By Dr Yeak Su Hoe, s.h.yeak@utm.my

SSCE1793, Differential Equation: Tutorial, wave equation (PDE)

The displacement of stretched string is represented by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

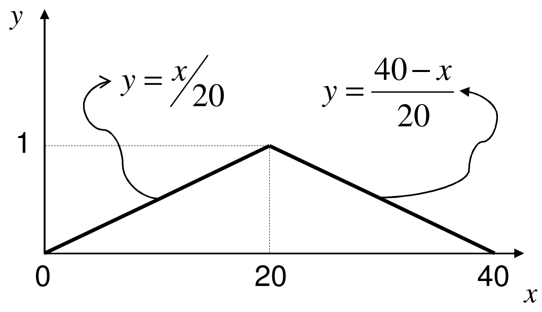
Subject to:

Condition (A): $u(0,t)=0$,

Condition (B): u(40,t)=0, [fixed end points]

Condition (C): Initial displacement:

$$u(x,0) = f(x) = \begin{cases} x/20 & ,0 \le x \le 20 \\ (40-x)/20 & ,20 \le x \le 40 \end{cases}$$



Condition (D): zero initial velocity

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = 0$$

Solution:

Using separation of variable, we assume that the solution is

$$u(x,t) = u = X(x)\cdot T(t) = XT, \tag{2}$$

Then, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} XT = T\frac{\partial}{\partial x} X = T\frac{d}{dx} X = TX', \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} TX' = TX'',$$

$$\frac{\partial u}{\partial t} = T'X, \quad \frac{\partial^2 u}{\partial t^2} = T''X.$$

After substitute the above into equation (1), we get

$$X''T = XT'', \tag{3}$$

After rearranging, we get

$$\frac{X''}{X} = \frac{T''}{T} = -p^2, \tag{4}$$

Case p=0,

We get

$\frac{X''}{X} = \frac{T''}{T} = 0$, or it can be break down into two equations as:

$\frac{X''}{X} = 0$, and $\frac{T''}{T} = 0$.

It can be further simplified as

$X'' = 0$, and $T'' = 0$, and we get

$X' = a$, and $T' = c$,

$X = \int a\,dx + b = ax + b$, and $T = \int c\,dt + d = ct + d$,

Finally, according to equation (2) we get

Solution, for p=0,

$$u_{p=0} = XT = (ax+b)(ct+d), \tag{5}$$

Now, apply condition (A) for (5), $u(0,t)=0$, we get

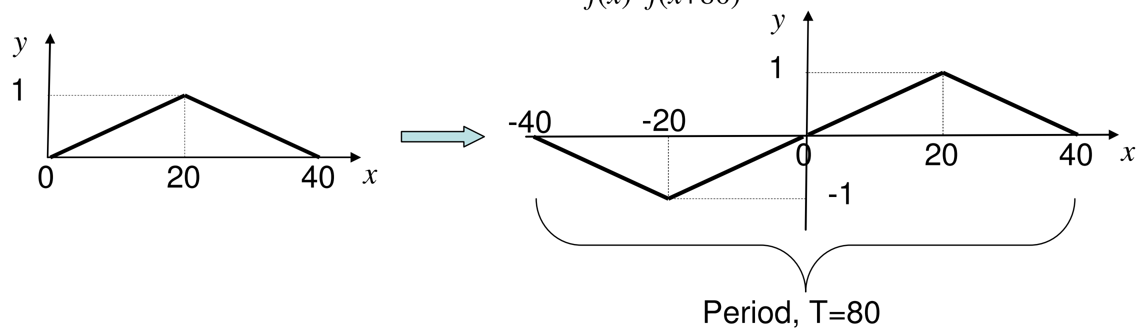
$0=(a\cdot 0+b)(ct+d)=b(ct+d)$, since $(ct+d)\neq 0$, we get $b=0$.
From equation (5), we get
$$u_{p=0}=ax(ct+d), \tag{5.a}$$
Now, apply condition (B) for (5.a), $u(40,t)=0$, we get
$0=40a(ct+d)$, since $(ct+d)\neq 0$, we get $a=0$.
So, $u_{p=0}=0$. (5.b)
And general solution is
$$u=u_{p=0}+u_{p\neq 0}. \tag{6}$$
Now, we study for
Case $p\neq 0$,
From equation (4), we get
$$\frac{X''}{X}=-p^2,\rightarrow X''=-p^2X \text{, and get homogeneous equation, } X''+p^2X=0, \tag{7}$$
Using the assumption, $X=e^{mx}$, $\rightarrow m^2+p^2=0$, $\rightarrow m=\pm pi$.
So, equation (7) has solution $X=A\cos px+B\sin px$,
Use same method, $\frac{T''}{T}=-p^2$, $\rightarrow T''+p^2T=0$, $\rightarrow m^2+p^2=0$, $\rightarrow m=\pm pi$.
So, we have $T=C\cos pt+D\sin pt$.

Finally, using **superposition principle**, the **general solution** of wave equation is
$$u=u_{p=0}+u_{p\neq 0}=0+(A\cos px+B\sin px)(C\cos pt+D\sin pt) \tag{8}$$
Now, apply condition (A) for (5), $u(0,t)=0$, we get
$0=(A\cos 0+B\sin 0)(C\cos pt+D\sin pt)=A(C\cos pt+D\sin pt)$,
Since $(C\cos pt+D\sin pt)\neq 0$, we get $A=0$.
Equation becomes
$$u=B\sin\lambda x(C\cos\lambda t+D\sin\lambda t)=\sin\lambda x(M\cos\lambda t+N\sin\lambda t), \tag{9}$$
where, $M=BC$, $N=BD$ and $\lambda=p$.
Now, apply condition (B) for (9), $u(40,t)=0$, we get
$0=\sin 40\lambda\,(M\cos\lambda t+N\sin\lambda t)$, since $(M\cos\lambda t+N\sin\lambda t)\neq 0$, we get
$\sin 40\lambda=0=\sin n\pi$, $n=1,2,3,\ldots$
So, we get $\lambda=n\pi/40$, $n=1,2,3,\ldots$
Or $\lambda_n=n\pi/40$, $n=1,2,3,\ldots$ (9.a)
λ_n is called eigenvalues.
So, for different n, from equation (9), we get
$u_n=\sin\lambda_n x(M_n\cos\lambda_n t+N_n\sin\lambda_n t)$, $n=1,2,3,\ldots$
Using the superposition principle, the general solution of wave equation becomes
$u=u_1+u_2+u_3+\cdots=\sum_{n=1}^{\infty}u_n$, or can be written as
$$u=\sum_{n=1}^{\infty}\sin(\lambda_n x)(M_n\cos\lambda_n t+N_n\sin\lambda_n t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{40}\right)\left(M_n\cos\left(\frac{n\pi t}{40}\right)+N_n\sin\left(\frac{n\pi t}{40}\right)\right), \tag{10}$$
Now, apply condition (C), initial conditions, $u(x,0)=f(x)$ or $t=0$, $u=f(x)$ we get
$f(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{40}\right)(M_n\cos(0)+N_n\sin(0))$, since $\cos(0)=1$, $\sin(0)=0$, we get
$$f(x)=\sum_{n=1}^{\infty}M_n\sin\left(\frac{n\pi x}{40}\right), \tag{11}$$

Equation (11) can be solved by using half-range sin series, where we have to transform the original $f(x)$ into an odd periodic $f(x)$.

Non periodic $f(x)$

Periodic odd function, $f(x)$
$f(x)=f(x+80)$



Using the Fourier series technique, M_n can be calculated as

$$M_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{n\pi x}{40}\right)dx = \frac{2}{T}\int_{-T/2}^{T/2} odd \cdot odd\, dx = \frac{2}{T}\int_{-T/2}^{T/2} even\, dx = \frac{4}{T}\int_{0}^{T/2} even\, dx,$$

So, we get

$$M_n = \frac{4}{T}\int_{0}^{T/2} f(x)\sin\left(\frac{n\pi x}{40}\right)dx = \frac{4}{80}\int_{0}^{40} f(x)\sin\left(\frac{n\pi x}{40}\right)dx,$$ or it can be written as

$$M_n = \frac{1}{20}\int_{0}^{20}\frac{x}{20}\sin\left(\frac{n\pi x}{40}\right)dx + \frac{1}{20}\int_{20}^{40}\frac{40-x}{20}\sin\left(\frac{n\pi x}{40}\right)dx,$$

Using the integration by parts, we get

differentiate		Integrate
$\frac{x}{20}$	+1	$\sin\left(\frac{n\pi x}{40}\right)$
$\frac{1}{20}$	-1	$\left(-\frac{40}{n\pi}\right)\cos\left(\frac{n\pi x}{40}\right)$
0		$\left(-\frac{40^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{40}\right)$

differentiate		Integrate
$\frac{40-x}{20}$	+1	$\sin\left(\frac{n\pi x}{40}\right)$
$\frac{-1}{20}$	-1	$\left(-\frac{40}{n\pi}\right)\cos\left(\frac{n\pi x}{40}\right)$
0		$\left(-\frac{40^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{40}\right)$

So, we get

$$M_n = \frac{1}{20}\left[-\tfrac{x}{20}\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi x}{40}\right) + \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi x}{40}\right)\right]_{x=0}^{x=20} + \frac{1}{20}\left[-\tfrac{40-x}{20}\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi x}{40}\right) - \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi x}{40}\right)\right]_{x=20}^{x=40}$$

$$M_n = \frac{1}{20}\left[-\tfrac{20}{20}\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi}{2}\right) + \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right)\right] - \frac{1}{20}\left[-0\left(\tfrac{40}{n\pi}\right)\cos(0) + \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin(0)\right]$$

$$+\frac{1}{20}\left[-\tfrac{40-40}{20}\left(\tfrac{40}{n\pi}\right)\cos(n\pi) - \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin(n\pi)\right] - \frac{1}{20}\left[-\tfrac{40-20}{20}\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi}{2}\right) - \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right)\right]$$

$$M_n = \frac{1}{20}\left[-\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi}{2}\right) + \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right)\right] - \frac{1}{20}[0+0]$$

$$+\frac{1}{20}[0-0] - \frac{1}{20}\left[-\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi}{2}\right) - \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right)\right]$$

$$M_n = \frac{1}{20}\left[-\left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi}{2}\right) + \left(\tfrac{40}{n\pi}\right)\cos\left(\tfrac{n\pi}{2}\right)\right] + \frac{1}{20}\left[\tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right) + \tfrac{1}{20}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right)\right] = \tfrac{1}{200}\left(\tfrac{1600}{n^2\pi^2}\right)\sin\left(\tfrac{n\pi}{2}\right).$$

Finally, we get $M_n = \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \frac{8}{1^2\pi^2}, -\frac{8}{3^2\pi^2}, \frac{8}{5^2\pi^2}, -\frac{8}{7^2\pi^2}, \cdots.$

Now, apply condition (D), $\left[\dfrac{\partial u}{\partial t}\right]_{t=0} = 0$, we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{40}\right)\left[M_n\left(-\sin\left(\frac{n\pi t}{40}\right)\right)\frac{n\pi}{40} + N_n\left(\cos\left(\frac{n\pi t}{40}\right)\right)\frac{n\pi}{40}\right].$$

$$0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{40}\right)\left[M_n(-\sin(0))\frac{n\pi}{40} + N_n(\cos(0))\frac{n\pi}{40}\right] = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{40}\right)\left[N_n \frac{n\pi}{40}\right].$$

Since $\sin\left(\dfrac{n\pi x}{40}\right) \neq 0$, So, we get $N_n=0$, $n=1,2,3,\ldots$

Finally, the general solution of wave equation, from Equation (10), become

$$u = \sum_{n=1}^{\infty} M_n \cos\left(\frac{n\pi t}{40}\right)\sin\left(\frac{n\pi x}{40}\right) = \sum_{n=1}^{\infty} \tfrac{8}{n^2\pi^2}\sin\left(\tfrac{n\pi}{2}\right)\cos\left(\frac{n\pi t}{40}\right)\sin\left(\frac{n\pi x}{40}\right). \qquad (12)$$

End of solution.

Note for integration by parts:

$\int_{x=a}^{x=b} x^2 \sin(nx)dx$ can be calculated as

differentiate		Integrate
x^2	+1	$\sin nx$
$2x$	-1	$\left(-\dfrac{1}{n}\right)\cos nx$
2	+1	$\left(-\dfrac{1}{n^2}\right)\sin nx$
0		$\left(\dfrac{1}{n^3}\right)\cos nx$

Finally, we get

$$\int_{x=a}^{x=b} x^2 \sin(nx)dx = \left[x^2\left(-\frac{1}{n}\right)\cos nx + 2x(-1)\left(-\frac{1}{n^2}\right)\sin nx + 2\left(\frac{1}{n^3}\right)\cos nx\right]_{x=a}^{x=b}$$

End.