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SSCE1793, Differential Equation: Tutorial, heat equation (PDE)

The heat conduction equation is represented by (u is temperature)

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0<x<10,\ t>0 \tag{1}$$

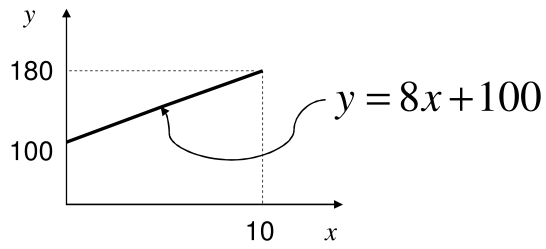
Subject to:

Condition (A): $u(0,t)=70$,

Condition (B): u(10,t)=120, [fix temperature]

Condition (C): Initial temperature:

$u(x,0)=f(x)=8x+100$, $0<x<10$.



Solution:

Equation (1) can be written as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}, \tag{2}$$

Using separation of variable, we assume that the solution is

$$u(x,t) = u = X(x)\cdot T(t) = XT, \tag{3}$$

Then, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}XT = T\frac{\partial}{\partial x}X = T\frac{d}{dx}X = TX', \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}TX' = TX'',$$

$$\frac{\partial u}{\partial t} = T'X,.$$

After substitute the above into equation (2), we get

$$X''T = \frac{1}{\alpha^2}XT', \tag{4}$$

After rearranging, we get

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -p^2, \tag{5}$$

Case p=0,

We get

$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = 0$, or it can be break down into two equations as:

$\frac{X''}{X} = 0$, and $\frac{1}{\alpha^2}\frac{T'}{T} = 0$.

It can be further simplified as

$X'' = 0$, and $T' = 0$, and we get

$X' = a$, and $T = c$,

$X = \int a\,dx + b = ax + b$, and $T = c$,

Finally, according to equation (3) we get

Solution, for p=0,

$$u_{p=0} = XT = c(ax+b) = Ax + B, \tag{6}$$

Where $A=ca$, $B=cb$.

Now, apply condition (A) for (6), $u(0,t)=70$, we get

$70=A\cdot 0+B= B$, we get $B=70$.

From equation (6), we get

$$u_{p=0} = Ax + 70, \tag{6.a}$$

Now, apply condition (B) for (6.a), $u(10,t)$=120, we get
120=10A+70, we get A=5.

So, $u_{p=0} = 5x + 70$. (6.b)

And general solution is

$$u = u_{p=0} + u_{p\neq 0}. \tag{7}$$

Now, we study for

Case $p \neq 0$,

From equation (5), we get

$\frac{X''}{X} = -p^2$, $\rightarrow$ $X'' = -p^2 X$, and get homogeneous equation, $X'' + p^2 X = 0$,

Using the assumption, $X = e^{mx}$, $\rightarrow$ $m^2 + p^2 = 0$, $\rightarrow$ $m = \pm pi$.

So, equation (7) has solution $X = A\cos px + B\sin px$,

Use same method, $\frac{1}{\alpha^2}\frac{T'}{T} = -p^2$, $\rightarrow$ $T' + \alpha^2 p^2 T = 0$, $\rightarrow$ $m + \alpha^2 p^2 = 0$, $\rightarrow$ $m = -\alpha^2 p^2$.

So, we have $T = Ce^{-\alpha^2 p^2 t}$.

Finally, the general solution of equation is

$$u = u_{p=0} + u_{p\neq 0} = 5x + 70 + (A\cos px + B\sin px)\left(Ce^{-\alpha^2 p^2 t}\right) = 5x + 70 + (M\cos px + N\sin px)e^{-\alpha^2 p^2 t}. \tag{8}$$

Let $\lambda = \alpha p$, we get

$$u = 5x + 70 + \left(M\cos\frac{\lambda x}{\alpha} + N\sin\frac{\lambda x}{\alpha}\right)e^{-\lambda^2 t}, \tag{8.a}$$

Now, apply condition (A) for (8.a), $u(0,t)$=70, we get

$70 = 0 + 70 + (M\cos 0 + N\sin 0)e^{-\lambda^2 t}$, or $0 = Me^{-\lambda^2 t}$,

Since $e^{-\lambda^2 t} \neq 0$, we get M=0.

Equation (8.a) becomes

$$u = 5x + 70 + \left(N\sin\frac{\lambda x}{\alpha}\right)e^{-\lambda^2 t}, \tag{9}$$

Now, apply condition (B) for (9), $u(10,t)$=120, we get

$120 = 50 + 70 + \left(N\sin\frac{10\lambda}{\alpha}\right)e^{-\lambda^2 t}$, since $e^{-\lambda^2 t} \neq 0$ and $N \neq 0$, we get

$\sin\frac{10\lambda}{\alpha} = 0 = \sin n\pi$, n=1,2,3,…

So, we get $\lambda = \frac{\alpha n\pi}{10}$, n=1,2,3,…

Or $\lambda_n = \frac{\alpha n\pi}{10}$, n=1,2,3,… (9.a)

λ_n is called eigenvalues.

So, for different n, from equation (9), we get

$u_n = \left(N_n \sin\frac{\lambda_n x}{\alpha}\right)e^{-\lambda_n^2 t}$, n=1,2,3,…

Using the superposition principle, the general solution of wave equation becomes

$$u = 5x + 70 + u_1 + u_2 + u_3 + \cdots = 5x + 70 + \sum_{n=1}^{\infty} u_n$$, or can be written as

$$u = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{\lambda_n x}{\alpha}\right) e^{-\lambda_n^2 t} = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right) \exp\left(-\frac{\alpha^2 n^2 \pi^2}{100} t\right), \qquad (10)$$

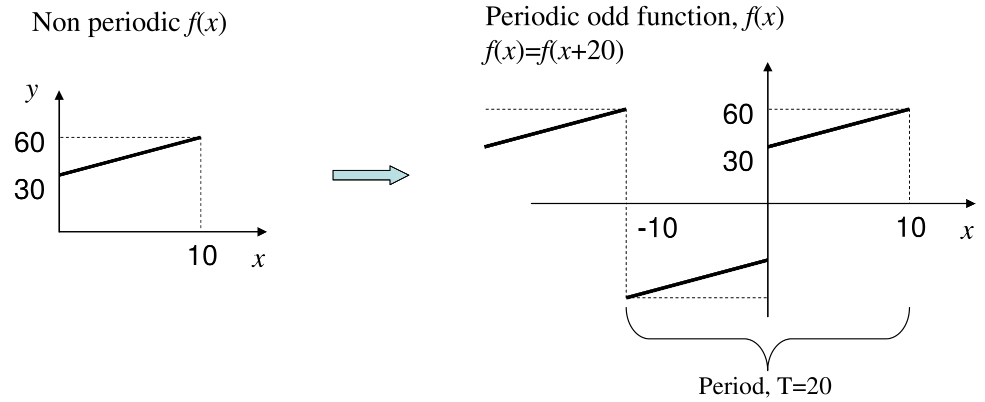
Now, apply condition (C), initial conditions, $u(x,0)=8x+100$, or [$t=0$, $u=8x+100$] we get

$$8x + 100 = 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right) e^0$$, since $e^0=1$, we get

$$3x + 30 = f(x) = \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right), \qquad (11)$$

where $f(x)=3x+30$.

Equation (11) can be solved by using half-range sin series, where we have to transform the original $f(x)$ into an odd periodic $f(x)$.



Using the Fourier series technique, N_n can be calculated as

$$N_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{n\pi x}{10}\right)dx = \frac{2}{T}\int_{-T/2}^{T/2} odd\cdot odd\; dx = \frac{2}{T}\int_{-T/2}^{T/2} even\, dx = \frac{4}{T}\int_{0}^{T/2} even\, dx ,$$

So, we get

$$N_n = \frac{4}{T}\int_{0}^{T/2} f(x)\sin\left(\frac{n\pi x}{10}\right)dx = \frac{4}{20}\int_{0}^{10} f(x)\sin\left(\frac{n\pi x}{10}\right)dx$$, or it can be written as

$$N_n = \frac{1}{5}\int_{0}^{10} (3x + 30)\sin\left(\frac{n\pi x}{10}\right)dx ,$$

Using the integration by parts, we get

differentiate	Integrate
$3x + 30$ +1	$\sin\left(\frac{n\pi x}{10}\right)$
3 -1	$\left(-\frac{10}{n\pi}\right)\cos\left(\frac{n\pi x}{10}\right)$
0	$\left(-\frac{100}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{10}\right)$

So, we get

$$N_n = \frac{1}{5}\left[(3x + 30)\left(-\tfrac{10}{n\pi}\right)\cos\left(\tfrac{n\pi x}{10}\right) + 3\left(\tfrac{100}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi x}{10}\right)\right]_{x=0}^{x=10}$$

$$N_n = \frac{1}{5}\left[60\left(-\tfrac{10}{n\pi}\right)\cos(n\pi) + \left(\tfrac{300}{n^2\pi^2}\right)\sin(n\pi)\right] - \frac{1}{5}\left[30\left(-\tfrac{10}{n\pi}\right)\cos(0) + \left(\tfrac{300}{n^2\pi^2}\right)\sin(0)\right]$$

Since sin($n\pi$)=0, cos(0)=1 and sin(0)=0, we get

$$N_n = \frac{60}{n\pi}\left(1 - 2\cos(n\pi)\right), \qquad (12)$$

The final solution is

$$\begin{aligned} u &= 5x + 70 + \sum_{n=1}^{\infty} N_n \sin\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{\alpha^2 n^2 \pi^2}{100} t\right) \\ &= 5x + 70 + \sum_{n=1}^{\infty} \tfrac{60}{n\pi}\left(1 - 2\cos(n\pi)\right)\sin\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{\alpha^2 n^2 \pi^2}{100} t\right). \end{aligned} \qquad (13)$$

.

End of solution.

Note for integration by parts: using trigonometry identities

I=$\int \sin x \cdot \sin nx\, dx$ can be calculated as

Using trigo identities, we get

cos(A+B)=cosAcosB-sinAsinB (a)

cos(A-B)=cosAcosB+sinAsinB, (b)

(b)-(a), we get

cos(A-B)-cos(A+B)= 2sinAsinB, → sinAsinB= ½ cos(A-B)- ½ cos(A+B),

Finally, we get

sin(x)sin(nx)= ½ cos([1-n]x)- ½ cos([1+n]x)

The final result is

$$\begin{aligned} I &= \int \sin x \cdot \sin nx\, dx = \frac{1}{2}\int \cos([1-n]x)dx - \frac{1}{2}\int \cos([1+n]x)dx \\ &= \frac{1}{2(1-n)}\sin([1-n]x) - \frac{1}{2(1+n)}\sin([1+n]x). \quad n \neq 1. \end{aligned}$$

For n=1, we get

$$I = \int \sin^2 x\, dx = \frac{1}{2}\int 1 dx - \frac{1}{2}\int \cos(2x)dx = \frac{x}{2} - \frac{1}{4}\sin(2x).$$

End.