$

Since $u_x=u_\xi\,\xi_x+u_\eta\eta_x$ and $u_y=u_\xi\,\xi_y+u_\eta\eta_y$, the transformed eq is:

$$\left(a\xi_x+b\xi_y\right)u_\xi+\left(a\eta_x+b\eta_y\right)u_\eta+cu=0 \qquad \text{(a)}$$

To simplify, then choose η such that $\boldsymbol{a\eta_x+b\eta_y=0}$ (b)

Now assume $a(x,y)\neq 0$ and let characteristic eq. $$\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)} \qquad \text{(c)}$$

Let the general solution of (c) be $\boldsymbol{\eta(x,y)=c}$ where $\eta_y\neq 0$ and c is arbitrary constant. (characteristic curve)

Then this function η(x,y) satisfies (b) since $d\eta=\eta_x dx+\eta_y dy=0$. ⟹ $$\frac{dy}{dx}=\frac{b}{a}=-\frac{\eta_x}{\eta_y}$$

Now choose $\boldsymbol{\xi(x,y)=x}$. Then $\xi_x\eta_y-\xi_y\eta_x\neq 0$, the transformation is invertible. Eq (a) become

$\boldsymbol{au_\xi+cu=0}$. **(d)** Hold η fixed and integrate w.r.t. ξ, we get: $u(\xi,\eta)=f(\eta)\exp\left[-\int c/a\,d\xi\right]$ f: arbitrary function

The function $\psi(\xi,\eta)=\exp\left[-\int\frac{c(\xi,\eta)}{a(\xi,\eta)}d\xi\right]$ Satisfies eq (d). Hence $u(x,y)=\psi[x,\eta(x,y)]$ is **particular solution** of eq (a0)

General solution: $\boldsymbol{u_h=u(x,y)f[\eta(x,y)]}$

First order Partial Differential Equations

The Linear Equation : Coordinate Transformation

e.g. $x^2u_x - xyu_y + yu = 0$. (a)

The coefs: $a(x,y)=x^2$, $b(x,y)= -xy$, and $c(x,y)=y$. ⇨ $\dfrac{dy}{dx} = \dfrac{-xy}{x^2} = -\dfrac{y}{x}$ ⇨ $\boldsymbol{\eta(x,y) = xy = c}$

The transformation is: $\xi = x$, $\eta = xy$. The coef: $a = \xi^2$, $c = \eta/\xi$, we get

(a) → $\xi^2 u_\xi + \dfrac{\eta}{\xi} u = 0 \rightarrow u_\xi + \dfrac{\eta}{\xi^3} u = 0$ ⇨ $\psi(\xi,\eta) = \exp\left[-\int \dfrac{c(\xi,\eta)}{a(\xi,\eta)} d\xi\right] = \exp\left[-\int \dfrac{\eta}{\xi^3} d\xi\right] = \exp\left[\dfrac{\eta}{2\xi^2}\right]$

particular solution → $u(x,y) = \psi[x, \eta(x,y)] = \exp(y/2x)$.

General solution**:** → $\boldsymbol{u_h = u(x,y) f\,[\eta(x,y)]} = \exp(y/2x)\, f\,(xy)$.

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

or

Integrating factor: $\rho = \exp\left(\int \dfrac{\eta}{\xi^3} d\xi\right) = \exp\left(-\dfrac{\eta}{2\xi^2}\right)$

$$\frac{\partial}{\partial \xi}(\rho u) = \frac{\partial}{\partial \xi}\left(\exp\left(-\frac{\eta}{2\xi^2}\right)u\right) = \rho 0 = 0$$

⇨ $\exp\left(-\dfrac{\eta}{2\xi^2}\right)u = 0 + f(\eta) \rightarrow u = \exp\left(\dfrac{\eta}{2\xi^2}\right) f(\eta)$

First order Partial Differential Equations

The Linear Equation: Coordinate Transformation (general)

Consider the linear first order partial differential equation in two independent variables:

$$au_x + bu_y + cu = f$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix} \rightarrow \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{\partial x(r,\theta)}{\partial r} = 1, \; \frac{\partial r(x,y)}{\partial x} = 4 \rightarrow \frac{\partial x}{\partial r} \neq 1 \Big/ \frac{\partial r}{\partial x}$$

1. **Find the characteristic equation:** $\dfrac{dy}{dx} = \dfrac{b}{a}$
2. **Find the general solution of the characteristic equation and put it in the form:** $\psi(x,y) = c$
3. **Use the transformation:** $\xi = x$ $\quad \eta = \psi(x,y)$
4. **To change PDE into this form:**

$$u_\xi + hu = F$$

First order Partial Differential Equations

The Linear Equation: Coordinate Transformation

Exercise: transform into simple form only

Consider the linear 1st PDE

$$x^2 u_x + y u_y + xyu = 1$$

$$a = x^2, \quad b = y, \quad c = xy$$

1 **characteristic equation:** $\frac{dy}{dx} = \frac{y}{x^2}$

2 **Solution:** $\ln(y) + \frac{1}{x} = c$

3 **transformation:** $\xi = x, \quad \eta = \ln(y) + \frac{1}{x}$

Ei(a,z) : exponential integral

4 $$u_\xi + \frac{1}{\xi} e^{\eta - 1/\xi} u = \frac{1}{\xi^2}$$

Integrating factor:

$$\rho = \exp\left(\int \frac{\exp(-1/\xi)}{\xi} d\xi \right) = \exp\left(Ei\left(1, \frac{1}{x}\right)\right)$$

First order Partial Differential Equations

The Linear Equation: Coordinate Transformation

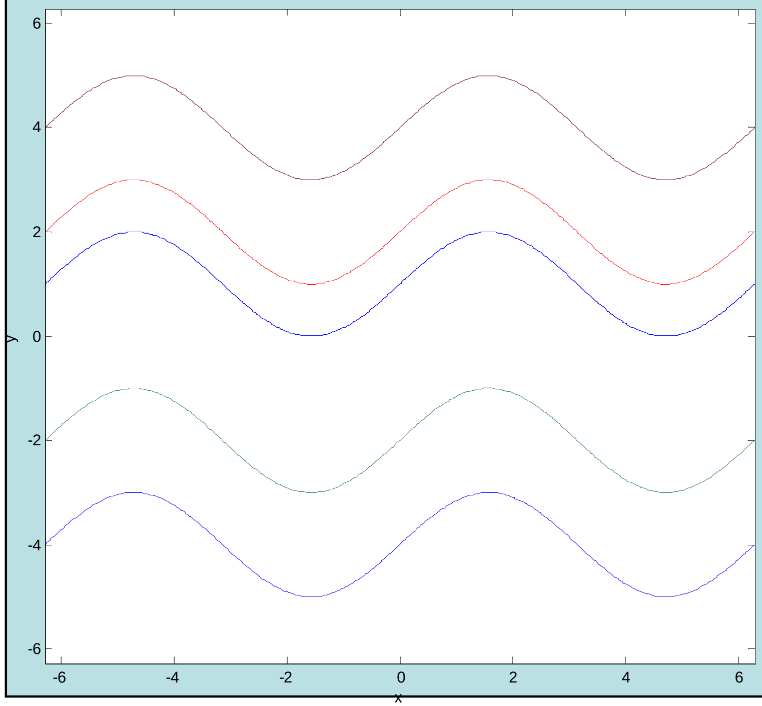
Exercise:

Consider the linear 1st PDE

$$u_x + \cos(x)u_y + u = xy$$

$$a = 1, \quad b = \cos(x), \quad c = 1$$

characteristic curves:



1 **characteristic equation:** $\frac{dx}{dy} = \cos x$

2 **Solution:** $y - \sin x = \hat{c}$

3 **transformation:** $\xi = x, \quad \eta = y - \sin x$

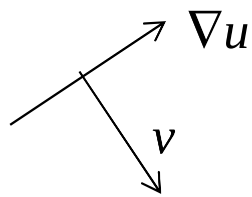
Integrating factor: $\rho = \exp\left(\int 1 d\xi\right) = e^{\xi}$

4 $u_{\xi} + u = \xi[\eta + \sin \xi]$

$$\frac{\partial}{\partial \xi}(\rho u) = \frac{\partial}{\partial x}\left(e^{\xi}u\right) = e^{\xi}\xi(\eta + \sin \xi)$$

First order Partial Differential Equations

The Linear Equation : characteristic method

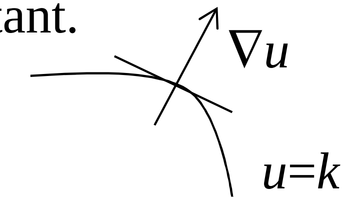


$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y) \quad \text{(a)}$$

Let $v = a\mathbf{i} + b\mathbf{j}$

$$\nabla u \cdot v = \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right)\cdot(a\mathbf{i} + b\mathbf{j}) = au_x + bu_y = 0$$

Characteristic curve is the curve where u is constant.
Characteristic curve is derived from $v = a\mathbf{i} + b\mathbf{j}$.



Along Characteristic curve $\rightarrow$ $du = u_x dx + u_y dy = 0$.

We consider the curves (x,y) whose tangents at each point have those directions. That is the **one family Curves (characteristic curve)** defined by ODE:

$$\frac{dy}{dx} = \frac{b}{a}, \quad or \quad \frac{dx}{dt} = a, \frac{dy}{dt} = b$$

The curve is : $\psi = \psi(x,y)$

We also get $$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = u_x + \frac{b}{a}u_y = \frac{au_x + bu_y}{a} = \frac{cu+d}{a}$$

Or we get $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = au_x + bu_y = cu + d$$

This means that along the one family curves (characteristic curves) have the property that along them $u(x,y)$ will satisfy

$$\frac{du}{dx} = \frac{cu+d}{a} \quad ,or \quad \frac{du}{dt} = cu + d$$

First order Partial Differential Equations

The Linear Equation : characteristic method

E.g. solve $xu_x+yu_y=2u$, with $u(x,1)=\cos x$.

Solution: the characteristic curve $\to$ $dy/dx=b/a=y/x$.
characteristic curve (one family curves) $\to$ $y=kx$. $\to k=y/x$.

Along characteristic curve, u satisfies $\dfrac{du}{dx}=\dfrac{cu+d}{a}=\dfrac{2u}{x}$ ⇨ $u=hx^2$.

Please note that h may **differ** from characteristic curve to characteristic curve, so h depends on k.
$\boldsymbol{u=h(k)x^2} \to \boldsymbol{u(x,y)=h(y/x)x^2}$.

Applying initial condition: $u(x,1)=\cos x$. $\to$ $\cos x=h(1/x)x^2$.
To find h, let $\theta=1/x$. $\to \cos(1/\theta)=h(\theta)/\theta^2$. $\to h(\theta)=\theta^2\cos(1/\theta)$. ⇨ $h\left(\dfrac{y}{x}\right)=\left(\dfrac{y}{x}\right)^2\cos\left(\dfrac{x}{y}\right)$

Hence, the solution is $u(x,y)=\left(\dfrac{y}{x}\right)^2\cos\left(\dfrac{x}{y}\right)x^2=y^2\cos\left(\dfrac{x}{y}\right)$.

Parameter as a constant

$\boldsymbol{u_{xx}-u=0.}$ ⇨ Let $u=e^{mx} \to m^2-1=0$. $u=Ae^x+Be^{-x}$, since A, B may differ from y value to new y, so A, B depends on y. Final solution : $\boldsymbol{u=A(y)e^x+B(y)e^{-x}}$.

Let y as parameter (constant)

First order Partial Differential Equations

The Linear Equation : characteristic method

E.g. solve $u_x+u_y+u=\exp(x+2y)$, with $u(x,0)=0$.

Solution: the characteristic curve $\rightarrow$ $dy/dx=b/a=1/1$.
characteristic curve (one family curves) $\rightarrow$ $k = y-x$.

Along characteristic curve, u satisfies $\dfrac{du}{dx}=\dfrac{cu+d}{a}=\dfrac{-u+\exp(x+2y)}{1}$

$$\frac{du}{dx}+u=\exp(x+2x+2k)=\exp(3x+2k)$$ Integrating factor: $\rho=\exp\left(\int 1dx\right)=e^x$

$$\frac{d}{dx}(\rho u)=\frac{d}{dx}(e^x u)=e^x e^{3x+2k}=e^{4x+2k} \quad \Rightarrow \quad e^x u=\frac{e^{2k}e^{4x}}{4}+f(k)=\frac{e^{2y}e^{2x}}{4}+f(y-x)$$

$$\Rightarrow \quad u=\frac{e^{2y+x}}{4}+e^{-x}f(y-x)$$

Let $y=0, u=0$: $\rightarrow$ $0=e^x/4+e^{-x}f(-x)$ $\rightarrow$ $f(-x)=-e^{2x}/4$ $\rightarrow$ $f(\theta)=-\exp(-2\theta)/4$.

$$\Rightarrow \quad u=\frac{e^{2y+x}}{4}-\frac{e^{-x}}{4}e^{-2(y-x)}=\frac{e^{2y+x}}{4}-\frac{e^{x-2y}}{4}$$

First order Partial Differential Equations

The Linear Equation: Method of Lagrange

$$a(x, y)u_x + b(x, y)u_y = d(x, y, u) \quad \text{(a)}$$

Theorem: if $\phi(v,w) = 0$
is a general solution of PDE (a) where ϕ is an arbitrary function, and
$v(x,y,u)=c_1$, $w(x,y,u)=c_2$, where c_1, c_2 are constants, then
$\phi(v,w) = 0$
Is a solution for

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{d(x, y, u)} = ds \qquad \text{(b)}$$

First order Partial Differential Equations

The Linear Equation: Method of Lagrange

Let solution, $u=f(x,y)$. Let solution as a surface of constant, $\psi(x,y,u)$. On this surface, we get

$$a(x,y)u_x + b(x,y)u_y = d(x,y,u) \quad \text{(a)}$$

$$\frac{\partial}{\partial x}\psi(x,y) = \frac{\partial}{\partial x}\psi(x,y,u) + \frac{\partial \psi}{\partial u}\cdot\frac{\partial u}{\partial x} = 0,$$
$$\frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}\psi(x,y,u) + \frac{\partial \psi}{\partial u}\cdot\frac{\partial u}{\partial y} = 0 \quad \text{(b)}$$

From (b), we get

$$\frac{\partial \psi}{\partial u}u_x = -\frac{\partial \psi}{\partial x};\ \frac{\partial \psi}{\partial u}u_y = -\frac{\partial \psi}{\partial y}. \quad \text{(c)}$$

(a)×ψ_u, we get

$$d\frac{\partial \psi}{\partial u} - a\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} - b\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y} = 0$$

Use (c) ⇒

$$a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial y} + d\frac{\partial \psi}{\partial u} = 0 \quad \text{(d)}$$

Any solution of (d) is solution of (a), and vice versa!

(d) → $$\frac{\partial \psi}{\partial x} + \frac{b}{a}\frac{\partial \psi}{\partial y} + \frac{d}{a}\frac{\partial \psi}{\partial u} = 0$$

Let ψ= ψ(x,y(x),u(x))=constant →will get characteristic eq.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} + \frac{\partial \psi}{\partial u}\frac{du}{dx} = 0$$

Compare: $$\frac{dy}{dx} = \frac{b}{a};\ \frac{du}{dx} = \frac{d}{a} \to \frac{dx}{a} = \frac{dy}{b} = \frac{du}{d} \quad \text{(e)}$$

$$\frac{\partial(v,w)}{\partial(x,y)} = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$$

There are 2 independent integrals of eq (e), we get : **c_1=v(x,y,u); c_2=w(x,y,u).**
Then, the general solution is **ϕ(v,w)=0, or v=F(w), or w=G(v).**

Each relation c1=v, c2=w is called an integral of the subsidiary eqs. It is assumed that the functions v,w are **functionally independent**; that is, the Jacobians ∂(v,w)/∂(x,y), ∂(v,w)/∂(x,z), ∂(v,w)/∂(y,z) are **not all zero** at any point.

First order Partial Differential Equations

Quasilinear PDE **The Linear Equation: Method of Lagrange**

E.g. $\boldsymbol{xuu_x + yuu_y = -(x^2+y^2)} \rightarrow a(x,y,u)=xu,\ b(x,y,u)=yu,\ d(x,y,u)=-(x^2+y^2)$

$$\frac{dy}{dx}=\frac{yu}{xu}=\frac{y}{x};\ \frac{du}{dx}=\frac{-\left(x^2+y^2\right)}{xu}$$

$$\frac{dy}{dx}=\frac{b}{a};\ \frac{du}{dx}=\frac{d}{a}\rightarrow\frac{dx}{a}=\frac{dy}{b}=\frac{du}{d}=ds$$

The first integral of subsidiary eqs is $\ln y=\ln x+c_1 \rightarrow v(x,y,u)=\frac{y}{x}=c_1$

$$\frac{\partial(v,w)}{\partial(x,y)}=\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$$

This relation is used to eliminate y from the second subsidiary eq:

$$\frac{du}{dx}=\frac{-\left(x^2+x^2c_1^2\right)}{xu} \rightarrow xdx=-\frac{udu}{1+c_1^2} \rightarrow \left(1+c_1^2\right)x^2+u^2=c_2$$

Replace c_1 by y/x, the second integral of subsidiary eqs is $\boldsymbol{w(x,y,u)=x^2+y^2+u^2=c_2}$**.**

The Jacobians are $\frac{\partial(v,w)}{\partial(x,y)}=-2\frac{x^2+y^2}{x^2},\ \frac{\partial(v,w)}{\partial(x,u)}=-2\frac{yu}{x^2},\ \frac{\partial(v,w)}{\partial(y,u)}=2\frac{u}{x}$

They are different from zero in any region of space in which $xyu\neq 0$. If Ω is such a region, then the General solution of the quasilinear eq is

$$F\left(\frac{y}{x},x^2+y^2+u^2\right)=0$$

the general solution is **ϕ(v,w)=0, or v=*F*(w), or w=*G*(v).**

Or can be written as

$$\frac{y}{x}=f\left(x^2+y^2+u^2\right),\quad x^2+y^2+u^2=G\left(\frac{y}{x}\right)$$

First order Partial Differential Equations

The Linear Equation: Method of Lagrange (Lagrange Multipliers)

E.g. $2xu_x+3yu_y = x+y \rightarrow a(x,y,u)=2x,\ b(x,y,u)=3y,\ d(x,y,u)= x+y$

$$\frac{dy}{dx}=\frac{b}{a};\ \frac{du}{dx}=\frac{d}{a}\rightarrow\frac{dx}{a}=\frac{dy}{b}=\frac{du}{d}=ds$$

$$\frac{dx}{ds}=2x \qquad \text{(a)}$$

$$\frac{dy}{ds}=3y \qquad \text{(b)} \qquad \frac{du}{ds}=x+y \qquad \text{(c)}$$

$$\frac{\partial(v,w)}{\partial(x,y)}=\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$$

(a)+(b) $\rightarrow$ $\dfrac{dx}{2x}=\dfrac{dy}{3y}\rightarrow 3\ln x=2\ln y+\ln c\rightarrow c_1=\dfrac{x^3}{y^2}=v$

(a)+(b) $\rightarrow$ $\dfrac{1}{2}\dfrac{dx}{ds}+\dfrac{1}{3}\dfrac{dy}{ds}=x+y$ +(c) $\rightarrow$ $\dfrac{1}{2}\dfrac{dx}{ds}+\dfrac{1}{3}\dfrac{dy}{ds}=\dfrac{du}{ds}\rightarrow\dfrac{d}{ds}\left(\dfrac{1}{2}x+\dfrac{1}{3}y-u\right)=0$

$\rightarrow$ $c_1=\dfrac{1}{2}x+\dfrac{1}{3}y-u=w$

the general solution is $\dfrac{1}{2}x+\dfrac{1}{3}y-u=G\left(\dfrac{x^3}{y^2}\right)$

the general solution is **ϕ(v,w)=0, or v=*F*(w), or w=*G*(v).**

First order Partial Differential Equations

The Linear Equation: Separation of variables

Extra notes

Solve using separation of variables

$$2\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} = 0 \qquad u(x,0) = 4e^x$$

Let $u(x,y) = X(x)Y(y)$ ⇨ $u_x(x,y) = X'Y$ $\quad u_y(x,y) = X\,Y'$

Substituting into the PDE, we have $2X'Y - 4X\,Y' = 0$ ⟶ $\frac{X'}{X} = 2\frac{Y'}{Y} = k$

The two ODE's are $X' - kX = 0$ $\quad Y' - \frac{1}{2}kY = 0$

The solutions of the two ODE's are $X = c_1 e^{kx}$ $\quad Y = c_2 e^{ky/2}$

⇨ $u(x,y) = c_1 e^{kx} c_2 e^{ky/2} = c\,e^{k\left(x+\frac{1}{2}y\right)}$ The general solution is $u(x,y) = \sum_i c_i \exp\left(k_i\left(x + \frac{1}{2}y\right)\right) = f\left(x + \frac{1}{2}y\right)$

If y=0, then

$$u(x,0) = c_1 e^{kx} c_2 e^{ky/2} = c\,e^{kx} = 4e^x$$

Then, c=4, k=1, we get

Therefore

$$u(x,y) = c\,e^{k\left(x+\frac{1}{2}y\right)} = 4e^{\left(x+\frac{1}{2}y\right)}$$

Second order Partial Differential Equations

Classification

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix} \rightarrow \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{\partial x(r,\theta)}{\partial r} = 1, \quad \frac{\partial r(x,y)}{\partial x} = 4 \rightarrow \frac{\partial x}{\partial r} \neq 1 \Big/ \frac{\partial r}{\partial x}$$

The general linear 2ed order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

The discriminant of the equation = $B^2 - 4AC$

Example :

$$u_{xx} + u_{yy} = \sin(x + y)$$

$$yu_{xx} - 2u_{xy} - xu_{yy} - u_x + \cos(y)u_y - 4 = 0$$

$$u_{tt} = c^2 u_{xx}$$

$$4u_{xx} + u_{xy} - 2u_{yy} - \cos(xy) = 0$$

$$u_t = ku_{xx}$$

Second order Partial Differential Equations

Classification

The general linear 2ed order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

$B^2 - 4AC > 0$ → hyperbolic

$B^2 - 4AC = 0$ → parabolic

$B^2 - 4AC < 0$ → elliptic

Example:

$$u_{tt} = c^2 u_{xx}$$

$$u_t = ku_{xx}$$

$$u_{xx} + u_{yy} = \sin(x + y)$$

$$4u_{xx} + u_{xy} - 2u_{yy} - \cos(xy) = 0$$

$$yu_{xx} - 2u_{xy} - xu_{yy} - u_x + \cos(y)u_y - 4 = 0$$

Second order Partial Differential Equations

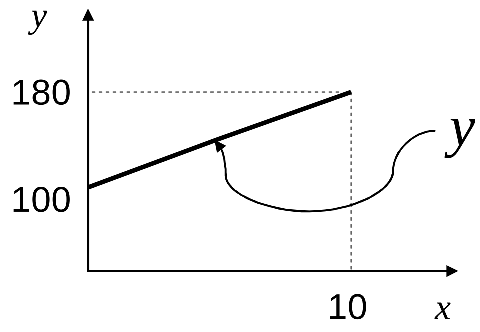
Separation of variables – Heat equation (nonzero temp at endpoints), Fourier Sinus series

The heat conduction equation is represented by (u is temperature)

$B^2 - 4AC = 0^2 - 4\alpha^2 \cdot 0 = 0$
parabolic

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0<x<10,\ t>0 \qquad (1)$$

Subject to:
Condition (A): $u(0,t)=70$,
Condition (B): u(10,t)=120, [fix temperature]
Condition (C): Initial temperature:
$u(x,0)=g(x)=8x+100$, $0<x<10$



Solution:
Equation (1) can be written as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \qquad (2)$$

α related with rate or time!

Using separation of variable, we assume that the solution is

$$u(x,t) = u = X(x)\cdot T(t) = XT \qquad (3)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}XT = T\frac{\partial}{\partial x}X = T\frac{d}{dx}X = TX' \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}TX' = TX'' \Rightarrow \frac{\partial u}{\partial t} = T'X$$

After substitute the above into equation (2), we get $X''T = \frac{1}{\alpha^2}XT'$ (4) $\Rightarrow$ $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -p^2$ (5)

Case p=0, $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = 0 \Rightarrow \frac{X''}{X} = 0 \qquad \frac{1}{\alpha^2}\frac{T'}{T} = 0 \qquad X = \int a\,dx + b = ax + b \qquad T = c$

$$\Rightarrow u_{p=0} = XT = c(ax+b) = Ax + B \qquad (6)$$

Second order Partial Differential Equations

Separation of variables – Heat equation (nonzero temp at endpoints), Fourier Sinus series

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \Longrightarrow u_{p=0} = XT = c(ax+b) = Ax+B \quad (6)$$

Now, apply condition (A) for (6), $u(0,t)=70$, we get
$70=A\cdot 0+B= B$, we get $B=70$.

From equation (6), we get $\boldsymbol{u_{p=0}=Ax+70}$. (6.a)

Now, apply condition (B) for (6.a), $u(10,t)=120$, we get
$120=10A+70$, we get $A=5$. $\rightarrow$ we get $\boldsymbol{u_{p=0}=5x+70}$. (6.b)

And general solution is $\boldsymbol{u} = \boldsymbol{u_{p=0}+u_{p\neq 0}}$. **(7)**

Now, we study for
Case $p\neq 0$,

$$\frac{X''}{X} = -p^2 \Longrightarrow X'' + p^2 X = 0 \quad \text{Using assumption: } X=e^{mx}$$

$$\longrightarrow X = A\cos px + B\sin px$$

$$\frac{1}{\alpha^2}\frac{T'}{T} = -p^2 \Longrightarrow T = Ce^{-\alpha^2 p^2 t}$$

the general solution of equation is

$$u = u_{p=0} + u_{p\neq 0} = 5x + 70 + (A\cos px + B\sin px)\left(Ce^{-\alpha^2 p^2 t}\right) =$$
$$5x + 70 + (M\cos px + N\sin px)e^{-\alpha^2 p^2 t}. \quad (8)$$

Let $\lambda=\alpha p$, we get

$$u = 5x + 70 + \left(M\cos\frac{\lambda x}{\alpha} + N\sin\frac{\lambda x}{\alpha}\right)e^{-\lambda^2 t} \quad (8.a)$$

Second order Partial Differential Equations

Separation of variables – Heat equation (nonzero temp at endpoints), Fourier Sinus series

$$u = 5x + 70 + \left(M\cos\frac{\lambda x}{\alpha} + N\sin\frac{\lambda x}{\alpha} \right) e^{-\lambda^2 t} \qquad (8.a)$$

Now, apply condition (A) for (8.a), $u(0,t)$=70, we get $70 = 0 + 70 + (M\cos 0 + N\sin 0)e^{-\lambda^2 t}$ ⇨ $0 = Me^{-\lambda^2 t}$

$e^{-\lambda^2 t} \neq 0$ ⇨ we get M=0. ⇨ $u = 5x + 70 + \left(N\sin\frac{\lambda x}{\alpha} \right) e^{-\lambda^2 t}$ (9)

Now, apply condition (B) for (9), $u(10,t)$=120, we get $120 = 50 + 70 + \left(N\sin\frac{10\lambda}{\alpha} \right) e^{-\lambda^2 t}$ ➞ $e^{-\lambda^2 t} \neq 0$

and N≠0, we get $\sin\frac{10\lambda}{\alpha} = 0 = \sin n\pi$ ⇨ $\lambda = \frac{\alpha n\pi}{10}$ ➞ $\lambda_n = \frac{\alpha n\pi}{10}$ λ_n is called eigenvalues

$u_n = \left(N_n \sin\frac{\lambda_n x}{\alpha} \right) e^{-\lambda_n^2 t}$ superposition principle, → $u = 5x + 70 + u_1 + u_2 + u_3 + \cdots = 5x + 70 + \sum_{n=1}^{\infty} u_n$

⇨ $$u = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{\lambda_n x}{\alpha}\right) e^{-\lambda^2 t} = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right) \exp\left(-\frac{\alpha^2 n^2 \pi^2}{100} t\right) \qquad (10)$$

apply condition (C), initial conditions, $u(x,0)=8x+100$, or
[t=0, u=8x+100] we get

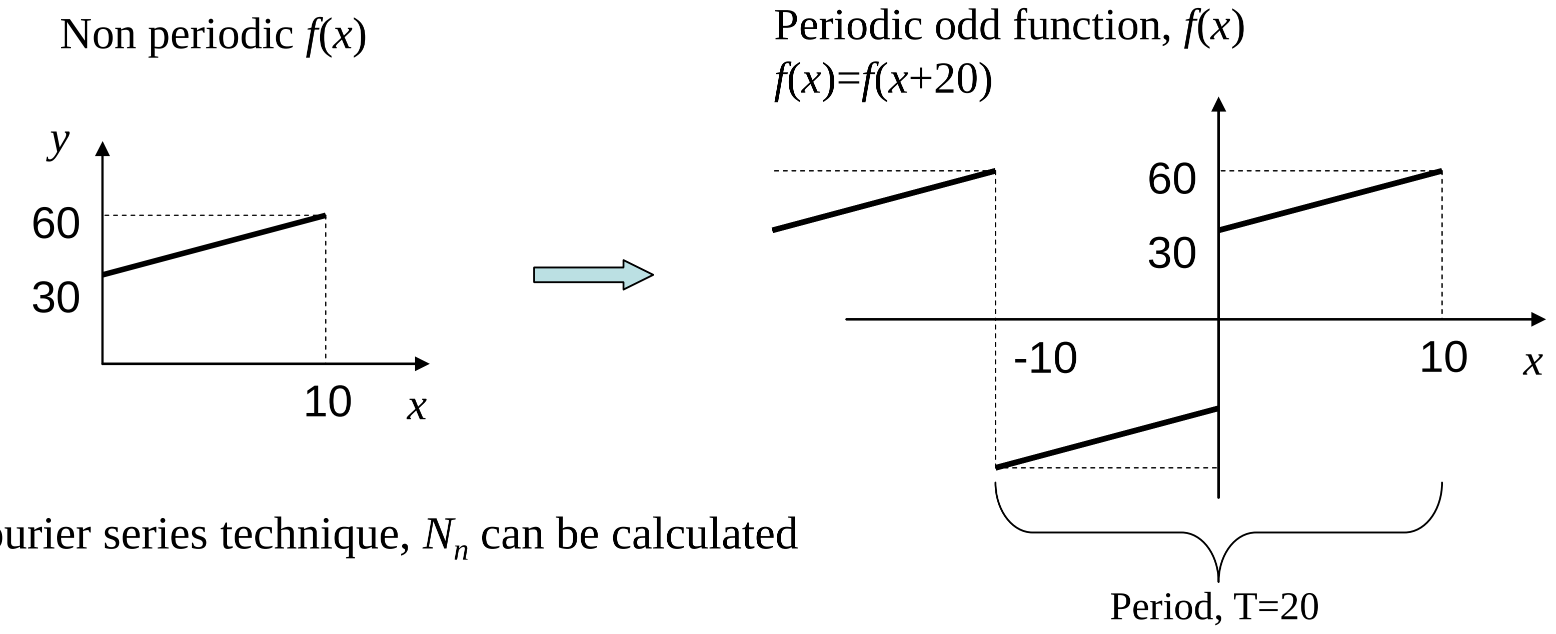
$8x + 100 = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right) e^{0}$ ➞ $$3x + 30 = f(x) = \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right) \qquad (11)$$

Second order Partial Differential Equations

Separation of variables – Heat equation (nonzero temp at endpoints), Fourier Sinus series

$$3x + 30 = f(x) = \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right) \qquad (11)$$

Equation (11) can be solved by using half-range sin series, where we have to transform the original *f*(*x*) into an odd periodic *f*(*x*).



Using the Fourier series technique, N_n can be calculated as

$$N_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{n\pi x}{10}\right)dx = \frac{2}{T}\int_{-T/2}^{T/2} odd \cdot odd\, dx = \frac{2}{T}\int_{-T/2}^{T/2} even\, dx = \frac{4}{T}\int_{0}^{T/2} even\, dx$$

$$N_n = \frac{4}{T}\int_{0}^{T/2} f(x)\sin\left(\frac{n\pi x}{10}\right)dx = \frac{4}{20}\int_{0}^{10} f(x)\sin\left(\frac{n\pi x}{10}\right)dx \Rightarrow N_n = \frac{1}{5}\int_{0}^{10}(3x+30)\sin\left(\frac{n\pi x}{10}\right)dx$$

Second order Partial Differential Equations

Separation of variables – Heat equation (nonzero temp at endpoints), Fourier Sinus series

Using the integration by parts, we get

$$N_n = \frac{1}{5}\left[(3x+30)\left(-\tfrac{10}{n\pi}\right)\cos\left(\tfrac{n\pi x}{10}\right) + 3\left(\tfrac{100}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi x}{10}\right)\right]_{x=0}^{x=10}$$

differentiate	Integrate
$3x+30$ +1	$\sin\left(\frac{n\pi x}{10}\right)$
3 -1	$\left(-\frac{10}{n\pi}\right)\cos\left(\frac{n\pi x}{10}\right)$
0	$\left(-\frac{100}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{10}\right)$

$$N_n = \frac{1}{5}\left[60\left(-\tfrac{10}{n\pi}\right)\cos(n\pi) + \left(\tfrac{300}{n^2\pi^2}\right)\sin(n\pi)\right] - \frac{1}{5}\left[30\left(-\tfrac{10}{n\pi}\right)\cos(0) + \left(\tfrac{300}{n^2\pi^2}\right)\sin(0)\right]$$

Since $\sin(n\pi)=0$, $\cos(0)=1$ and $\sin(0)=0$, we get $N_n = \dfrac{60}{n\pi}(1-2\cos(n\pi)) \quad n=1,2,3,\ldots$

The final solution is

$$u = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{\alpha^2 n^2\pi^2}{100}t\right)$$

$$= 5x + 70 + \sum_{n=1}^{\infty} \tfrac{60}{n\pi}(1-2\cos(n\pi))\sin\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{\alpha^2 n^2\pi^2}{100}t\right).$$

$$= 5x + 70 + \frac{60}{\pi}\left(\frac{3}{1}\sin\left(\frac{1\pi x}{10}\right)\exp\left(-\frac{\alpha^2 1^2\pi^2}{100}t\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{10}\right)\exp\left(-\frac{\alpha^2 2^2\pi^2}{100}t\right) + \frac{3}{3}\sin\left(\frac{3\pi x}{10}\right)\exp\left(-\frac{\alpha^2 3^2\pi^2}{100}t\right) + \cdots\right)$$

Second order Partial Differential Equations

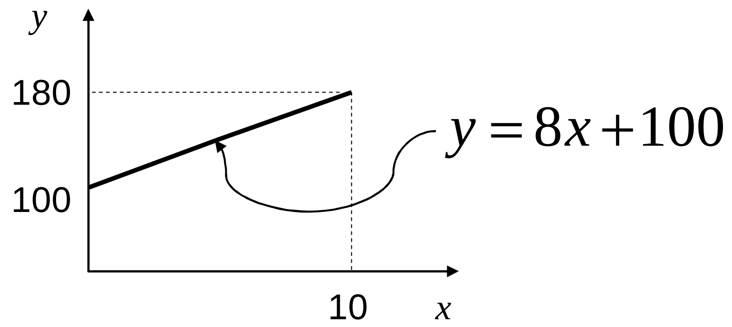
Separation of variables – Heat equation (insulated at endpoints), Fourier Cos series

The heat conduction equation is represented by (u is temperature)

$B^2 - 4AC = 0^2 - 4\alpha^2 \cdot 0 = 0$
parabolic

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0<x<10,\ t>0 \qquad (1)$$

Subject to:
Condition (A): $u_x(0,t)=0$,
Condition (B): $u_x(10,t)=0$, [both insulated]
Condition (C): Initial temperature:
$u(x,0)=g(x)=8x+100$, $0<x<10$



Solution:
Equation (1) can be written as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \qquad (2)$$

Using separation of variable, we assume that the solution is

$$u(x,t) = u = X(x)\cdot T(t) = XT \qquad (3)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}XT = T\frac{\partial}{\partial x}X = T\frac{d}{dx}X = TX' \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}TX' = TX'' \Rightarrow \frac{\partial u}{\partial t} = T'X$$

After substitute the above into equation (2), we get $X''T = \frac{1}{\alpha^2}XT'$ (4) $\Rightarrow$ $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -p^2$ (5)

Case p=0, $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = 0 \Rightarrow \frac{X''}{X} = 0 \qquad \frac{1}{\alpha^2}\frac{T'}{T} = 0 \qquad X = \int a\,dx + b = ax + b \qquad T = c$

$$\Rightarrow u_{p=0} = XT = c(ax+b) = Ax + B \qquad (6)$$

Second order Partial Differential Equations

Separation of variables – Heat equation (insulated at endpoints), Fourier Cos series

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \quad \Longrightarrow \quad u_{p=0} = XT = c(ax+b) = Ax + B \quad (6)$$

Now, apply condition (A) for (6), $u_x(0,t)=0$, we get
0=A, we get A=0.

From equation (6), we get $\boldsymbol{u_{p=0}=B}$. (6.a)

Now, apply condition (B) for (6.a), $u_x(10,t)=0$, we get
0=0, remain unchanged. $\rightarrow$ we get $\boldsymbol{u_{p=0}=B}$. (6.b)

And general solution is $\boldsymbol{u} = \boldsymbol{u_{p=0}+u_{p\neq 0}}$. **(7)**

Now, we study for
Case $p\neq 0$,

$$\Longrightarrow X'' + p^2 X = 0 \quad \text{Using assumption: } X=e^{mx}$$

$$X = A\cos px + B\sin px \qquad \frac{1}{\alpha^2}\frac{T'}{T} = -p^2 \quad \Longrightarrow \quad T = Ce^{-\alpha^2 p^2 t}$$

the general solution of equation is

$$u = u_{p=0} + u_{p\neq 0} = B + (A\cos px + B\sin px)\left(Ce^{-\alpha^2 p^2 t}\right) =$$

$$B + (M\cos px + N\sin px)e^{-\alpha^2 p^2 t}.$$

Second order Partial Differential Equations

Separation of variables – Heat equation (insulated at endpoints), Fourier Cos series

$$u = u_{p=0} + u_{p\neq 0} = B + (M\cos px + N\sin px)e^{-\alpha^2 p^2 t} \qquad (8.a)$$

Now, apply condition (A) for (8.a), $u_x(0,t)=0$, we get $0 = 0 + (-Mp\sin 0 + Np\cos 0)e^{-\alpha^2 p^2 t}$ ⇨ $0 = Npe^{-\alpha^2 p^2 t}$

$e^{-\alpha^2 p^2 t} \neq 0$ ⇨ we get N=0. ⇨ $u = B + M\cos px \cdot e^{-\alpha^2 p^2 t}$ (9)

Now, apply condition (B) for (9), $u_x(10,t)=0$, we get $0 = 0 - Mp\sin(10p)\cdot e^{-\alpha^2 p^2 t}$ → $e^{-\alpha^2 p^2 t} \neq 0$

and $M\neq 0$, we get $\sin(10p) = 0 = \sin n\pi$ ⇨ $p = \dfrac{n\pi}{10}$ → $p_n = \dfrac{n\pi}{10}$

$u_n = M_n \cos\left(\dfrac{n\pi x}{10}\right)e^{-\alpha^2 p_n^2 t}$ superposition principle, → $u = B + u_1 + u_2 + u_3 + \cdots = B + \sum_{n=1}^{\infty} u_n$

⇨
$$u = B + \sum_{n=1}^{\infty} M_n \cos\left(\frac{n\pi x}{10}\right)e^{-\alpha^2 p_n^2 t} = B + \sum_{n=1}^{\infty} M_n \cos\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{\alpha^2 n^2 \pi^2}{100}t\right) \qquad (10)$$

apply condition (C), initial conditions, $u(x,0)=8x+100$, or
[t=0, u=8x+100] we get

$$8x + 100 = B + \sum_{n=1}^{\infty} M_n \cos\left(\frac{n\pi x}{10}\right)e^{0} \quad \rightarrow \quad 8x + 100 = B + \sum_{n=1}^{\infty} M_n \cos\left(\frac{n\pi x}{10}\right) \qquad (11)$$

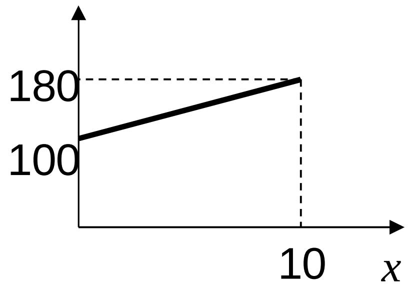
Second order Partial Differential Equations

Separation of variables – Heat equation (insulated at endpoints), Fourier Cos series

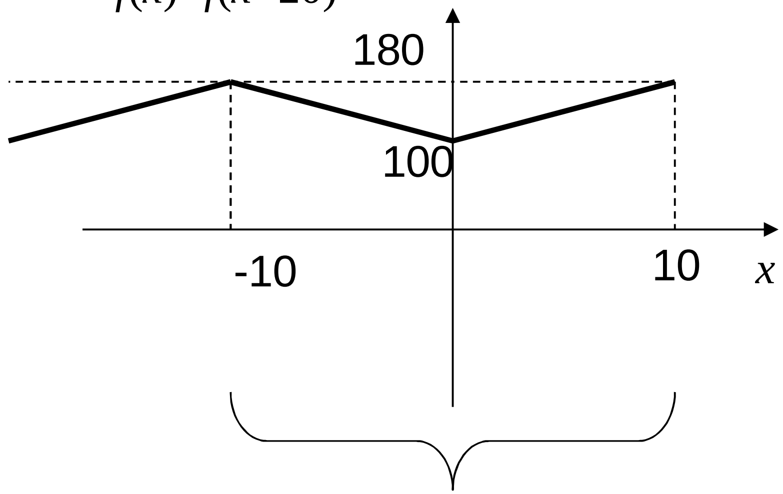
$$8x + 100 = B + \sum_{n=1}^{\infty} M_n \cos\left(\frac{n\pi x}{10}\right) \qquad (11)$$

Equation (11) can be solved by using half-range Fourier cos series, where we have to transform the original $f(x)$ into an even periodic $f(x)$.

Non periodic $f(x)$



Periodic odd function, $f(x)$
$f(x)=f(x+20)$



Period, T=20

Using the Fourier series technique, B can be calculated as

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx = 2\frac{2}{20}\int_0^{T/2} even\,dx = \frac{1}{5}\int_0^{10} 8x + 100\,dx = 280 \;\rightarrow\; B = \frac{a_0}{2} = 140$$

$$a_n = M_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{n\pi x}{10}\right)dx = 2\frac{2}{20}\int_0^{T/2} even\,dx = \frac{1}{5}\int_0^{10}(8x+100)\cos\left(\frac{n\pi x}{10}\right)dx$$

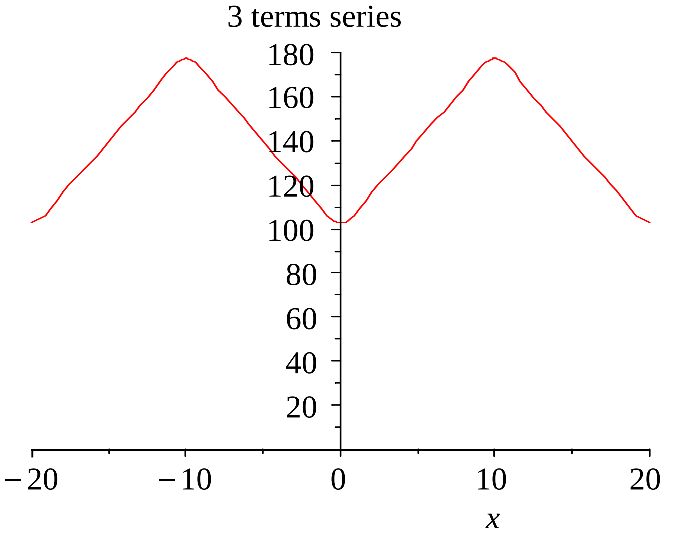
$$= \frac{1}{5}\left[\frac{40}{\pi^2 n^2}\left(\pi n(2x+25)\sin\left(\frac{n\pi x}{10}\right) + 20\cos\left(\frac{n\pi x}{10}\right)\right)\right] = \frac{1}{5}\left[\frac{200}{\pi^2 n^2}\left(9\pi n\sin(n\pi) + 4\cos(n\pi) - 4\right)\right]$$

Second order Partial Differential Equations

Separation of variables – Heat equation (insulated at endpoints), Fourier Cos series

$$a_n = M_n = \frac{40}{\pi^2 n^2}\left(9\pi n \sin(n\pi) + 4\cos(n\pi) - 4\right) = \frac{160}{\pi^2 n^2}\left(\cos(n\pi) - 1\right) = \frac{-320}{\pi^2 1^2}, \frac{-320}{\pi^2 3^2}, \cdots$$

half-range Fourier cos series



$$g(x) = 8x + 100 = 140 + \sum_{n=1}^{\infty} \frac{160}{\pi^2 n^2}\left(\cos(n\pi) - 1\right)\cos\left(\frac{n\pi x}{10}\right)$$

$$= 140 - \frac{320}{\pi^2}\left(\frac{1}{1^2}\cos\left(\frac{1\pi x}{10}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{10}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{10}\right) + \cdots\right)$$

The final solution is

$$u = 140 + \sum_{n=1}^{\infty} \frac{160}{\pi^2 n^2}\left(\cos(n\pi) - 1\right)\cos\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{\alpha^2 n^2 \pi^2}{100} t\right)$$

$$u = 140 - \frac{320}{\pi^2}\left(\frac{1}{1^2}\cos\left(\frac{1\pi x}{10}\right)\exp\left(-\frac{\alpha^2 1^2 \pi^2}{100} t\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{10}\right)\exp\left(-\frac{\alpha^2 3^2 \pi^2}{100} t\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{10}\right)\exp\left(-\frac{\alpha^2 5^2 \pi^2}{100} t\right) + \cdots\right)$$

Second order Partial Differential Equations

Separation of variables – Heat equation – linear property

$$\alpha^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} \qquad 0<x<10,\ t>0 \qquad (1)$$

Subject to:
Condition (A): $w(0,t)=a$,
Condition (B): $w(10,t)=b$,
Condition (C): Initial temperature:
$w(x,0)=f(x)$, $0<x<10$

nonhomogeneous

$$\alpha^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \qquad 0<x<10,\ t>0 \qquad (2)$$

Subject to:
Condition (A): $v(0,t)=0$,
Condition (B): $v(10,t)=0$,
Condition (C): Initial temperature:
$v(x,0)=g(x)$, $0<x<10$

homogeneous

Linear property: $w = v + h(x)$,

Where: $h''(x) = 0$
$f(x) = g(x)+h(x)$
$w(0,t)=v(0,t)+h(0)$,
$w(10,t)=v(10,t)+h(10)$,

Second order Partial Differential Equations

Transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} r \\ \theta \end{bmatrix} \rightarrow \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{\partial x(r,\theta)}{\partial r} = 1, \ \frac{\partial r(x,y)}{\partial x} = 4 \rightarrow \frac{\partial x}{\partial r} \neq 1 \Big/ \frac{\partial r}{\partial x}$$

The general linear 2[ed] order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Use the transformation: $\xi = \xi(x,y), \ \eta = \eta(x,y)$

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

One-to-one

$u_x = u_\xi \xi_x + u_\eta \eta_x, \ u_y = u_\xi \xi_y + u_\eta \eta_y$ $\quad u(x,y)=w(x,y)$ or **remain**

$$u_{xx} = u_\xi \xi_{xx} + \xi_x (u_{\xi\xi}\xi_x + u_{\eta\xi}\eta_x) + \eta_{xx} u_\eta + \eta_x (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)$$

$$u_{yy} = u_\xi \xi_{yy} + \xi_y (u_{\xi\xi}\xi_y + u_{\eta\xi}\eta_y) + \eta_{yy} u_\eta + \eta_y (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

Substitute we get:

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu + g = 0$$

where:

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$b^2 - 4ac = (B^2 - 4AC)J^2$$

The sign of the discriminant is **invariant** of the transformation.

Second order Partial Differential Equations

Canonical Form

The general linear 2[ed] order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

1 hyperbolic

$$u_{\xi\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

$B^2 - 4AC > 0$

2 parabolic

$$u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

$B^2 - 4AC = 0$

3 elliptic

$$au_{\xi\xi} + bu_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

$B^2 - 4AC < 0$

$$u_{\xi\xi} + u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

Canonical Form

Second order Partial Differential Equations

Canonical Form

The general linear 2ed order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

$B^2 - 4AC > 0$

1 hyperbolic

$$u_{\xi\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

$$\xi = \xi(x, y), \;\; \eta = \eta(x,y)$$

Use the transformation:

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu + g = 0$$

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \quad b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y \quad c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$a = 0 \longrightarrow A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \longrightarrow A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \longrightarrow \left(\frac{\xi_x}{\xi_y}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Second order Partial Differential Equations

Canonical Form

Hyperbolic: $B^2 - 4AC > 0$

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_{\xi} + eu_{\eta} + fu + g = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \quad b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y \quad c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$a = 0$ → $A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$ → $A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$ → $\left(\frac{\xi_x}{\xi_y}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$c = 0$ → $A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ → $A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$ → $\left(\frac{\eta_x}{\eta_y}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

Second order Partial Differential Equations

Canonical Form

The general linear 2ed order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

1 hyperbolic

$$u_{\xi\eta} + \Phi(\xi,\eta,u,u_\xi,u_\eta) = 0$$

Step 1 **Form characterestic equation:**

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B-\sqrt{B^2-4AC}}{2A} \qquad \frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B+\sqrt{B^2-4AC}}{2A}$$

Step 2 **Solve these equations**

$$\xi(x,y) = c_1 \qquad \eta(x,y) = c_2$$

Example:

$$u_{xx} + 2\cos(x)u_{xy} - \sin^2(x)u_{yy} - \sin(x)u_y = 0$$

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{2\cos x - 2}{2} \rightarrow \xi = y - \sin x + x = c_1$$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{2\cos x + 2}{2} \rightarrow \eta = y - \sin x - x = c_2$$

$\rightarrow -4u_{\xi\eta} = 0$

$\rightarrow u = f(\xi) + g(\eta)$

Second order Partial Differential Equations

Canonical Form

Hyperbolic: $B^2 - 4AC > 0$

e.g.: $3u_{xx}+10u_{xy}+3u_{yy}=0.$ **$10^2 - 4(3)(3) > 0$**

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

a=0 $\rightarrow$ $$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B-\sqrt{B^2-4AC}}{2A} = \frac{1}{3} \rightarrow \xi = 3y - x = c_1$$

c=0 $\rightarrow$ $$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B+\sqrt{B^2-4AC}}{2A} = 3 \rightarrow \eta = y - 3x = c_2$$

$$3u_{xx}+10u_{xy}+3u_{yy}=0 \rightarrow -64u_{\xi\eta}=0. \quad \rightarrow u = f(\xi)+g(\eta) \rightarrow u(x,y)=f(3y-x)+g(y-3x).$$

e.g.: $u_{xx}-(2\sin x)u_{xy}-(\cos^2 x)u_{yy}-(\cos x)u_y=0.$

a=0 $\rightarrow$ $$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B-\sqrt{B^2-4AC}}{2A} = -\sin x - 1 \rightarrow \xi = x + y - \cos x = c_1$$

c=0 $\rightarrow$ $$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B+\sqrt{B^2-4AC}}{2A} = -\sin x + 1 \rightarrow \eta = -x + y - \cos x = c_2$$

$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y = -4$$

$$-4u_{\xi\eta}=0 \rightarrow u = f(\xi)+g(\eta) \rightarrow u(x,y)=f(x+y-\cos x)+g(-x+y-\cos x)$$

Second order Partial Differential Equations

Canonical Form

***Parabolic:* $B^2 - 4AC = 0$**

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_{\xi} + eu_{\eta} + fu + g = 0$$

The general linear 2ed order PDE in two variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

2 parabolic

$$u_{\eta\eta} + \Phi(\xi, \eta, u, u_{\xi}, u_{\eta}) = 0$$

$b^2 - 4ac = 0 \rightarrow$ set $a=0 \rightarrow$ $b=0$. ($c \neq 0$)

$$A \neq 0, C \neq 0$$
$$B^2 - AC = 0$$

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B - \sqrt{B^2 - 4AC}}{2A} \quad a=0$$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B + \sqrt{B^2 - 4AC}}{2A} \quad c=0 \text{ (no !)}$$

$$\frac{dy}{dx} = \frac{B}{2A}$$

Step 1 **Form characterestic equation:** $\dfrac{dy}{dx} = \dfrac{B}{2A}$

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

Step 2 **Solve this equation** $\xi(x, y) = k$

Choose any cont func with cont 1st 2ed partial derivatives with

$\xi(x,y)$ & $\eta(x,y)$ are independent:

e.g. $\eta = y$ or $\eta = x$.

Example: $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

$$\frac{dy}{dx} = \frac{4}{2} = 2 \rightarrow \xi = y - 2x; \eta = x;$$

$$\rightarrow u_{\eta\eta} = 0 \rightarrow u = \eta F(\xi) + g(\xi)$$

Second order Partial Differential Equations

Canonical Form

Parabolic: $B^2 - 4AC = 0$

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_{\xi} + eu_{\eta} + fu + g = 0$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

2 parabolic

$$u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

e.g.: $9u_{xx} - 12u_{xy} + 4u_{yy} = 0$. $\quad$ $12^2 - 4(9)(4) = 0$

***a*=0 (and *b*=0)** $\quad \dfrac{dy}{dx} = \dfrac{B}{2A} = -\dfrac{2}{3} \rightarrow \xi = 2x + 3y \text{ , } \eta = x$ $\quad \dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \neq 0$

$9u_{xx} - 12u_{xy} + 4u_{yy} = 0 \rightarrow \mathbf{9u_{\eta\eta} = 0} \rightarrow \boldsymbol{u = \eta F(\xi) + g(\xi) = xF(2x+3y) + g(2x+3y)}$

Second order Partial Differential Equations

Canonical Form

Elliptic: $B^2 - 4AC < 0$

The general linear 2ed order PDE in two variables x, y.

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

3 elliptic

$$au_{\xi\xi} + bu_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

$$u_{\xi\xi} + u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

Canonical form is not solved!

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu + g = 0$$

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \qquad b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y \qquad c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$a=0 \rightarrow \frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B - \sqrt{B^2 - 4AC}}{2A} \rightarrow \xi = \alpha + i\beta = c_1 \qquad c=0 \rightarrow \frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B + \sqrt{B^2 - 4AC}}{2A} \rightarrow \eta = \alpha - i\beta = c_2$$

Second transformation $\alpha = \dfrac{\zeta + \eta}{2}, \beta = \dfrac{\zeta - \eta}{2i} \qquad \alpha_x = \dfrac{\zeta_x + \eta_x}{2}, \beta_x = \dfrac{\zeta_x - \eta_x}{2i}$

$$b = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \beta_x\alpha_y) + 2C\alpha_y\beta_y$$

$$\rightarrow b = 0, a \neq 0, c \neq 0.$$

$$= 2A\left(-\tfrac{i}{4}\right)(\xi_x + \eta_x)(\xi_x - \eta_x) + B\left(-\tfrac{i}{4}\right)\left[(\xi_x + \eta_x)(\xi_y - \eta_y) + (\xi_x - \eta_x)(\xi_y + \eta_y)\right] + 2C\left(-\tfrac{i}{4}\right)(\xi_y + \eta_y)(\xi_y - \eta_y)$$

$$= 2\left(-\tfrac{i}{4}\right)A\left(\xi_x^2 - \eta_x^2\right) + 2\left(-\tfrac{i}{4}\right)C\left(\xi_y^2 - \eta_y^2\right) + B\left(-\tfrac{i}{4}\right)2\left(\xi_x\xi_y - \eta_x\eta_y\right) = \left(-\tfrac{i}{4}\right)2(a - c) = 0$$

Second order Partial Differential Equations

Canonical Form

Elliptic: $B^2 - 4AC < 0$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

3 elliptic

$$u_{\xi\xi} + u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu + g = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

Canonical form is not solved!

e.g.: $u_{xx} + x^2 u_{yy} = 0$. **$0^2 - 4(1)(x^2) < 0$**

$$a=0 \rightarrow \frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -ix \rightarrow \xi = \frac{x^2}{2} - iy = \alpha + i\beta = c_1$$

$$c=0 \rightarrow \frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B + \sqrt{B^2 - 4AC}}{2A} = ix \rightarrow \eta = \frac{x^2}{2} + iy = \alpha - i\beta = c_2$$

Second transformation $\alpha = \frac{\zeta + \eta}{2} = \frac{x^2}{2}, \beta = \frac{\zeta - \eta}{2i} = -y$ $\rightarrow$ **$b = 0, a \neq 0, c \neq 0.$**

$$\longrightarrow \quad x^2 u_{\alpha\alpha} + x^2 u_{\beta\beta} + u_\alpha = 0$$

$$\longrightarrow \quad u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{x^2} \rightarrow u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}$$

Second order Partial Differential Equations

Canonical Form

Elliptic: $B^2 - 4AC < 0$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

3 elliptic

$$u_{\xi\xi} + u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

Transformation $\xi(x,y)$, $\eta(x,y)$ are given!

Example :

$$u_{xx} + 2u_{xy} + 3u_{yy} + 4u = 0$$

$2^2 - 4(1)(3) = -8 < 0$

Canonical form is not solved!

$$\boldsymbol{a=0} \rightarrow \frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B - \sqrt{B^2 - 4AC}}{2A} = 1 - i\sqrt{2} \rightarrow \xi = y - x + x\sqrt{2}i = \alpha + i\beta = c_1$$

$$\boldsymbol{c=0} \rightarrow \frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B + \sqrt{B^2 - 4AC}}{2A} == 1 + i\sqrt{2} \rightarrow \eta = y - x - x\sqrt{2}i = \alpha - i\beta = c_2$$

Second transformation $\alpha = \dfrac{\zeta + \eta}{2} = y - x,\ \beta = \dfrac{\zeta - \eta}{2i} = x\sqrt{2}$ $\rightarrow$ ***b* = 0, *a* ≠ 0, *c* ≠ 0.**

$\rightarrow$

$$2u_{\alpha\alpha} + 2u_{\beta\beta} + 4u = 0$$

Second order Partial Differential Equations

d'Alembert's Solution

1 The Wave Equation :

$$u_{tt} = c^2 u_{xx}$$

$-\infty < x < \infty.$

1 hyperbolic

$$w_{\xi\eta} + \Phi(\xi, \eta, w, w_\xi, w_\eta) = 0$$

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B + \sqrt{B^2 - AC}}{A} \qquad \frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = \frac{B - \sqrt{B^2 - AC}}{A}$$

$$\xi(x, y) = k \qquad \eta(x, y) = K$$

$$\frac{dt}{dx} = \frac{1}{c} \longrightarrow \xi = x - ct$$

$$\frac{dt}{dx} = -\frac{1}{c} \longrightarrow \eta = x + ct$$

$$\longrightarrow w_{\xi\eta} = 0 \longrightarrow (w_\eta)_\xi = 0$$

$$\downarrow$$

$$w_\eta = v(\eta)$$

$$w(\xi, \eta) = \int v(\eta) d\eta + F(\xi)$$

$$w(\xi, \eta) = F(\xi) + G(\eta)$$

$$u(x, t) = F(x - ct) + G(x + ct)$$

d'Alembert's Solution of the wave equation

Second order Partial Differential Equations

d'Alembert's Solution

$$u_{tt} = c^2 u_{xx}$$

$-\infty < x < \infty.$

Initial conditions:

$u(x,0)=f(x)$ (a)

$u_t(x,0)=g(x)$ (b)

→ **$u(x,t) = \phi(x+ct)+\psi(x-ct)$ (c)**

From eq (c) and (a), we get $u(x,0) = f(x) = \phi(x)+\psi(x)$ (d)

From eq (b), we get $\frac{\partial u(x,0)}{\partial t} = c\frac{d\phi(x)}{dx} - c\frac{d\psi(x)}{dx} = g(x)$ (e)

⇨ $\phi(x)-\psi(x) = \frac{1}{c}\int_{x_0}^{x} g(\tau)d\tau + k$ (f) , x_0 and k are arbitrary constants.

Solve (d) and (f) together, we get

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)d\tau + \frac{k}{2},$$
$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)d\tau - \frac{k}{2}. \quad \text{(g)}$$

Using transformation in (c), we get

$$\phi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(\tau)d\tau + \frac{k}{2},$$
$$\psi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(\tau)d\tau - \frac{k}{2}. \quad \text{(h)}$$

Following eq. (c), we get

$$u(x,t) = \phi(x+ct)+\psi(x-ct) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\tau)d\tau - \int_{x_0}^{x-ct} g(\tau)d\tau\right]$$
$$= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\tau)d\tau + \int_{x-ct}^{x_0} g(\tau)d\tau\right] = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

Second order Partial Differential Equations

d'Alembert's Solution

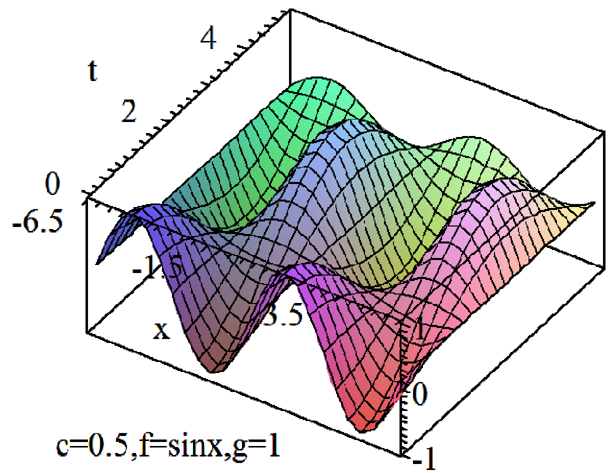
E.g.: $u_{tt}=c^2u_{xx}$, $-\infty<x<\infty$.
IC: $u(x,0)=\sin(x)=f(x)$, $u_t(x,0)=0=g(x)$.

Using d'Alembert method, we get

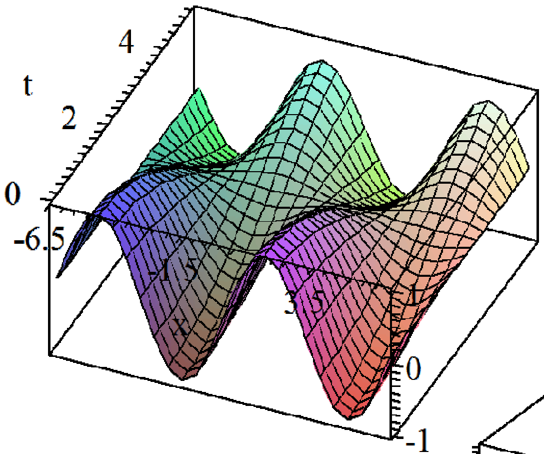
$$u(x,t)=\phi(x+ct)+\psi(x-ct)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\tau)d\tau$$

$$=\frac{1}{2}[f(x+ct)+f(x-ct)]=\frac{1}{2}[\sin(x+ct)+\sin(x-ct)]$$

c=1,f=sinx,g=0



c=0.5,f=sinx,g=0



E.g.: $u_{tt}=c^2u_{xx}$, $-\infty<x<\infty$.
IC: $u(x,0)=\sin(x)=f(x)$, $u_t(x,0)=1=g(x)$.

$$u(x,t)=\phi(x+ct)+\psi(x-ct)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}1d\tau$$

$$=\frac{1}{2}[\sin(x+ct)+\sin(x-ct)]+t$$

c=0.5,f=sinx,g=1