Numerical Methods II
SSCM 3423

Part 2
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August 2013

Shooting Method
Finite Element Method (FEM)
Cubic Spline Interpolation

Shooting method – linear problem

BVP - Dirichlet Boundary condition

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y(b) = \beta.$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} & y'' = p(x)y' + q(x)y + r(x), \\ & y(a) = \alpha, \quad y'(a) = 0. \end{aligned}\right\} \quad \text{Solution: } y_1(x)$$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} & y'' = p(x)y' + q(x)y, \\ & y(a) = 0, \quad y'(a) = 1. \end{aligned}\right\} \quad \text{Solution: } y_2(x)$$

Linearity of De: $\boldsymbol{y = y_1(x) + c y_2(x)}$ is the solution for $\boldsymbol{y'' = p(x)y' + q(x)y + r(x)}$.

⇨ $y(a) = y_1(a) + c y_2(a) = \alpha + c\cdot 0 = \alpha$ ⇨ $y(b) = y_1(b) + c y_2(b) = \beta$

⇨ $$c = \frac{\beta - y_1(b)}{y_2(b)}$$

Shooting method – linear problem

Optimal RK2 method, $y'=f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

BVP - Dirichlet Boundary condition

$$-u'' + \pi^2 u = 2\pi^2 \sin(\pi x), \quad u(0)=\alpha=0, \ u(1)=\beta=0.$$

BVP

General solution: $ae^{-\pi x} + be^{\pi x} + \sin(\pi x)$

$$v'' = \pi^2 v - 2\pi^2 \sin(\pi x), \quad v(0)=\alpha=\mathbf{0}, \ v'(0)=0.$$

IVP1, solution: v

To system

$$\frac{d}{dx}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ \pi^2 v_1 - 2\pi^2 \sin(\pi x) \end{bmatrix}$$

$$w'' = \pi^2 w, \quad w(0)=0, \ w'(0)=1.$$

IVP2, (homogeneous) solution: w

To system

$$\frac{d}{dx}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ \pi^2 w_1 \end{bmatrix}$$

Solve using RK4

$$\boldsymbol{y(x+h) \approx y(x) + 1/6[k_1 + 2k_2 + 2k_3 + k_4]}$$

$k_1 = hf(x,y)$, $k_2 = hf(x + ½h, y + ½ k_1)$,

$k_3 = hf(x + ½h, y + ½k_2)$, $k_4 = hf(x+h, y+k_3)$

x_i	$v_i(x_i)$	$w_i(x_i)$
0.00	0.00000	0.000000
0.25	-0.157372	0.275702
0.50	-1.290357	0.730213
0.75	-4.490694	1.657343
1.00	-11.466375	3.656793

Not satisfy right BC!

Shooting method – linear problem

BVP - Dirichlet Boundary condition

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y(b) = \beta.$$

$$\left.\begin{aligned} -u'' + \pi^2 u &= 2\pi^2 \sin(\pi x), \\ u(0) &= u(1) = 0. \end{aligned}\right\} \text{BVP}$$

$\boldsymbol{y = y_1(x) + c y_2(x)}$ is the solution for $\boldsymbol{y'' = p(x)y' + q(x)y + r(x)}$.

$$c = \frac{\beta - v(b)}{w(b)} = \frac{0 - (-11.466375)}{3.656793} = 3.135637$$

$$c = \frac{\beta - y_1(b)}{y_2(b)}$$

$\boldsymbol{y_i = y_1(x_i) + c y_2(x_i) = v(x_i) + 3.135637 w(x_i)}$

Solve using RK4

x_i	$v_i(x_i)$	$w_i(x_i)$	y_i	***Exact, sin(πx)***
0.00	0.00000	0.000000	0.00000	0.00000
0.25	-0.157372	0.275702	0.707129	0.707107
0.50	-1.290357	0.730213	0.999327	1.00000
0.75	-4.490694	1.657343	0.706132	0.707107
1.00	-11.466375	3.656793	0.000000	0.000000

Optimal RK2 method, $y' = f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

Shooting method – linear problem

BVP – Left: Dirichlet BC, right: Robin BC

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \;\; \beta_1 y(b) + \beta_2 y'(b) = \beta_3.$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y + r(x),\\ &y(a) = \alpha, \;\; y'(a) = 0. \end{aligned}\right\}$$

Solution: $y_1(x)$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y,\\ &y(a) = 0, \;\; y'(a) = 1. \end{aligned}\right\}$$

Solution: $y_2(x)$

Linearity of De: $\boldsymbol{y = y_1(x) + c y_2(x)}$ is the solution for $\boldsymbol{y'' = p(x)y' + q(x)y + r(x)}$.

⇒ $y(a) = y_1(a) + c y_2(a) = \alpha + c \cdot 0 = \alpha$ ⇒ $\beta_1[y_1(b) + c y_2(b)] + \beta_2[y'_1(b) + c y'_2(b)] = \beta_3$

⇒ $$c = \frac{\beta_3 - \beta_1 y_1(b) - \beta_2 y_1'(b)}{\beta_1 y_2(b) + \beta_2 y_2'(b)}$$

Shooting method – linear problem

BVP – Left: Dirichlet BC, right: Neumann BC

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y'(b) = \beta \ .$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y + r(x), \\ &y(a) = \alpha, \quad y'(a) = 0. \end{aligned}\right\} \quad \text{Solution: } y_1(x)$$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y, \\ &y(a) = 0, \quad y'(a) = 1. \end{aligned}\right\} \quad \text{Solution: } y_2(x)$$

Linearity of De: $\boldsymbol{y = y_1(x) + c y_2(x)}$ is the solution for $\boldsymbol{y'' = p(x)y' + q(x)y + r(x)}$.

⇒ $y(a) = y_1(a) + c y_2(a) = \alpha + c \cdot 0 = \alpha$ ⇒ $[y'_1(b) + c y'_2(b)] = \beta$

⇒ $$c = \frac{\beta - y_1'(b)}{y_2'(b)}$$

Shooting method – linear problem

BVP – Left: Neumann BC, right: Dirichlet BC (or others)

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y'(a) = \alpha, \;\; y(b) = \beta \;.$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} & y'' = p(x)y' + q(x)y + r(x), \\ & y(a) = 0, \;\; y'(a) = \alpha. \end{aligned}\right\} \quad \text{Solution: } y_1(x)$$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} & y'' = p(x)y' + q(x)y, \\ & y(a) = 1, \;\; y'(a) = 0. \end{aligned}\right\} \quad \text{Solution: } y_2(x)$$

Linearity of De: $\boldsymbol{y = y_1(x) + c y_2(x)}$ is the solution for $\boldsymbol{y'' = p(x)y' + q(x)y + r(x)}$.

⇨ $y'(a) = y'_1(a) + c y'_2(a) = \alpha + c \cdot 0 = \alpha$ ⇨ $y(b) = y_1(b) + c y_2(b) = \beta$

⇨ $$c = \frac{\beta - y_1(b)}{y_2(b)}$$

Shooting method – linear problem

BVP – Left: Neumann BC, right: Robin BC

$$u'' + u = \sin(3x),$$
$$u'(0) = \alpha = 1,\ u\left(\pi/2\right) + u'\left(\pi/2\right) = \beta = -1.$$

Exact solution: $u = a\sin(x) + b\cos(x) - 1/8\sin(3x)$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} &v'' + v = \sin(3x),\\ &v(0) = 0,\ v'(0) = \alpha = 1. \end{aligned}\right\}$$

Solution: $v(x)$ $\frac{d}{dx}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ \sin(3x) - v_1 \end{bmatrix}$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} &w'' + w = 0,\\ &w(0) = 1,\ w'(0) = 0. \end{aligned}\right\}$$

Solution: $w(x)$ $\frac{d}{dx}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ -w_1 \end{bmatrix}$

Optimal RK2 method, $y' = f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}\left(K_1 + 3K_2\right)$$
$$K_1 = hf\left(t_i, y_i\right)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

Linearity of De: $\boldsymbol{u = v(x) + cw(x)}$ is the solution for $\boldsymbol{u'' + u = \sin(3x)}$.

⇨ $u'(0) = v'(0) + cw'(0) = \alpha + c\cdot 0 = \alpha = 1$

⇨ $u(\pi/2) + u'(\pi/2) =$
$v(\pi/2) + v'(\pi/2) + cw(\pi/2) + cw'(\pi/2) = \beta = -1$

⇨ $$c = \frac{\beta - v\left(\pi/2\right) - v'\left(\pi/2\right)}{w\left(\pi/2\right) + w'\left(\pi/2\right)}$$

Shooting method – linear problem

BVP – Left: Neumann BC, right: Robin BC

$$u'' + u = \sin(3x),$$
$$u'(0) = \alpha = 1,\ u\left(\pi/2\right) + u'\left(\pi/2\right) = \beta = -1.$$

general solution: $u=a\sin(x)+b\cos(x) - 1/8\sin(3x)$

$$c = \frac{\beta - v\left(\pi/2\right) - v'\left(\pi/2\right)}{w\left(\pi/2\right) + w'\left(\pi/2\right)} = \frac{-1 - 1.499586 - 0.000195}{0.000294 - 0.999900} = 2.500766$$

$$u_i = v(x_i) + c\,w(x_i) = v(x_i) + 2.500766\,w(x_i)$$

Solve using RK4

Exact solution: $u=(11/8)\sin(x)+(5/2)\cos(x) - 1/8\sin(3x)$

x_i	$v(x_i)$	$v'(x_i)$	$w(x_i)$	$w'(x_i)$	*Approx,* $u(x_i)$	*Exact,* $u(x_i)$
0.00	0.00000	1.000000	1.00000	0.00000	2.500766	2.5
π/8	0.411165	1.126997	0.923885	-0.382606	2.721585	2.720404
π/4	0.884311	1.237806	0.707176	-0.706967	2.652793	2.65165
3π/8	1.318095	0.873002	0.382859	-0.923726	2.275536	2.274878
π/2	1.499586	0.000195	0.000294	-0.999900	1.500321	1.5

Optimal RK2 method, $y'=f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}\left(K_1 + 3K_2\right)$$
$$K_1 = hf\left(t_i, y_i\right)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h,\ y_i + \frac{2}{3}K_1\right)$$

Shooting method – linear problem

BVP – Left: Robin BC, right: Dirichlet BC

$$y'' = p(x)y' + q(x)y + r(x),$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3,\ y(b) = \beta.$$

Let IVP1 (initial value problem)

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = 0,\quad y'(a) = 0.$$

Solution: $y_1(x)$

Let IVP2 (initial value problem) (homogeneous)

$$y'' = p(x)y' + q(x)y,$$
$$y(a) = 1,\quad y'(a) = 0.$$

Solution: $y_2(x)$

Let IVP3 (initial value problem)

$$y'' = p(x)y' + q(x)y,$$
$$y(a) = 0,\quad y'(a) = 1.$$

Solution: $y_3(x)$

Linearity of De: $\boldsymbol{y = y_1(x) + c_1 y_2(x) + c_2 y_3(x)}$ is the solution for $\boldsymbol{y'' = p(x)y' + q(x)y + r(x)}$.

$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$ ⇒ $[\alpha_1 y_2(a) + \alpha_2 y'_2(a)]c_1 + [\alpha_1 y_3(a) + \alpha_2 y'_3(a)]c_2 = \alpha_3 - \alpha_1 y_1(a) - \alpha_2 y'_1(a)$

⇒ $\alpha_1 c_1 + \alpha_2 c_2 = \alpha_3$

$y(b) = \beta$ ⇒ $y_1(b) + c_1 y_2(b) + c_2 y_3(b) = \beta$ ⇒ System of 2 equations!

Shooting method – nonlinear problem

BVP - Dirichlet Boundary condition

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1, \quad y(2) = 2.$$

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1.$$

General solution

$$y = \sqrt{2a - 2xa + 2x - x^2}$$

Let IVP (initial value problem)

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1, \quad y'(1) = p = p_i.$$

General solution

$$y = \sqrt{-2p + 2xp + 2x - x^2}$$

Solution: $y(x;p)$

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ {-1-(y_2)^2}\Big/{y_1} \end{bmatrix}$$

Get p^*, $\rightarrow$ $y(2;p^*)=2$.

p_0 & p_1 $\rightarrow$ arbitrary value

Rootfinding, objective function, $\rightarrow F(p)=2-y(2;p)$.

$$F(p_n) \approx 0$$

Avoid using Newton's method $\rightarrow$ $F' = \frac{d}{dp} y(2;p)$

Use Secant method $\rightarrow$ $p_n = p_{n-1} - F(p_{n-1})\dfrac{p_{n-1} - p_{n-2}}{F(p_{n-1}) - F(p_{n-2})}$

Start with p_0 & p_1 $\rightarrow$ get p_2 …

$$p_2 = p_1 - F(p_1)\frac{p_1 - p_0}{F(p_1) - F(p_0)}$$
$$= 1 - 0.585803\frac{1-0}{0.585803 - 1.895899} = 1.447145$$

Use RK4 $\rightarrow p_0 = 0 = y'(1)$

i	p_i	$y(2;p_i)$	$F(p_i)$ (error)
0	0.000000	0.104101	1.895899
1	1.0	1.414197	0.585803
2	1.447145	1.701210	0.298790
3	1.912638	1.955692	0.044308

Shooting method – nonlinear problem

BVP - Dirichlet Boundary condition

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1, \quad y(2) = 2.$$

IVP (initial value problem)

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1, \quad y'(1) = p = p_i.$$

Solution: $y(x;p)$

General solution

$$y = \sqrt{-2p + 2xp + 2x - x^2}$$

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1.$$

General solution

$$y = \sqrt{2a - 2xa + 2x - x^2}$$

Rootfinding, objective function, $\rightarrow F(p) = 2 - y(2;p)$.

$F(p_n) \approx 0$ Use Secant method

$$p_n = p_{n-1} - F(p_{n-1}) \frac{p_{n-1} - p_{n-2}}{F(p_{n-1}) - F(p_{n-2})}$$

Use RK4

$\rightarrow p_0 = 0 = y'(1)$

i	p_i	$y(2;p_i)$	$F(p_i)$ (error)
3	1.912638	1.955692	0.044308
4	1.993685	1.996677	0.003322
5	2.000256	1.999963	3.698×10^{-5}
6	2.000330	1.999999969	3.088×10^{-8}

x_i	$y(x_i;p_6)$	$y(x_i)$ exact	Absolute error
1.00	1.000000	1.000000	
1.10	1.178956	1.178983	0.000027
1.20	1.326623	1.326650	0.000027
1.30	1.452560	1.452584	0.000024

Shooting method – nonlinear problem

Try this!

$$2xy'' + (y')^2 - 4y = 4x,$$
$$y'(2) = 6, \ \ y'(2.1) = 6.2$$

BVP – Left Neumann, right Robin BC

$$2xy'' + (y')^2 - 4y = 4x,$$
$$y'(1) = 4, \ \ y(3) + 2y'(3) = 32.$$

Exact solution
***y*(*x*)=(*x*+1)²**

IVP (initial value problem)

$$2xy'' + (y')^2 - 4y = 4x,$$
$$y'(1) = 4, \ \ y(1) = p = p_i$$

Solution: $y(x;p)$

$$y_1 = y(x), \quad y_2 = y'(x).$$

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ \left(4x + 4y_1 - (y_2)^2\right)/2x \end{bmatrix}$$

$$y'' = -2yy', \ 0 \le x \le 1,$$
$$y(0) = 1, \ \ y(0.2) = 0.8333$$

Exact solution: *y*=1/(*x*+1).

Rootfinding, objective function, $\rightarrow F(p)=32-y(3;p)-2y'(3;p)$.

Optimal RK2 method, $y'=f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

Use RK4
$\rightarrow p_0=0=y'(1)$

$F(p_n) \approx 0$ Use Secant method

$$p_n = p_{n-1} - F(p_{n-1})\frac{p_{n-1} - p_{n-2}}{F(p_{n-1}) - F(p_{n-2})}$$

i	p_i	$y(3;p_i)$	$y'(3;p_i)$	$F(p_i)$ (error)
1	0	8.303482	5.781270	12.133978
2	1	10.361601	6.445059	8.748282
3	3.583894	15.254059	7.811455	1.123031
4	3.964445	15.936660	7.984070	0.0952008
5	3.999693	15.999486	7.999773	9.677×10^{-4}

$\Rightarrow$ p_6**=4.000055**

x_i	$y(x_i;p_6)$	$y(x_i)$ exact	Absolute error
1.00	4.000055	4.00	0.000055
1.20	4.840102	4.84	0.000102
1.40	5.760118	5.76	0.000118
…	…	…	…
3.00	16.000132	16.00	0.000132

Shooting method – nonlinear problem

BVP – Left Robin, right Dirichlet

$$y'' = 3y^3,$$

$$3y(0) - 9y'(0) = 2, \quad y(1) = \frac{1}{4}$$

IVP (initial value problem)

$$y'' = 3y^3,$$

$$3y(0) - 9p_i = 2, \quad y'(0) = p_i$$

IVP (initial value problem)

$$y'' = 3y^3,$$

$$3p_i - 9y'(0) = 2, \quad y(0) = p_i$$

Try this!!

$$y'' = -2yy', \; 0 \le x \le 1,$$

$$y(0) = 1, \quad y(0.2) = 0.8333$$

Exact solution: *y*=1/(*x*+1).

$$\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix} \rightarrow \frac{d}{dx}\mathbf{y} = \frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -2y_1y_2 \end{bmatrix}$$

Finite Element Method (FEM)

Comparison with the finite difference method (FDM)

The finite difference method (FDM) is an alternative way of approximating solutions of PDEs.
The differences between FEM and FDM are:

- The finite difference method is an approximation to the differential equation; the finite element method is an approximation to its solution.
- The most attractive feature of the FEM is its ability to handle complex geometries (and boundaries) with relative ease. While FDM in its basic form is restricted to handle rectangular shapes and simple alterations.
- The most attractive feature of finite differences is that it can be very easy to implement.
- The quality of the approximation between grid points is poor in FDM comparing to FEM.
- The quality of a FEM approximation is often higher than in the corresponding FDM approach, but this is extremely problem dependent and several examples to the contrary can be provided.

Generally, FEM is the method of choice in all types of analysis in **structural mechanics** while computational fluid dynamics (CFD) tends to use FDM or other methods (e.g., finite volume method). CFD problems usually require discretization of the problem into a **large number** of cells/gridpoints (millions and more), therefore cost of the solution favors **simpler**, **lower order approximation** within each cell.

Finite Element Method

Steady-state 2-D heat conduction

The heat flow through the wall of a heated room on a winter day is an example of conduction. In a thermally isotropic medium, Fourier's law for 2-D heat flow is:

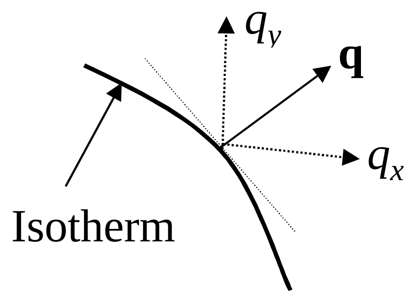
Index, not partial derivative!

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}$$

$T=T(x,y)$=temperature, q_x and q_y are components of heat flux (W/m^2), k is thermal conductivity (W/m. °C). (1W=1J/s=1Nm/s). Minus sign: heat is transferred in direction of decreasing temperature. k is material property.

$\mathbf{q}=q_x\mathbf{i}+q_y\mathbf{j}$, resultant heat flux (at right angles to an isotherm or a line of constant temperature.

$\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ are temperature gradients along x and y.



Constitutive relation- contains a material property.

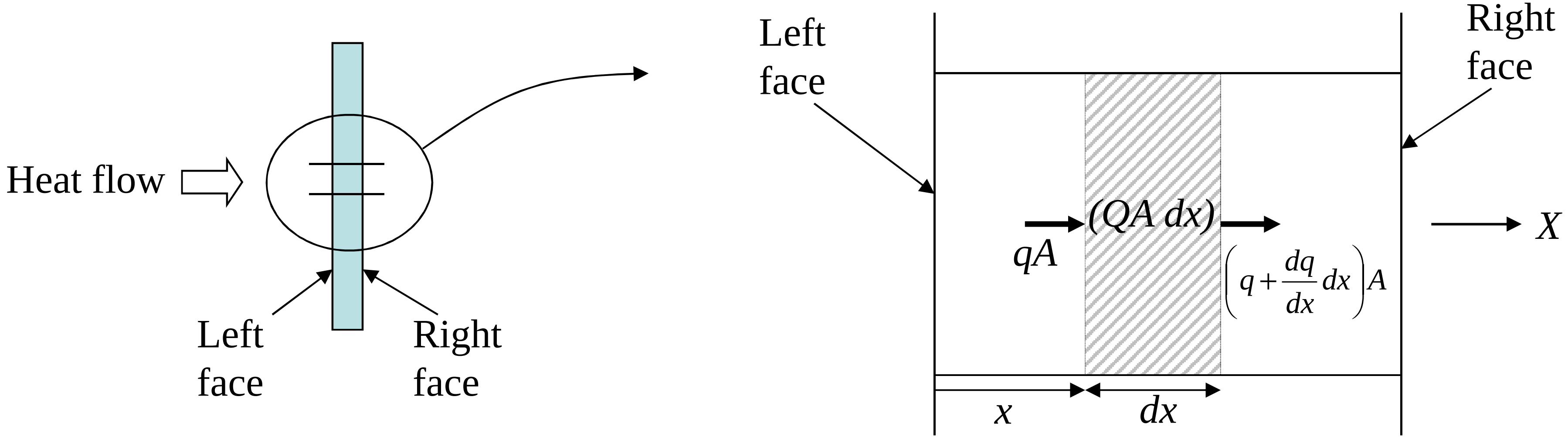
Convection-the flow of heat through a gas or a liquid

$q = h(T_s - T_\infty)$, q is convective heat flux (W/m^2), h is convection heat transfer coefficient or film coef (W/m$^2\cdot$°C), T_s and T_∞ are surface and fluid temperature.

Finite Element Method

Steady-state 1-D heat conduction

Governing equation (heat conduction in plane wall with uniform heat generation)



Let A =area normal to direction of heat flow,

Q(W/m^3) =internal heat generated per unit volume.

Heat rate (heat flux × area) enter the control volume + heat rate generated = Heat rate leaving control volume.

$$qA + QAdx = \left(q + \frac{dq}{dx}dx\right)A \xrightarrow{\text{simplify}} Q = \frac{dq}{dx}$$

$$q = -k\frac{small - big}{dx} = +ve$$

+ve=heat flux same direction with x-axis

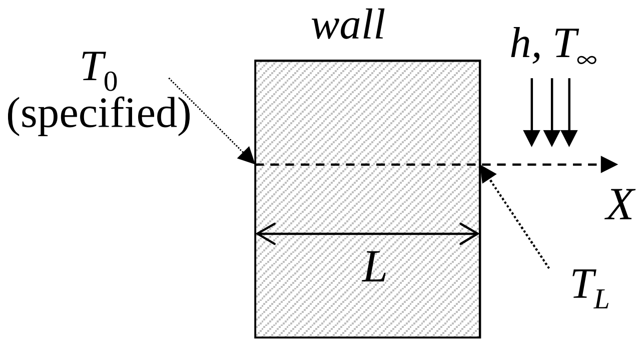
Substitute Fourier's law $q = -k\dfrac{dT}{dx}$ $\Rightarrow$ $$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0$$

Q is called source when +ve (heat is generated) and is called sink when –ve (heat is consumed)
Here, Q is referred as source.

Finite Element Method

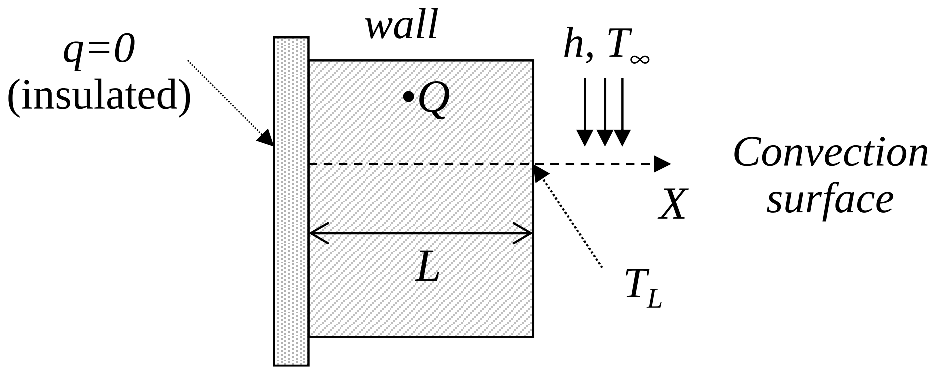
Steady-state 1-D heat conduction, Boundary conditions

Specified temperature



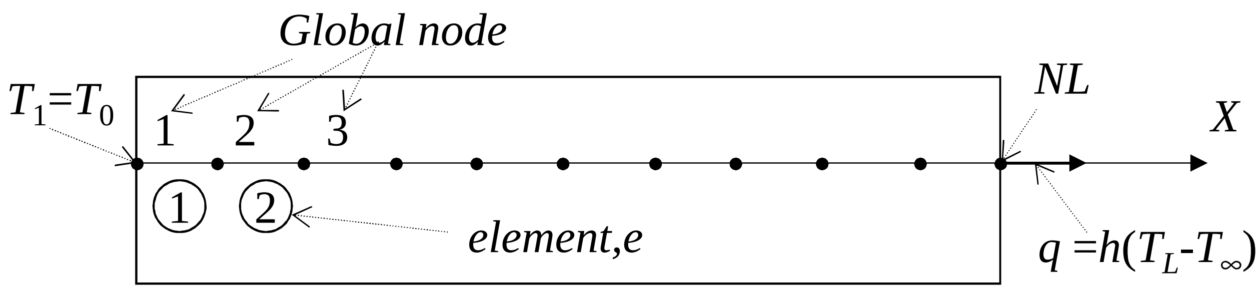
Wall of tank contain hot liquid at T_0,
airstream of T_∞ passed on outside,
maintain T_L at boundary.
$T|_{x=0} = T_0$, $q|_{x=L} = h(T_L - T_\infty)$. [note: $T_L > T_\infty$]

Specified heat flux



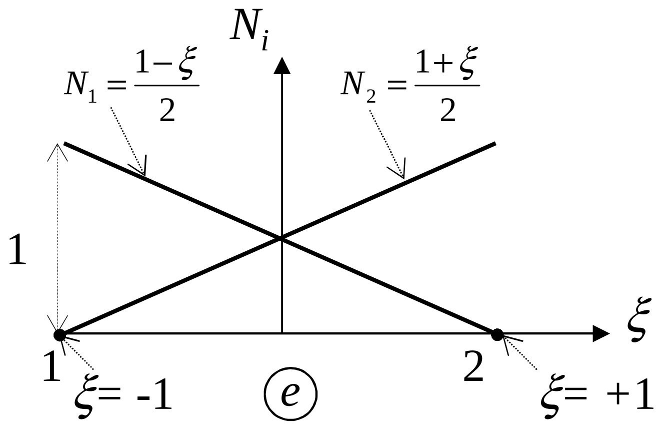
A wall where the inside surface is insulated
And outside is convection surface.
$q|_{x=0} = 0$, $q|_{x=L} = h(T_L - T_\infty)$.

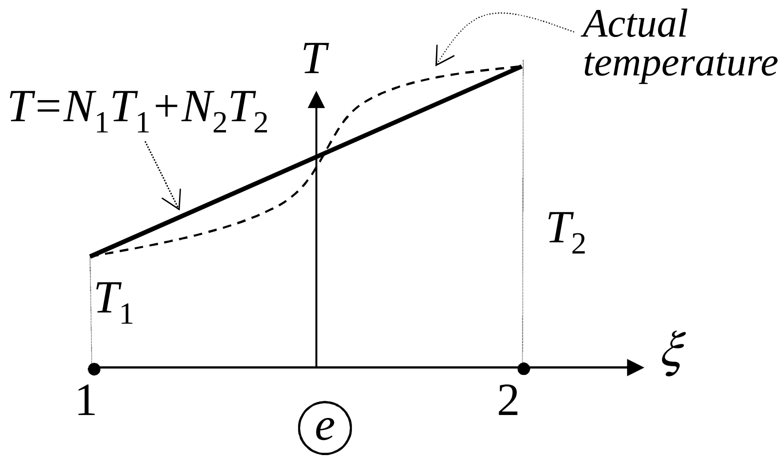
1-D element : two-node element with linear shape functions



Finite Element Method

1-D element





$$T(\xi) = N_1 T_1 + N_2 T_2 = \mathbf{N}\mathbf{T}^e$$

where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{T}^e=[T_1, T_2]^T$.

Please note $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1, \ d\xi = \frac{2}{x_2 - x_1} dx = \frac{2}{l_e} dx.$

$$x = N_1 x_1 + N_2 x_2$$

$$x = \frac{(1-\xi)}{2} x_1 + \frac{(1+\xi)}{2} x_2$$

Use chain rule, $$\frac{dT}{dx} = \frac{dT}{d\xi} \cdot \frac{d\xi}{dx} = \frac{2}{x_2 - x_1} \frac{d\mathbf{N}}{d\xi} \cdot \mathbf{T}^e = \frac{1}{x_2 - x_1}[-1,1]\mathbf{T}^e = \mathbf{B}_T \mathbf{T}^e.$$

where $$\mathbf{B}_T = \frac{d}{dx}\mathbf{N} = \frac{1}{x_2 - x_1}[-1, 1] = \frac{1}{l_e}[-1 \quad 1]$$

$$\int_e f dx = \int_{-1}^{1} f \ J d\xi \ , \quad J = \frac{l_e}{2} = Jacobian$$

Finite Element Method

Galerkin's approach for heat conduction

y=T=temperature

Problem: $$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q=0 \qquad y|_{x=0}=y_0,\quad q|_{x=L}=h(y_L\text{-}y_\infty).$$

Assume: $$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q\right]dx=0$$

$\phi(x)$ constructed from same basis function of y, with $\phi(0)$=0. ϕ as a virtual temperature change that is consistent with boundary conditions.

Weighted-Residual Method

First term use integration by part: $$\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b}-\int_{x=a}^{x=b} v\,du \quad\Longrightarrow\quad \phi k\frac{dy}{dx}\Big|_{x=0}^{x=L}-\int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx+\int_0^L \phi Q dx=0$$

Now, $$\phi k\frac{dy}{dx}\Big|_0^L=\phi(L)k(L)\frac{dy}{dx}(L)-\phi(0)k(0)\frac{dy}{dx}(0)$$

Since, $q=-k\dfrac{dy}{dx}$ So, $\phi(0)$=0, $q(L)$ =-$k(L)$[d$y(L)$/dx]=$h(y_L\text{-}y_\infty)$, we get $\phi k\dfrac{dy}{dx}\Big|_0^L=-\phi(L)h(y_L-y_\infty).$

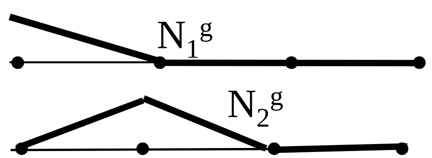
Finally, we get $$-\phi(L)h(y_L-y_\infty)-\int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx+\int_0^L \phi Q dx=0$$

Weak form – reduced (weakened) continuity of y

A global virtual-temperature vector is denoted: $\psi=[\psi_1, \psi_2,\ldots, \psi_{NL}]^T$, or element-wise: $\psi^e=[\psi_i, \psi_{i+1}]^T$.

The test function within each element is interpolated as: (global nodes) $\phi=\mathbf{N}\psi$, or element-wise $\phi^e=\mathbf{N}^e\psi^e$.

$$\frac{d\phi^e}{dx}=\frac{d}{dx}\phi^e=\frac{d}{dx}\left(\mathbf{N}^e\boldsymbol{\psi}^e\right)=\left(\frac{d\mathbf{N}^e}{d\xi}\frac{d\xi}{dx}\right)\cdot\boldsymbol{\psi}^e=\mathbf{B}_T\boldsymbol{\psi}^e$$



Finite Element Method

Galerkin's approach for heat conduction

<u>Some matrix concept</u>: $(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$.
Let **A,B,C,D**=row vector, $\mathbf{AB}^T$=scalar $\rightarrow$ $\mathbf{AB}^T=(\mathbf{AB}^T)^T$.
$\mathbf{AB}^T\mathbf{CD}^T=(\mathbf{AB}^T)^T\mathbf{CD}^T=\mathbf{B}^{TT}\mathbf{A}^T\mathbf{CD}^T=\mathbf{B}(\mathbf{A}^T\mathbf{C})\mathbf{D}^T$=scalar

$N_i(x_j)=\delta_{ij}$ (Kronecker delta function,global)

We get,

$$-\phi(L)h(y_L-y_\infty)-\int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx+\int_0^L \phi Q dx=0=-(\mathbf{N}(L)\boldsymbol{\psi})h(y_L-y_\infty)-\sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}}k\frac{d\phi}{dx}\frac{dy}{dx}dx+\sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}}\phi Q dx$$

$$=-\psi_{NL}h(y_L-y_\infty)-\sum_e\frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dy^e}{dx}d\xi+\sum_e\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi=0$$

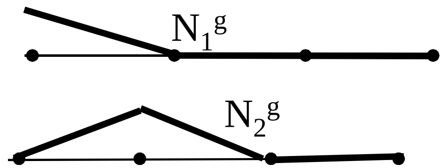
$$d\xi=\frac{2}{l_e}dx$$

Note that, $\frac{d\phi^e}{dx}\frac{dy^e}{dx}=(\mathbf{B}_T\boldsymbol{\psi}^e)(\mathbf{B}_T\mathbf{y}^e)=(\mathbf{B}_T\boldsymbol{\psi}^e)^T(\mathbf{B}_T\mathbf{y}^e)=\boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{y}^e$ and, $\mathbf{N}^e\boldsymbol{\psi}^e$=scalar=$(\mathbf{N}^e\boldsymbol{\psi}^e)^T=\boldsymbol{\psi}^T\mathbf{N}^T$.

$$0=-\psi_{NL}h(y_L-y_\infty)-\sum_e\frac{k_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{y}^e d\xi+\sum_e\frac{Q_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T\mathbf{N}^T d\xi$$

$$\mathbf{k}_T=k_e\int\mathbf{B}_T^T\mathbf{B}_T dx=k_e\int\frac{d}{dx}(\mathbf{N}^T)\frac{d}{dx}(\mathbf{N})dx$$

$$0=-\psi_{NL}h(y_L-y_\infty)-\sum_e\boldsymbol{\psi}^T\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi\mathbf{y}^e+\sum_e\boldsymbol{\psi}^T\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi$$



Note that: $\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{1}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int_{-1}^{1}d\xi=\frac{2}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\int_{-1}^{1}\mathbf{N}^T d\xi=\int_{-1}^{1}\begin{Bmatrix}(1-\xi)/2\\ (1+\xi)/2\end{Bmatrix}d\xi=\begin{bmatrix}1\\ 1\end{bmatrix}$.

Finally, $$\boxed{0=-\psi_{NL}h(y_L-y_\infty)-\sum_e\boldsymbol{\psi}^T\mathbf{k}_T\mathbf{y}^e+\sum_e\boldsymbol{\psi}^T\mathbf{r}_Q}$$ where, $\mathbf{k}_T=\frac{k_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\mathbf{r}_Q=\frac{Q_e l_e}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$.

Finite Element Method

Galerkin's approach for heat conduction

<u>Some matrix concept</u>: **AMC**+**ANC**=(**AM**+**AN**)**C**=**A**(**M**+**N**)**C**.

Let: $\mathbf{k}_T^{e=i} = \begin{bmatrix} k_{i,i} & k_{i,i+1} \\ k_{i+1,i} & k_{i+1,i+1} \end{bmatrix}, \quad \mathbf{r}_Q^{e=i} = \begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}.$

Please note: k_T are symmetry! $\rightarrow k_{i,j}=k_{j,i}$

2 elements example:

$$\boldsymbol{\psi}^T\mathbf{k}_T^{e=1}\mathbf{y}^{e=1} + \boldsymbol{\psi}^T\mathbf{k}_T^{e=2}\mathbf{y}^{e=2} = [\psi_1 \quad \psi_2]\begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} k_{2,2} & k_{2,3} \\ k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & 2k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{\Psi}^T\mathbf{K}_T\mathbf{y}.$$

We also get:

$$\boldsymbol{\psi}^T\mathbf{r}_Q^{e=1} + \boldsymbol{\psi}^T\mathbf{r}_Q^{e=2} = [\psi_1 \quad \psi_2]\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + [\psi_2 \quad \psi_3]\begin{bmatrix} r_2 \\ r_3 \end{bmatrix} = [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} + [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} 0 \\ r_2 \\ r_3 \end{bmatrix}$$

$$= [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} r_1 \\ 2r_2 \\ r_3 \end{bmatrix} = \boldsymbol{\Psi}^T\mathbf{R} = [\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}.$$

Finite Element Method

Galerkin's approach for heat conduction

Finally, we get:

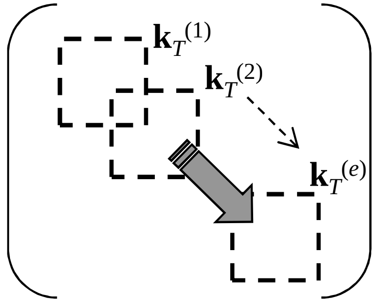
$$0 = -\psi_{NL} h\left(y_L - y_\infty\right) - \sum_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{y}^e + \sum_e \boldsymbol{\psi}^T \mathbf{r}_Q$$

$$0 = -\psi_{NL} h y_L + \psi_{NL} h y_\infty - \left(\boldsymbol{\psi}^T_{e=1} \mathbf{k}_T^{e=1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \cdots + \boldsymbol{\psi}^T_{e=NL-1} \mathbf{k}_T^{e=NL-1} \begin{bmatrix} y_{NL-1} \\ y_{NL} \end{bmatrix} \right)$$

$$+ \left(\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \cdots + \begin{bmatrix} \psi_{NL-1} & \psi_{NL} \end{bmatrix} \begin{bmatrix} r_{NL-1} \\ r_{NL} \end{bmatrix} \right)$$

$$0 = -\psi_{NL} h y_L + \psi_{NL} h y_\infty - \boldsymbol{\Psi}^T \mathbf{K}_T \mathbf{y} + \boldsymbol{\Psi}^T \mathbf{R}.$$

$\mathbf{K}_T =$



The global matrices $\mathbf{K}_T$ and $\mathbf{R}$ are assembled from element matrices $\mathbf{k}_T$ and $\mathbf{r}_Q$.

Now, let $\boldsymbol{\Psi}^T=[\psi_1, \psi_2, \ldots, \psi_{NL}]=[0,1,0,\ldots,0]$, and $y_1=y_0$, we get

$$-0+0-\begin{bmatrix} K_{21} & K_{22} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{NL} \end{bmatrix} + R_2 = 0 \rightarrow \begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_{NL} \end{bmatrix} = R_2 - K_{21} y_0.$$

Continue the process, finally let $\boldsymbol{\Psi}^T=[\psi_1, \psi_2, \ldots, \psi_{NL}]=[0,0,\ldots,1]$, we get ($y_L=y_{NL}$)

$$-1\cdot hy_L + 1\cdot hy_\infty - \begin{bmatrix} K_{NL,1} & K_{NL,2} & \cdots & K_{NL,NL} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{NL} \end{bmatrix} + R_{NL} = 0 \rightarrow \begin{bmatrix} K_{NL,2} & K_{NL,3} & \cdots & \left(K_{NL,NL} + h\right) \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_{NL} \end{bmatrix} = \left(R_{NL} + hy_\infty\right) - K_{NL,1} y_0.$$

Finite Element Method

Galerkin's approach for heat conduction

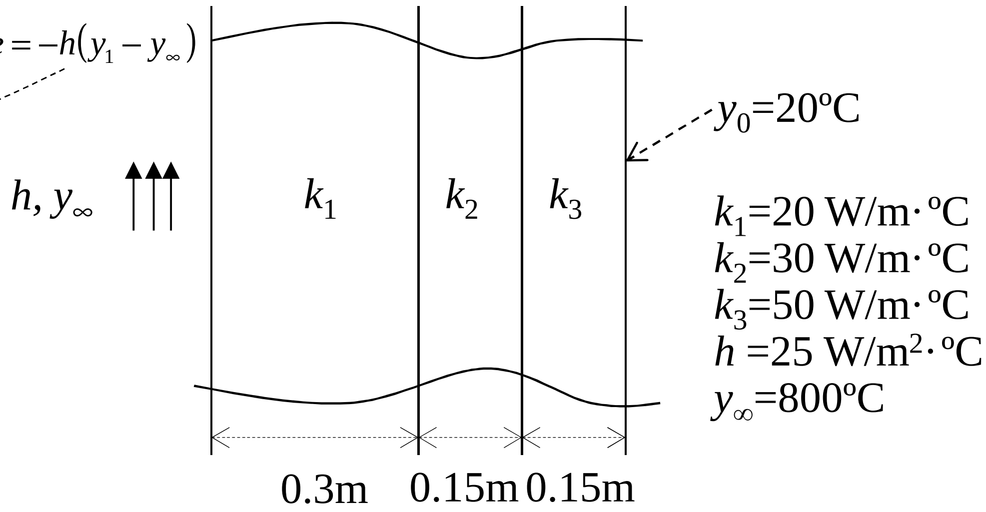
Finally, the compact form is given:

$$\begin{bmatrix} K_{2,2} & K_{2,3} & \cdots & K_{2,NL} \\ K_{3,2} & K_{3,3} & \cdots & K_{3,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,2} & K_{NL,3} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ \vdots \\ y_{NL} \end{Bmatrix} = \begin{Bmatrix} R_2 \\ R_3 \\ \vdots \\ R_{NL}+hy_\infty \end{Bmatrix} - \begin{Bmatrix} K_{2,1}y_0 \\ K_{3,1}y_0 \\ \vdots \\ K_{NL,1}y_0 \end{Bmatrix}.$$

Try insulation at $x=L$, $\phi(L)=0$
Try $Q=2$

Problem: A composite wall consists of 3 materials. The outer temperature is y_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with y_∞=800ºC and h=25 W/m²·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k\left.\frac{\partial y}{\partial x}\right|_{x_1} = -k\frac{y_{1.5}-y_1}{\Delta x} \approx -k\frac{400-500}{+0.1} = +ve = -h(y_1-y_\infty)$$



Solution: we use 3 elements of linear element.

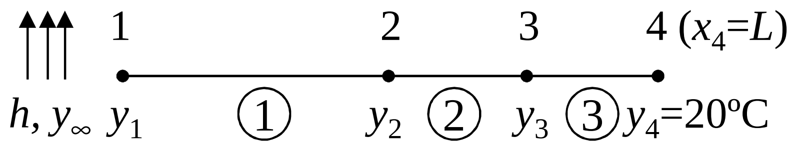
B.C.: $y_4 = y_0$=20, $q|_{x=0} = -h(y_1-y_\infty)$. [$y_\infty > y_1$] We get

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q\right]dx = 0 \rightarrow \phi k\left.\frac{dy}{dx}\right|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx + \int_0^L \phi Q dx = 0$$

$$\phi k\left.\frac{dy}{dx}\right|_0^L = \phi(L)k(L)\frac{dy}{dx}(L) - \phi(0)k(0)\frac{dy}{dx}(0)$$

So, let $\boldsymbol{\phi(L)=0}$, $q(0) = -k(0)[dy(0)/dx] = -h(y_1-y_\infty)$, we get

$$\phi k\left.\frac{dy}{dx}\right|_0^L = -\phi(0)h(y_1-y_\infty).$$



3 elements of linear FE

Finite Element Method

Galerkin's approach for heat conduction

Let ϕ=$\mathbf{N\Psi}$, we get

$$0=-\psi_1 h(y_1-y_\infty)-\sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dy^e}{dx}d\xi+\sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi$$

Finally,

$$\boxed{0=-\psi_1 h(y_1-y_\infty)-\sum_e \boldsymbol{\psi}^T\mathbf{k}_T\mathbf{y}^e+\sum_e \boldsymbol{\psi}^T\mathbf{r}_Q} \Rightarrow \boxed{0=-\psi_1 h y_1+\psi_1 h y_\infty-\mathbf{\Psi}^T\mathbf{K}_T\mathbf{y}+\mathbf{\Psi}^T\mathbf{R}.}$$

Now, let $\mathbf{\Psi}^T=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[1,0,0,0]$, and $y_4=y_0$, we get

$$-h(y_1-y_\infty)-\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+R_1=0\rightarrow\begin{bmatrix}(K_{11}+h) & K_{12} & K_{13}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=R_1+hy_\infty-K_{14}y_0$$

let $\mathbf{\Psi}^T=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[0,1,0,0]$, we get

$$-0+0-\begin{bmatrix}K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+R_2=0\rightarrow\begin{bmatrix}K_{2,1} & K_{2,2} & K_{2,3}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=R_2-K_{2,4}y_0.$$

Finally, let $\mathbf{\Psi}^T=[\psi_1,\ \psi_2,\ \ldots,\ \psi_{NL}]=[0,0,1,0]$, we get

$$-0+0-\begin{bmatrix}K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+R_3=0\rightarrow\begin{bmatrix}K_{3,1} & K_{3,2} & K_{3,3}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=R_3-K_{3,4}y_0.$$

Finite Element Method

Galerkin's approach for heat conduction

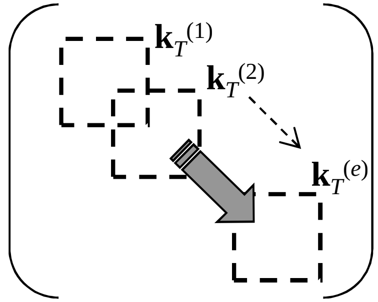
Finally, we get

$$\begin{bmatrix} (K_{11}+h) & K_{1,2} & K_{1,3} \\ K_{2,1} & K_{2,2} & K_{2,3} \\ K_{3,1} & K_{3,2} & K_{3,3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} R_1 + hy_\infty \\ R_2 \\ R_3 \end{bmatrix} - \begin{bmatrix} K_{1,4}y_0 \\ K_{2,4}y_0 \\ K_{3,4}y_0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \ (a)$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(1,\,2)},\ \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(2,\,3)},\ \mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(3,\,4)}$$

$\mathbf{K}_T =$



The global $\mathbf{K}_T = \Sigma\, \mathbf{k}_T$ is obtained

$$\mathbf{K}_T = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, we get $\mathbf{r}_Q=[0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0]^T$.

Given y_0=20ºC, y_∞=800ºC and h=25 W/m$^2\cdot$ºC, eq. (a) becomes

$$66.7\begin{bmatrix} 1.375 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0+25(800) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -5(66.7)(20) \end{bmatrix} = \begin{bmatrix} 20{,}000 \\ 0 \\ 6670 \end{bmatrix}$$

This linear system can be solved using Thomas algorithm and we get $[y_1, y_2, y_3]$=**[304.6, 119.0, 57.1] ºC**

$$\mathbf{Ax}=\mathbf{b},\ \mathbf{A}=\mathbf{LU} \rightarrow \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & \cdots & 0 & c_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ c_2 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & \alpha_n \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

The whole Thomas algorithm can be summarized :

1. $\alpha_1=d_1$
2. $\alpha_i=d_i-c_i\beta_{i-1}$, $i=2,3,\ldots,n$
3. $\beta_i=e_i/\alpha_i$, $i=1,2,\ldots,n-1$.
4. $w_1=b_1/\alpha_1$
5. $w_i=(b_i-c_iw_{i-1})/\alpha_i$, $i=2,3,\ldots,n$.
6. $x_n=w_n$
7. $x_i=w_i-\beta_ix_{i+1}$, $i=n-1, n-2,\ldots,1$.

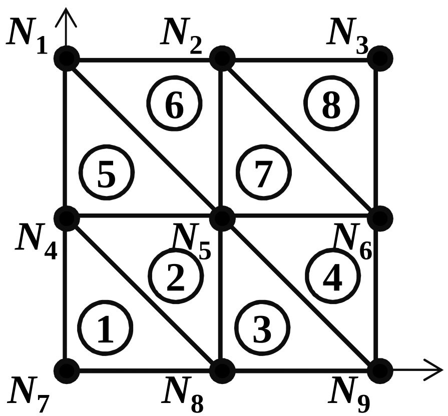
Finite Element Method

Galerkin's approach for heat conduction

Preprocessing

Preprocessing of the problem includes one or more of the following tasks:

- Read geometry and material data (E), and boundary and initial conditions of the problem.
- Mesh generation.
- Generation of node numbers.
- Generation of coordinates and connectivity.



element	1	2	3	← local
1	7	8	4	
2	8	5	4	
3	8	9	5	↑
4	9	6	5	Global
5	4	5	1	
6	5	2	1	↓
7	5	6	2	
8	6	3	2	

Linear triangular element

Processing of FEM

Processing of the FEM includes one or more of the following tasks:

- Calculate element matrices.
- Assemble element equations.
- Solve the system of equations.

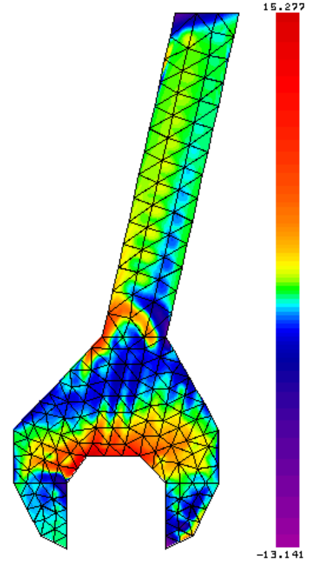
Finite Element Method

Galerkin's approach for heat conduction

Postprocessing

Postprocessing of the FEM includes one or more of the following tasks:

- Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
- Interpretation of the results to check whether the solution makes sense (based on physical Process and experience when other solutions are not available.
- Tabular and/or graphical presentation of the results. Contour plotting uses $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$

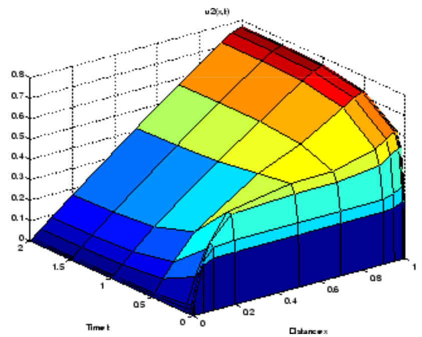


Contour plot for stress

Interpolation of temperature within each element is given

$$y(\xi) = N_1 y_1 + N_2 y_2 = \mathbf{N}\mathbf{y}^e$$

where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{y}^e=[y_1, y_2]^T$.


Contour plot for $u_2(x,t)$

The derivative of the solution is obtained by differentiation

Use chain rule,
$$\frac{dy}{dx} = \frac{dy}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{y}^e = \frac{1}{l_e}[-1, 1]\mathbf{y}^e = \mathbf{B}_T\mathbf{y}^e.$$

For element 1, we get
$$\frac{dy^{e=1}}{dx} = \mathbf{B}_T\mathbf{y}^{e=1} = \frac{1}{l_e}[-1, 1]\mathbf{y}^{e=1} = \frac{1}{0.3}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}304.6\\119.0\end{bmatrix} = -618.67$$

For element 2, we get
$$\frac{dy^{e=2}}{dx} = \mathbf{B}_T\mathbf{y}^{e=2} = \frac{1}{l_e}[-1, 1]\mathbf{y}^{e=2} = \frac{1}{0.15}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}119.0\\57.1\end{bmatrix} = -412.67$$

Note that the derivative above is discontinuous, for any order element, at the nodes connecting the different elements because the continuity of the derivative of FE solution at the connecting nodes is not imposed.

Finite Element Method

Galerkin's method with penalty approach

Calculus of variations

Let $F(x,u,u')$ with fixed value of independent variable x, F depends on u and u'. The change εv in u, where ε is a constant, v is a function, is called **variation** of u (denoted by δu):

$\delta u = \varepsilon v$. →(**variation** of u), operator δ is called **variational operator**.

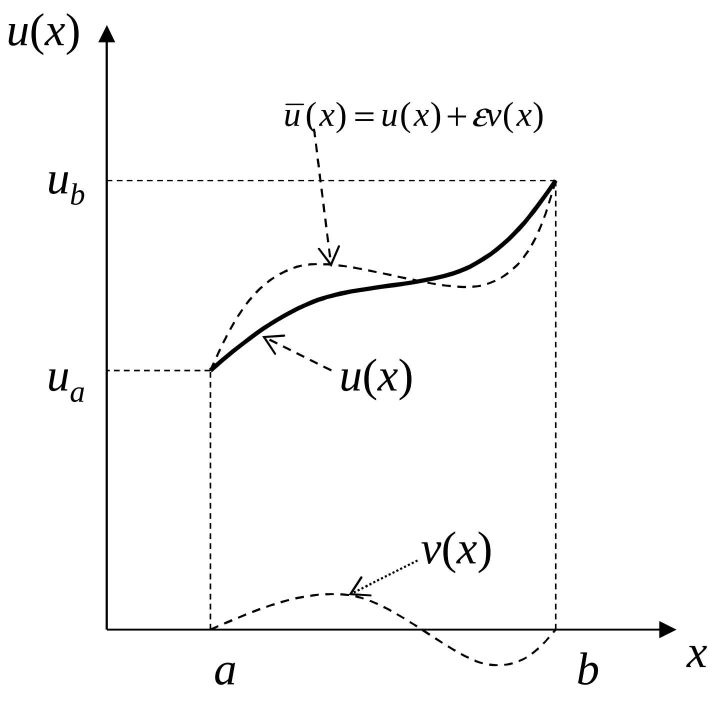
The variation δu represents an admissible change in function $u(x)$ at fixed value of x.

Expand in powers of ε gives ($[u+\varepsilon v]$, $[u'+\varepsilon v']$ are dependent functions)

$$\Delta F = F(x,u+\varepsilon v,u'+\varepsilon v') - F(x,u,u') = F(x,u,u') + \varepsilon v\frac{\partial F}{\partial u} + \varepsilon v'\frac{\partial F}{\partial u'} +$$

$$\frac{(\varepsilon v)^2}{2!}\frac{\partial^2 F}{\partial u^2} + \frac{(\varepsilon v)(\varepsilon v')}{2!}\frac{\partial^2 F}{\partial u \partial u'} + \frac{(\varepsilon v')^2}{2!}\frac{\partial^2 F}{\partial u'^2} + \cdots - F(x,u,u')$$

$$= \varepsilon v\frac{\partial F}{\partial u} + \varepsilon v'\frac{\partial F}{\partial u'} + O(\varepsilon^2). \quad where \lim_{\varepsilon\to 0} O(\varepsilon^2) = 0.$$



The **first variation** of F is

$$\delta F = \varepsilon\left[\lim_{\varepsilon\to 0}\frac{F(x,u+\varepsilon v,u'+\varepsilon v') - F(x,u,u')}{\varepsilon}\right] = \varepsilon\left[\lim_{\varepsilon\to 0}\frac{\Delta F}{\varepsilon}\right]$$

$$= \varepsilon\left[\frac{d}{d\varepsilon}\big(F(u+\varepsilon v)\big)\right]_{\varepsilon=0} = \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'.$$

Analogy for total differential of F with fixed x, $dx=0$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'.$$

Finite Element Method

Galerkin's method with penalty approach

Calculus of variations

Let $F=F(x,y,u,v,u_x,v_x,u_y,v_y)$, where $u=u(x,y)$ and $v=v(x,y)$ are dependent variables,

The first variation of F is $\delta F=\delta_u F+\delta_v F$, where $\delta_u F=\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u_x}\delta u_x+\frac{\partial F}{\partial u_y}\delta u_y,\ \delta_v F=\frac{\partial F}{\partial v}\delta v+\frac{\partial F}{\partial v_x}\delta v_x+\frac{\partial F}{\partial v_y}\delta v_y.$

If $F_1=F_1(u)$ and $F_2=F_2(u)$, then

$$\delta(F_1\pm F_2)=\delta F_1\pm\delta F_2,\ \ \delta(F_1\cdot F_2)=(\delta F_1)F_2+F_1(\delta F_2)$$

$$\delta\left(\frac{F_1}{F_2}\right)=\frac{(\delta F_1)F_2-F_1(\delta F_2)}{F_2^2},\ \ \delta(F_1)^n=n(F_1)^{n-1}\delta F_1$$

$$\frac{d}{dx}(\delta u)=\frac{d}{dx}(\varepsilon v)=\varepsilon\frac{dv}{dx}=\delta\left(\frac{du}{dx}\right)$$

$$\delta\int_a^b u(x)dx=\int_a^b \delta u(x)dx,\quad \textit{where } a,b \textit{ are fixed.}$$

Some examples

$$I(u)=\int_a^b F(x,u,u')dx\rightarrow\delta I(u)=\delta\int_a^b F(x,u,u')dx=\int_a^b\delta F(x,u,u')dx=\int_a^b\left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u'}\delta u'\right)dx$$

$$I(u)=\int_a^b\left(p(x)\frac{du}{dx}+q(x)u^2\right)dx+Pu(a)\rightarrow\delta I(u)=\int_a^b\left(p(x)\frac{d\delta u}{dx}+2q(x)u\delta u\right)dx+P\delta u(a)$$

$$I(u,v)=\int_\Omega\left(p(x,y)\frac{du}{dx}\frac{dv}{dx}+q(x,y)v\right)dxdy+\int_\Gamma Quds\rightarrow\delta I=\int_\Omega\left(p(x,y)\left(\frac{d\delta u}{dx}\frac{dv}{dx}+\frac{du}{dx}\frac{d\delta v}{dx}\right)+q(x,y)\delta v\right)dxdy+\int_\Gamma Q\delta uds$$

$\rightarrow$functions of position, p and q, do not undergo variation since not functions of dependent variables.

Finite Element Method

Galerkin's method with penalty approach

Euler equations

Find a function $u=u(x)$ such that $u(a)=u_a$, $u(b)=u_b$, and

$$I(u)=\int_a^b F(x,u(x),u'(x))dx$$

is extremum.

The second variation $\delta^2 I(u)$ of functional $I(u)$ is given, $\delta^2 I(u)=\frac{\varepsilon^2}{2}\left[\frac{d^2}{d\varepsilon^2}I(u+\varepsilon v)\right]_{\varepsilon=0}$
sufficient condition for $I(u)$ relative minimum (max) is $\delta^2 I(u)$ is greater (less) than zero.

We get,

$$0=\delta I(u)=\left.\frac{dI(u+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0}=\int_a^b \delta F dx=\int_a^b\left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u'}\delta u'\right)dx$$

$$=\varepsilon\int_a^b\left(\frac{\partial F}{\partial u}v+\frac{\partial F}{\partial u'}v'\right)dx$$

Integration by part: $\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$

$$\rightarrow 0=\int_a^b v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left.\left(\frac{\partial F}{\partial u'}v\right)\right|_a^b$$

Without constraint on u', the boundary term vanished if v zero at $x=a$ & b.

Finally, we get **Euler equation**

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0\ ,\ a<x<b.$$

Finite Element Method

Galerkin's method with penalty approach

Natural and essential boundary conditions

Find the extremum of $I(u)$ subject to no end conditions [the set of v is arbitrary even at end point, i.e., $v(a)\neq 0$ and $v(b)\neq 0$], the functional has the form

$$I(u)=\int_a^b F\big(x,u(x),u'(x)\big)dx-Q_a u(a)-Q_b u(b)$$

where Q_a and Q_b are known values.

We get,

$$0=\delta I(u)=\int_a^b \varepsilon v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left(\frac{\partial F}{\partial u'}\varepsilon v\right)\Bigg|_a^b-Q_a\varepsilon v(a)-Q_b\varepsilon v(b)$$

To eliminate boundary conditions, let $\left(-\frac{\partial F}{\partial u'}-Q_a\right)v\Bigg|_{x=a}=0,\quad \left(\frac{\partial F}{\partial u'}-Q_b\right)v\Bigg|_{x=b}=0.$

And we get,

(1) $v(a)=0, v(b)=0,$

(2) $v(a)=0,\ \frac{\partial F}{\partial u'}\Big|_{x=b}-Q_b=0,$

(3) $-\frac{\partial F}{\partial u'}\Big|_a-Q_a=0,\ v(b)=0,$

(4) $-\frac{\partial F}{\partial u'}\Big|_a-Q_a=0,\quad \frac{\partial F}{\partial u'}\Big|_{x=b}-Q_b=0.$

u is fixed at x=0, L

Note that v=0 at end point is equivalent to requirement that u is specified (some value) at that point.

Essential boundary conditions: v (and its derivatives) to vanish at boundary. E.g. v=0 on boundary.

Natural boundary conditions: coefficient of v (and its derivatives) is specified some value.

e.g. $\frac{\partial F}{\partial u'}=Q$ on boundary.

Finite Element Method

Galerkin's method with penalty approach

$$\frac{\partial F}{\partial y} = -Q, \ \frac{\partial F}{\partial y'} = ky'$$

Examples , let total potential energy is

$$I(u) = \int_0^L \left[\frac{A}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{h}{2}[u(L)]^2$$

Use principle of minimum total potential energy, we need to find the minimum with $\delta I(u)=0$,

$$\delta I(u) = \int_0^L \left(A\frac{du}{dx}\frac{d\delta u}{dx} - f\delta u \right) dx + hu(L)\delta u(L) = \int_0^L \left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] \delta u \, dx + \left[A\frac{du}{dx}\delta u \right]_0^L + hu(L)\delta u(L)$$

$$0 \quad = \int_0^L \delta u \left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[A\frac{du}{dx} + hu(L) \right]_{x=L} - \delta u(0)\left[A\frac{du}{dx} \right]_{x=0}$$

Let δu is arbitrary in $0<x\leq L$ but with **$\delta u(0)=0$**, the above boundary term vanishes and we get

Euler equation $\quad -\frac{d}{dx}\left(A\frac{du}{dx}\right) - f = 0, \quad 0<x<L$ **Natural boundary condition** $\quad A\frac{du}{dx} + hu(L)\Big|_{x=L} = 0$

Based on the above example, the problem of

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 = -\left[\frac{d}{dx}\left(k\frac{dy}{dx}\right) + Q \right]$$

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right) + Q = 0 \qquad y|_{x=0} = y_0, \quad q|_{x=L} = h(y_L - y_\infty).$$

Is equivalent to the minimizing the functional with **$\delta y(0)=0$** or y is fixed at $x=0$.

$$I(y) = \int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx - \int_0^L Qy\,dx + \frac{h}{2}(y_L - y_\infty)^2$$

$$I(u) = \int_0^L \left[\frac{A}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{h}{2}[u(L)]^2 + \frac{h}{2}[u(0)]^2 \Longrightarrow 0 = \delta I(u) = \int_0^L \delta u \left[-\frac{d}{dx}\left(A\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[A\frac{du}{dx} + hu(L) \right]_{x=L} - \delta u(0)\left[A\frac{du}{dx} - hu(0) \right]_{x=0}$$

with **$\delta u(0)\neq 0$, $\delta u(L)\neq 0$** ,and $\quad A\frac{du}{dx} + hu(L)\Big|_{x=L} = 0 \qquad A\frac{du}{dx} - hu(0)\Big|_{x=0} = 0$

Finite Element Method

Galerkin's method with penalty approach

Problem: **minimize** the quadratic function

$f(x,y)=4x^2-3y^2+2xy+6x-3y+5$

subject to the **constraint** $G(x,y)=2x+3y=0$

Lagrange Multiplier Method. The modified functional is

$$F(x,y) = f(x,y) + \lambda G(x,y)$$
$$\frac{\partial F}{\partial x} = 8x + 2y + 6 + 2\lambda = 0$$
$$\frac{\partial F}{\partial y} = -6y + 2x - 3 + 3\lambda = 0$$
$$\frac{\partial F}{\partial \lambda} = 2x + 3y = 0$$

Solve 3 algebraic equations, we get $x = -3$, $y = 2$, $\lambda = 7$.

Penalty Function Method. The modified functional is

$$F(x,y) = f(x,y) + \frac{\gamma}{2} G^2(x,y)$$
$$\frac{\partial F}{\partial x} = 8x + 2y + 6 + 2\gamma(2x+3y) = 0$$
$$\frac{\partial F}{\partial y} = -6y + 2x - 3 + 3\gamma(2x+3y) = 0$$

In the limit $\gamma \rightarrow \infty$, the results approaches the exact solution

γ	**x**	**y**	**G(x,y)**
0	-0.5769	-0.6923	-3.2308
1	1.5	-3	-6
10	-3.6702	2.7447	0.8936
100	-3.0537	2.0596	0.0716
1000	-3.0053	2.0058	0.0068
10000	-3.0005	2.0006	0.0008
∞	-3	2	0

Finite Element Method

Galerkin's method with penalty approach

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q=0 \qquad y|_{x=0}=y_0,\quad q|_{x=L}=h(y_L\text{-}y_\infty).$$

equivalent $\longrightarrow$

$$I(y)=\int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx-\int_0^L Qy\,dx+\frac{h}{2}\left(y_L-y_\infty\right)^2$$

Let

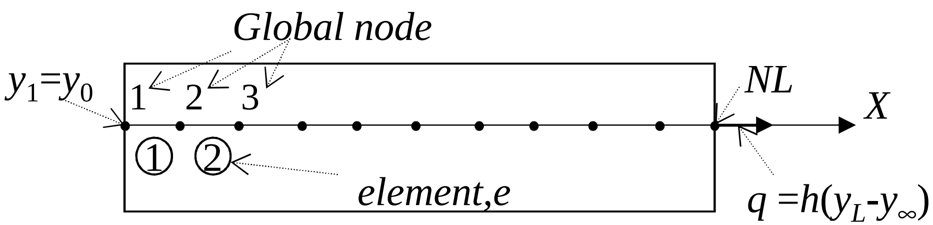
$$I(y)=\int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx-\int_0^L Qy\,dx+\frac{h}{2}\left(y_L-y_\infty\right)^2+\frac{\gamma}{2}\left(y_1-y_0\right)^2$$

Use global node: $y=N_1^g y_1+\cdots+N_{NL}^g y_{NL}=\mathbf{Ny}$

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i\frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g y_i\right)dx+\frac{h}{2}\left(y_{NL}-y_\infty\right)^2+\frac{\gamma}{2}\left(y_1-y_0\right)^2$$

The minimization of energy is equivalent to $\frac{\partial I(y)}{\partial y_i}=0\ ,\quad i=1,2,\cdots,NL.$

For i=1, involve e=1 only



Local node $\rightarrow N_i$

$$\frac{\partial I(y)}{\partial y_1}=\int_0^L k\left(\sum_i y_i\frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_1^g dx\bigg|_{e=1}+\gamma\left(y_1-y_0\right)$$

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}y_1\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_2}{dx}y_2\right)\frac{dN_1}{dx}dx-r_1^{e=1}+\gamma\left(y_1-y_0\right)$$

$$=k_{11}^{e=1}y_1+k_{12}^{e=1}y_2-r_1^{e=1}+\gamma\left(y_1-y_0\right)=\left[k_{11}\quad k_{12}\right]^{e=1}\begin{bmatrix}y_1\\ y_2\end{bmatrix}-r_1^{e=1}+\gamma\left(y_1-y_0\right).$$

$$\mathbf{k}_T=k_e\int \mathbf{B}_T^T\mathbf{B}_T dx=k_e\int \tfrac{d}{dx}\left(\mathbf{N}^T\right)\tfrac{d}{dx}\left(\mathbf{N}\right)dx$$

Finite Element Method

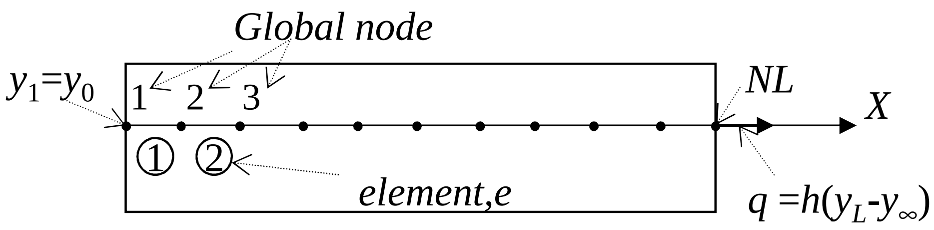
Galerkin's method with penalty approach

For i=2, involve e= 1& 2.

$$\frac{\partial I(y)}{\partial y_2} = \int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}\,dx\Bigg|_{e=1,2} - \int_0^L QN_2^g\,dx\Big|_{e=1,2} + 0$$

$$0 = \int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}y_1 + \frac{dN_2}{dx}y_2\right)\frac{dN_2}{dx}dx + \int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}y_2 + \frac{dN_2}{dx}y_3\right)\frac{dN_1}{dx}dx - \int_{e=1} QN_2 dx - \int_{e=2} QN_1 dx$$

$$0 = \begin{bmatrix} k_{21} & k_{22} \end{bmatrix}^{e=1}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \end{bmatrix}^{e=2}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix} - r_2^{e=1} - r_1^{e=2}$$



For i=NL, involve e= NL-1.

$$\frac{\partial I(y)}{\partial y_{NL}} = \int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_{NL}^g}{dx}\,dx\Bigg|_{e=NL-1} - \int_0^L QN_{NL}^g\,dx\Big|_{e=NL-1} + h\left(y_{NL} - y_\infty\right)$$

$$0 = \int_{e=NL-1} k_{e=NL-1}\left(\frac{dN_1}{dx}y_{NL-1} + \frac{dN_2}{dx}y_{NL}\right)\frac{dN_2}{dx}dx - \int_{e=NL-1} QN_2 dx + h\left(y_{NL} - y_\infty\right)$$

$$0 = \begin{bmatrix} k_{21} & k_{22} \end{bmatrix}^{e=NL-1}\begin{bmatrix} y_{NL-1} \\ y_{NL} \end{bmatrix} - r_2^{e=NL-1} + h\left(y_{NL} - y_\infty\right).$$

Finite Element Method

Galerkin's method with penalty approach

Combining all *NL* equations, we finally get

$$\begin{bmatrix} (K_{11}+\gamma) & K_{12} & \cdots & K_{1,NL} \\ K_{21} & K_{22} & \cdots & K_{2,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,1} & K_{NL,2} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{NL} \end{bmatrix} = \begin{bmatrix} (R_1+\gamma y_0) \\ R_2 \\ \vdots \\ (R_{NL}+h y_\infty) \end{bmatrix}$$

In the limit $\gamma \rightarrow \infty$, the results approaches the exact solution. A simple scheme suggests that $\gamma = \max |K_{ij}| \times 10^4$

Note: The minimization of the functional $I(y) = \int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx - \int_0^L Qy\,dx + \frac{h}{2}(y_L - y_\infty)^2$

$$\delta I(y) = 0 = \int_0^L k\left(\frac{dy}{dx}\right)\delta\left(\frac{dy}{dx}\right)dx - \int_0^L Q\delta y\,dx + h(y_L - y_\infty)\delta(y_L)$$

$$0 = \int_0^L k\frac{dy}{dx}\frac{d\delta y}{dx}dx - \int_0^L Q\delta y\,dx + h(y_L - y_\infty)\delta(y_L)$$

Set the arbitrary function $\boldsymbol{\delta y} \rightarrow \phi$, where $\boldsymbol{\phi(0)=0}$ or $\delta(y_0=0)=0$.

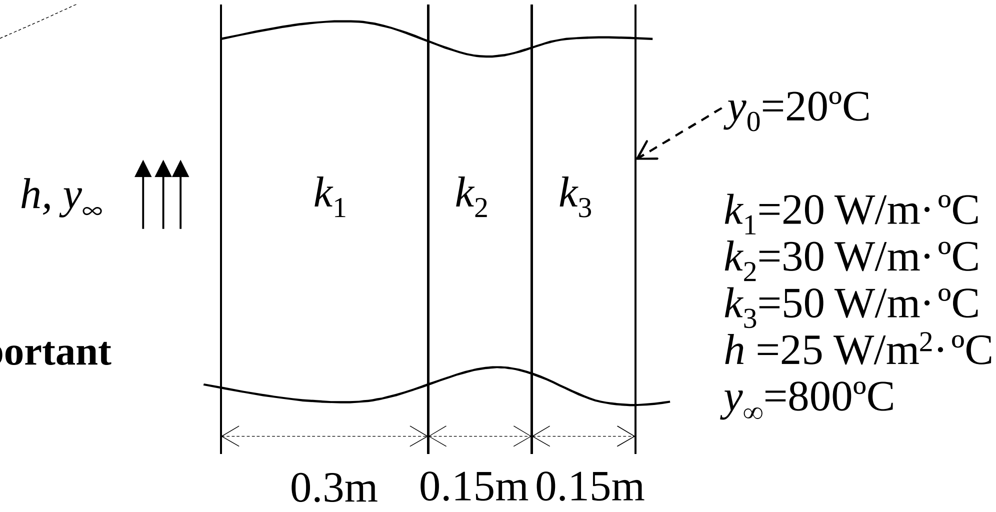
$$0 = \int_0^L k\frac{dy}{dx}\frac{d\phi}{dx}dx - \int_0^L Q\phi\,dx + h(y_L - y_\infty)\phi(L).$$

Finite Element Method

Galerkin's method with penalty approach

Problem: A composite wall consists of 3 materials. The outer temperature is y_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with y_∞=800ºC and h=25 W/m²·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k\left.\frac{\partial y}{\partial x}\right|_{x_1} = -k\,\frac{y_{1.1}-y_1}{\Delta x} \approx -k\,\frac{400-500}{+0.1} = +ve = -h(y_1 - y_\infty)$$



Solution: we use 3 elements of linear element.

B.C.: $y_4 = y_0$=20, $q|_{x=0} = -h(y_1 - y_\infty)$. We get

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right) + Q = 0$$

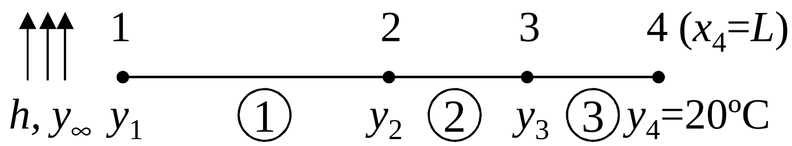
$$y|_{x=L} = y_0 = 20, \quad q|_{x=0} = -h(y_1 - y_\infty).$$

⇩ equivalent

Sign not important

$$I(y) = \int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx - \int_0^L Qy\,dx + \frac{h}{2}(y_\infty - y_1)^2$$

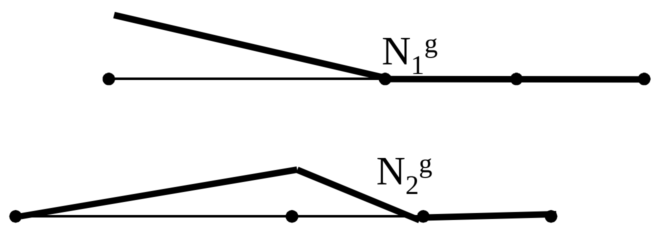
Now, let
$$I(y) = \int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx - \int_0^L Qy\,dx + \frac{h}{2}(y_\infty - y_1)^2 + \frac{\gamma}{2}(y_4 - y_0)^2$$



3 elements of linear FE

Use global node: $y = N_1^g y_1 + \cdots + N_4^g y_4$

$$I(y) = \int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g y_i\right)dx + \frac{h}{2}(y_\infty - y_1)^2 + \frac{\gamma}{2}(y_4 - y_0)^2$$

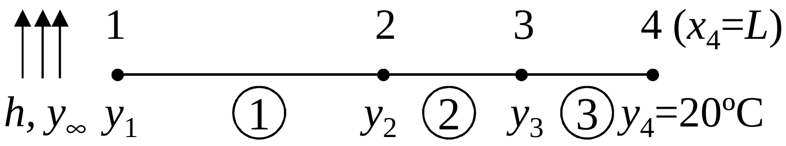


The minimization of energy is equivalent to $\frac{\partial I(y)}{\partial y_i} = 0$, $i = 1, 2, 3, 4.$

Finite Element Method

Galerkin's method with penalty approach

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g y_i\right)dx+\frac{h}{2}\left(y_\infty-y_1\right)^2+\frac{\gamma}{2}\left(y_4-y_0\right)^2$$



3 elements of linear FE

For i=1, involve e=1 only

$$\frac{\partial I(y)}{\partial y_1}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_1^g dx\bigg|_{e=1}+h\left(y_\infty-y_1\right)(-1)$$

Same final form

$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}T_1\right)\frac{dN_1}{dx}dx+\int_{e=1} k_{e=1}\left(\frac{dN_2}{dx}y_2\right)\frac{dN_1}{dx}dx-r_1^{e=1}+h\left(y_1-y_\infty\right)$$

$$=k_{11}^{e=1}y_1+k_{12}^{e=1}y_2-r_1^{e=1}+h\left(y_1-y_\infty\right)=\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=1}\begin{bmatrix}y_1\\ y_2\end{bmatrix}-r_1^{e=1}+h\left(y_1-y_\infty\right).$$

For i=2, involve e= 1& 2.

$$\frac{\partial I(y)}{\partial y_2}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\bigg|_{e=1,2}-\int_0^L QN_2^g dx\bigg|_{e=1,2}+0$$

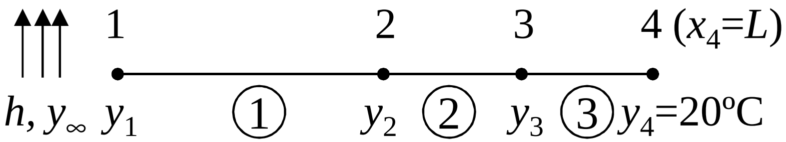
$$0=\int_{e=1} k_{e=1}\left(\frac{dN_1}{dx}y_1+\frac{dN_2}{dx}y_2\right)\frac{dN_2}{dx}dx+\int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}y_2+\frac{dN_2}{dx}y_3\right)\frac{dN_1}{dx}dx-\int_{e=1} QN_2 dx-\int_{e=2} QN_1 dx$$

$$0=\begin{bmatrix}k_{21} & k_{22}\end{bmatrix}^{e=1}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12}\end{bmatrix}^{e=2}\begin{bmatrix}y_2\\ y_3\end{bmatrix}-r_2^{e=1}-r_1^{e=2}$$

Finite Element Method

Galerkin's method with penalty approach

$$I(y) = \int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g y_i\right)dx + \frac{h}{2}(y_\infty - y_1)^2 + \frac{\gamma}{2}(y_4 - y_0)^2$$



3 elements of linear FE

For i=3, involve e= 2 & 3.

$$\frac{\partial I(y)}{\partial y_3} = \int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_3^g}{dx}dx\Bigg|_{e=2,3} - \int_0^L QN_3^g\, dx\Bigg|_{e=2,3} + 0$$

$$0 = \int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}y_2 + \frac{dN_2}{dx}y_3\right)\frac{dN_2}{dx}dx + \int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}y_3 + \frac{dN_2}{dx}y_4\right)\frac{dN_1}{dx}dx - \int_{e=2} QN_2 dx - \int_{e=3} QN_1 dx$$

$$0 = \begin{bmatrix} k_{21} & k_{22} \end{bmatrix}^{e=2}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \end{bmatrix}^{e=3}\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} - r_2^{e=2} - r_1^{e=3}$$

For i=4, involve e= 3.

$$\frac{\partial I(y)}{\partial y_4} = \int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_4^g}{dx}dx\Bigg|_{e=3} - \int_0^L QN_4^g\, dx\Bigg|_{e=3} + \gamma(y_4 - y_0)$$

$$0 = \int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}y_3 + \frac{dN_2}{dx}y_4\right)\frac{dN_2}{dx}dx - \int_{e=3} QN_2 dx + \gamma(y_4 - y_0)$$

$$0 = \begin{bmatrix} k_{21} & k_{22} \end{bmatrix}^{e=3}\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} - r_2^{e=3} + \gamma(y_4 - y_0).$$

Finite Element Method

Galerkin's method with penalty approach

Combining all 4 equations, we finally get

$$\begin{bmatrix} (K_{11}+h) & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & (K_{44}+\gamma) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} (R_1 + hy_\infty) \\ R_2 \\ R_3 \\ (R_4 + \gamma y_0) \end{bmatrix}$$

and let γ= max $|K_{ij}|\times 10^4 = 66.7 \times 8 \times 10^4$

We get

$$66.7 \begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 80{,}005 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 25\times 800 \\ 0 \\ 0 \\ 10{,}672\times 10^4 \end{bmatrix}$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global $\mathbf{K}_T = \Sigma\, \mathbf{k}_T$ is obtained

$$\mathbf{K}_T = 66.7 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, we get $\mathbf{r}_Q=[0\ 0]^T$, $\mathbf{R}=[0\ 0\ 0]^T$.

Given y_0=20ºC, y_∞=800ºC and h=25 W/m²·ºC,

a tridiagonal linear system can be solved using Thomas algorithm and we get

$[y_1, y_2, y_3, y_4]$=**[304.6, 119.0, 57.1, 20.0] ºC**

$$\mathbf{Ax}=\mathbf{b},\ \mathbf{A}=\mathbf{LU}\rightarrow \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & \cdots & 0 & c_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ c_2 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & \alpha_n \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

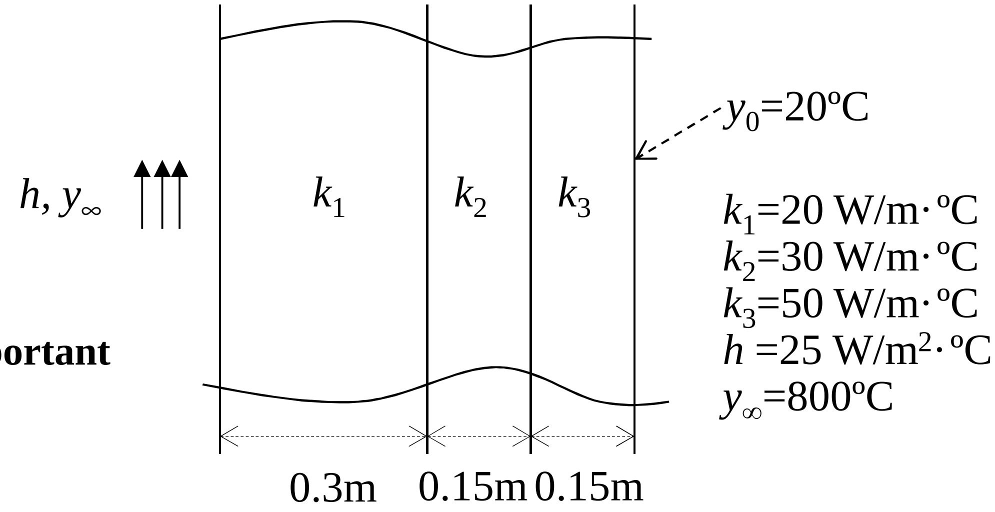
The whole Thomas algorithm can be summarized :

1. $\alpha_1=d_1$
2. $\alpha_i=d_i-c_i\beta_{i-1}$, i=2,3,…,n
3. $\beta_i=e_i/\alpha_i$, i=1,2,…,n-1.
4. $w_1=b_1/\alpha_1$
5. $w_i=(b_i-c_i w_{i-1})/\alpha_i$, i=2,3,…,n.
6. $x_n=w_n$
7. $x_i=w_i-\beta_i x_{i+1}$, i=n-1, n-2,…,1.

Extra notes

Galerkin's method with penalty approach– quadratic shape functions

Problem: A composite wall consists of 3 materials. The outer temperature is y_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with y_∞=800ºC and h=25 W/m²·ºC. Determine the temperature distribution in the wall using quadratic shape functions.



Solution: we use 3 elements of quadratic shape functions.

B.C.: $y_7 = y_0$=20, $q|_{x=0} = -h(y_1 - y_\infty)$. We get

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q=0$$

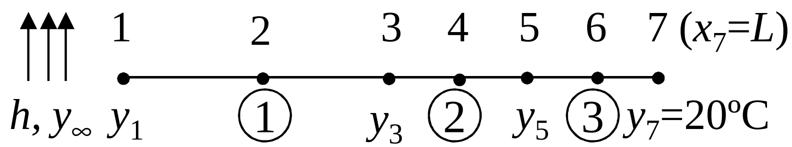
$$y|_{x=L} = y_0=20, \quad q|_{x=0} = -h(y_1-y_\infty).$$

⇓ equivalent

Sign not important

$$I(y)=\int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx-\int_0^L Qy\,dx+\frac{h}{2}(y_\infty - y_1)^2$$

Now, let
$$I(y)=\int_0^L \frac{k}{2}\left(\frac{dy}{dx}\right)^2 dx-\int_0^L Qy\,dx+\frac{h}{2}(y_\infty - y_1)^2+\frac{\gamma}{2}(y_7-y_0)^2$$



3 elements of quadratic FE

Use global node: $y = N_1^g y_1 + \cdots + N_7^g y_7$

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g y_i\right)dx+\frac{h}{2}(y_\infty - y_1)^2+\frac{\gamma}{2}(y_7-y_0)^2$$

The minimization of energy is equivalent to $\dfrac{\partial I(y)}{\partial y_i}=0\,, \quad i=1,\cdots,7.$

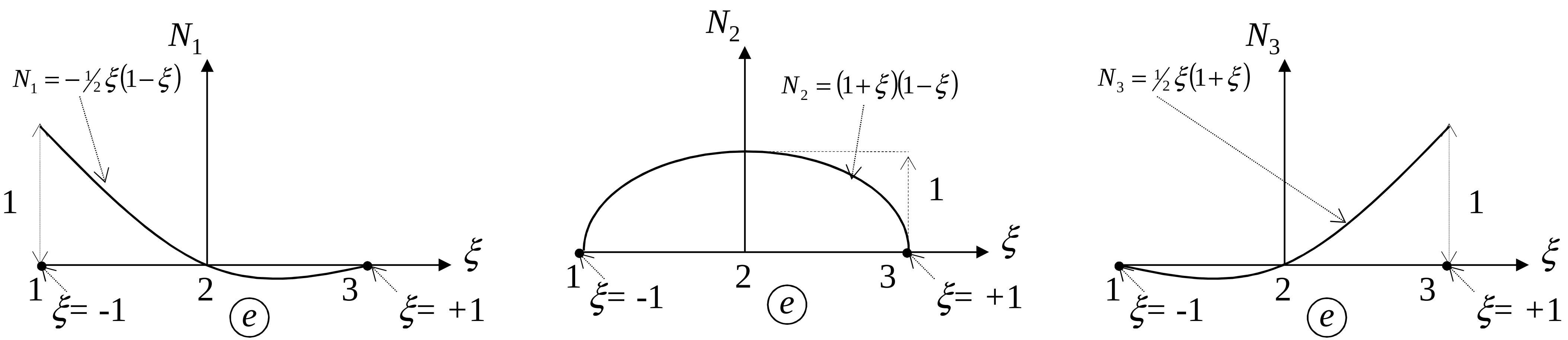
Galerkin's method with penalty approach– quadratic shape functions

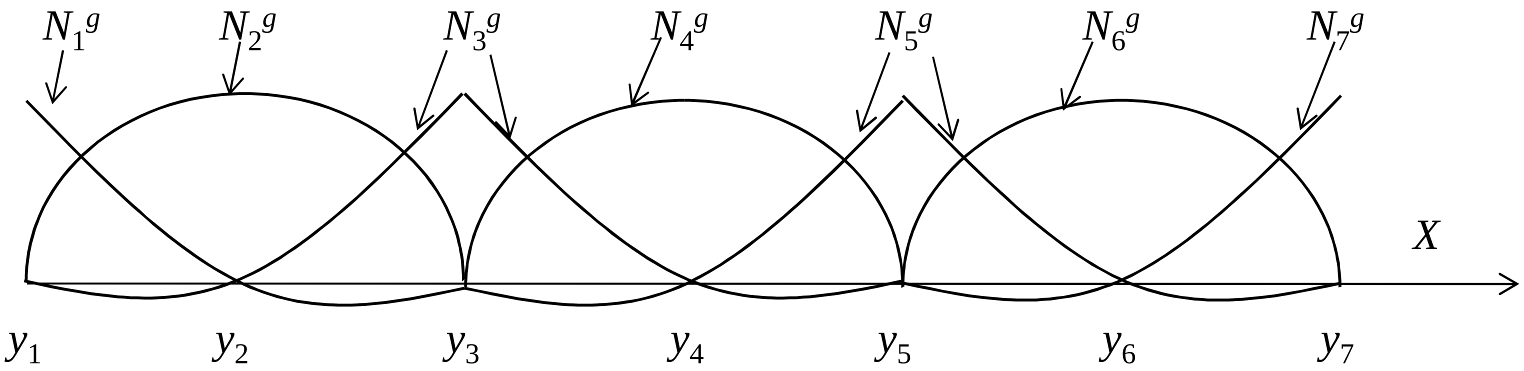
Extra notes

The temperature field within the element is written in terms of the nodal temperature as

$$y(\xi)=N_1y_1+N_2y_2+N_3y_3=\mathbf{N}\mathbf{y}^e$$

Where $N_1(\xi)=-\tfrac{1}{2}\xi(1-\xi)$, $N_2(\xi)=(1+\xi)(1-\xi)$, $N_3(\xi)=\tfrac{1}{2}\xi(1+\xi)$, ξ varies from – 1 to +1, $\mathbf{N}=[N_1, N_2, N_3]$, $\mathbf{y}^e=[y_1, y_2, y_3]^T$.





Please note

$$\xi=\frac{2(x-x_2)}{x_3-x_1},\quad d\xi=\frac{2}{x_3-x_1}dx=\frac{2}{l_e}dx.$$

$$\int_{-1}^{1}\mathbf{N}^T d\xi=\int_{-1}^{1}\begin{bmatrix}-\tfrac{1}{2}\xi(1-\xi)\\(1+\xi)(1-\xi)\\\tfrac{1}{2}\xi(1+\xi)\end{bmatrix}d\xi=\begin{bmatrix}1/3\\4/3\\1/3\end{bmatrix}$$

Use chain rule, $$\frac{dy}{dx}=\frac{dy}{d\xi}\cdot\frac{d\xi}{dx}=\frac{2}{x_3-x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{y}^e=\frac{2}{x_3-x_1}[-{}^{1-2\xi}\!/\!_2,-2\xi,{}^{1+2\xi}\!/\!_2]\mathbf{y}^e=\mathbf{B}_T\mathbf{y}^e.$$

$$\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{2}{3l_e^2}\begin{bmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{bmatrix}$$

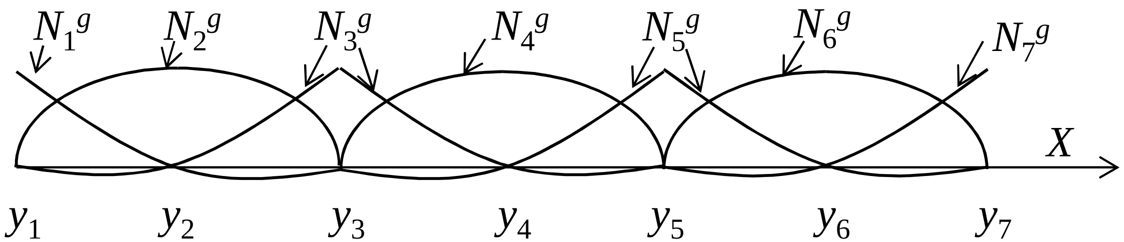
$$\mathbf{B}_T=\frac{d}{dx}\mathbf{N}=\frac{2}{x_3-x_1}[-{}^{1-2\xi}\!/\!_2,-2\xi,{}^{1+2\xi}\!/\!_2]=\frac{2}{l_e}[-{}^{1-2\xi}\!/\!_2,-2\xi,{}^{1+2\xi}\!/\!_2]$$

Galerkin's method with penalty approach– quadratic shape functions

Extra notes

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g y_i\right)dx+\frac{h}{2}(y_\infty-y_1)^2+\frac{\gamma}{2}(y_7-y_0)^2$$

Use global node: $y=N_1^g y_1+\cdots+N_7^g y_7$



For *i*=1, involve *e*=1 only

$$\frac{\partial I(y)}{\partial y_1}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_1^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_1^g dx\bigg|_{e=1}+h(y_\infty-y_1)(-1)$$

$$0=\int_{e=1}k_{e=1}\left(\frac{dN_1}{dx}y_1\right)\frac{dN_1}{dx}dx+\int_{e=1}k_{e=1}\left(\frac{dN_2}{dx}y_2\right)\frac{dN_1}{dx}dx+\int_{e=1}k_{e=1}\left(\frac{dN_3}{dx}y_3\right)\frac{dN_1}{dx}dx-r_1^{e=1}+h(y_1-y_\infty)$$

$$=k_{11}^{e=1}y_1+k_{12}^{e=1}y_2+k_{13}^{e=1}y_3-r_1^{e=1}+h(y_1-y_\infty)=\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=1}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}-r_1^{e=1}+h(y_1-y_\infty).$$

For *i*=2, involve *e*= 1.

$$\frac{\partial I(y)}{\partial y_2}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_2^g}{dx}dx\bigg|_{e=1}-\int_0^L QN_2^g\,dx\bigg|_{e=1}+0$$

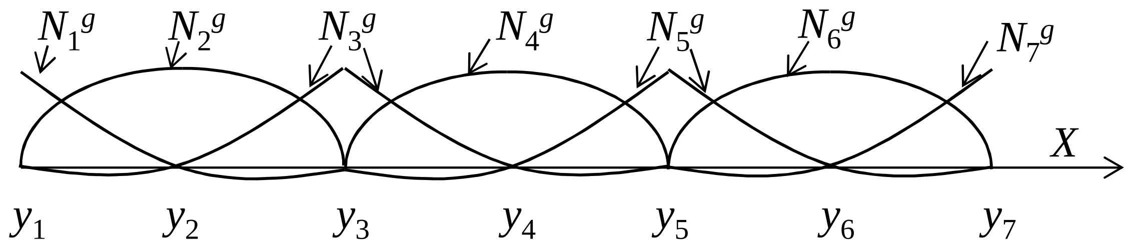
$$0=\int_{e=1}k_{e=1}\left(\frac{dN_1}{dx}y_1+\frac{dN_2}{dx}y_2+\frac{dN_3}{dx}y_3\right)\frac{dN_2}{dx}dx-\int_{e=1}QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=1}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}-r_2^{e=1}$$

Galerkin's method with penalty approach– quadratic shape functions

Extra notes

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g y_i\right)dx+\frac{h}{2}(y_\infty-y_1)^2+\frac{\gamma}{2}(y_7-y_0)^2$$



Use global node: $y=N_1^g y_1+\cdots+N_7^g y_7$

For i=3, involve e= 1& 2.

$$\frac{\partial I(y)}{\partial y_3}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_3^g}{dx}dx\bigg|_{e=1,2}-\int_0^L QN_3^g\,dx\bigg|_{e=1,2}+0$$

$$0=\int_{e=1}k_{e=1}\left(\frac{dN_1}{dx}y_1+\frac{dN_2}{dx}y_2+\frac{dN_3}{dx}y_3\right)\frac{dN_3}{dx}dx+\int_{e=2}k_{e=2}\left(\frac{dN_1}{dx}y_3+\frac{dN_2}{dx}y_4+\frac{dN_3}{dx}y_5\right)\frac{dN_1}{dx}dx-\int_{e=1}QN_3dx-\int_{e=2}QN_1dx$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=1}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12} & k_{13}\end{bmatrix}^{e=2}\begin{bmatrix}y_3\\y_4\\y_5\end{bmatrix}-r_3^{e=1}-r_1^{e=2}$$

For i=4, involve e= 2.

$$\frac{\partial I(y)}{\partial y_4}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_4^g}{dx}dx\bigg|_{e=2}-\int_0^L QN_4^g\,dx\bigg|_{e=2}+0$$

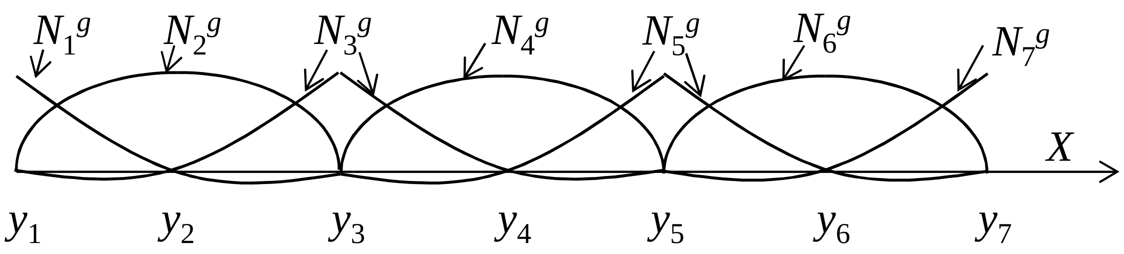
$$0=\int_{e=2}k_{e=2}\left(\frac{dN_1}{dx}y_3+\frac{dN_2}{dx}y_4+\frac{dN_3}{dx}y_5\right)\frac{dN_2}{dx}dx-\int_{e=2}QN_2dx$$

$$0=\begin{bmatrix}k_{21} & k_{22} & k_{23}\end{bmatrix}^{e=2}\begin{bmatrix}y_3\\y_4\\y_5\end{bmatrix}-r_2^{e=2}$$

Galerkin's method with penalty approach– quadratic shape functions

Extra notes

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx - \int_0^L Q\left(\sum_i N_i^g y_i\right)dx + \frac{h}{2}(y_\infty - y_1)^2 + \frac{\gamma}{2}(y_7 - y_0)^2$$



Use global node: $y = N_1^g y_1 + \cdots + N_7^g y_7$

For i=5, involve e=2 & 3.

$$\frac{\partial I(y)}{\partial y_5} = \int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_5^g}{dx}dx\Bigg|_{e=2,3} - \int_0^L QN_5^g\,dx\Bigg|_{e=2,3} + 0$$

$$0 = \int_{e=2} k_{e=2}\left(\frac{dN_1}{dx}y_3 + \frac{dN_2}{dx}y_4 + \frac{dN_3}{dx}y_5\right)\frac{dN_3}{dx}dx + \int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}y_5 + \frac{dN_2}{dx}y_6 + \frac{dN_3}{dx}y_7\right)\frac{dN_1}{dx}dx - \int_{e=2} QN_3 dx - \int_{e=3} QN_1 dx$$

$$0 = \begin{bmatrix} k_{31} & k_{32} & k_{33} \end{bmatrix}^{e=2}\begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \end{bmatrix}^{e=3}\begin{bmatrix} y_5 \\ y_6 \\ y_7 \end{bmatrix} - r_3^{e=2} - r_1^{e=3}$$

For i=6, involve e= 3.

$$\frac{\partial I(y)}{\partial y_6} = \int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_6^g}{dx}dx\Bigg|_{e=3} - \int_0^L QN_6^g\,dx\Bigg|_{e=3} + 0$$

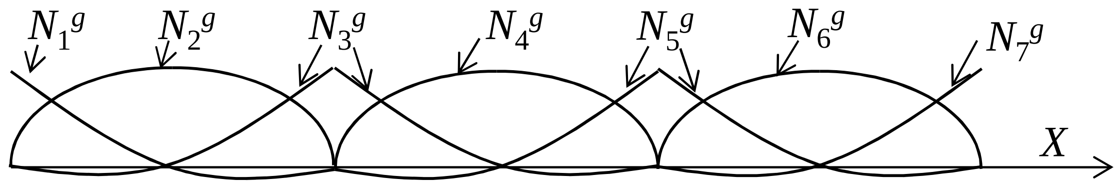
$$0 = \int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}y_5 + \frac{dN_2}{dx}y_6 + \frac{dN_3}{dx}y_7\right)\frac{dN_2}{dx}dx - \int_{e=3} QN_2 dx$$

$$0 = \begin{bmatrix} k_{21} & k_{22} & k_{23} \end{bmatrix}^{e=3}\begin{bmatrix} y_5 \\ y_6 \\ y_7 \end{bmatrix} - r_2^{e=3}$$

Galerkin's method with penalty approach– quadratic shape functions

Extra notes

$$I(y)=\int_0^L \frac{k}{2}\left(\sum_i y_i \frac{dN_i^g}{dx}\right)^2 dx-\int_0^L Q\left(\sum_i N_i^g y_i\right)dx+\frac{h}{2}(y_\infty-y_1)^2+\frac{\gamma}{2}(y_7-y_0)^2$$



Use global node: $y=N_1^g y_1+\cdots+N_7^g y_7$

For i=7, involve e= 3.

$$\frac{\partial I(y)}{\partial y_7}=\int_0^L k\left(\sum_i y_i \frac{dN_i^g}{dx}\right)\frac{dN_7^g}{dx}dx\Bigg|_{e=3}-\int_0^L QN_7^g\,dx\Bigg|_{e=3}+\gamma(y_7-y_0)$$

$$0=\int_{e=3} k_{e=3}\left(\frac{dN_1}{dx}y_5+\frac{dN_2}{dx}y_6+\frac{dN_3}{dx}y_7\right)\frac{dN_3}{dx}dx-\int_{e=3} QN_3 dx+\gamma(y_7-y_0)$$

$$0=\begin{bmatrix}k_{31} & k_{32} & k_{33}\end{bmatrix}^{e=3}\begin{bmatrix}y_5\\ y_6\\ y_7\end{bmatrix}-r_3^{e=3}+\gamma(y_7-y_0)$$

Combining all 7 equations, we finally get

$$\begin{bmatrix}(K_{11}+h) & K_{12} & K_{13} & 0 & 0 & 0 & 0\\ K_{21} & K_{22} & K_{23} & 0 & 0 & 0 & 0\\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & 0 & 0\\ 0 & 0 & K_{43} & K_{44} & K_{45} & 0 & 0\\ 0 & 0 & K_{53} & K_{54} & K_{55} & K_{56} & K_{57}\\ 0 & 0 & 0 & 0 & K_{65} & K_{66} & K_{67}\\ 0 & 0 & 0 & 0 & K_{75} & K_{76} & (K_{77}+\gamma)\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\end{bmatrix}=\begin{bmatrix}(R_1+hy_\infty)\\ R_2\\ R_3\\ R_4\\ R_5\\ R_6\\ (R_7+\gamma y_0)\end{bmatrix}$$

$$\mathbf{k}_T=k_e\int \mathbf{B}_T^T\mathbf{B}_T dx=\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi=\frac{k_e}{3l_e}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}$$

The element conductivity matrices are

$$\mathbf{k}_T^{(1)}=\frac{20}{0.9}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix},\ \mathbf{k}_T^{(2)}=\frac{30}{0.45}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix},\ \mathbf{k}_T^{(3)}=\frac{50}{0.45}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}$$

Galerkin's method with penalty approach– quadratic shape functions

Extra notes

let γ= max $|K_{ij}|\times 10^4$=80$\times$(200/9)$\times 10^4$

$$\frac{200}{9}\begin{bmatrix} 8.125 & -8 & 1 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 \\ 1 & -8 & 28 & -24 & 1 & 0 & 0 \\ 0 & 0 & -24 & 48 & -24 & 0 & 0 \\ 0 & 0 & 3 & -24 & 56 & -40 & 5 \\ 0 & 0 & 0 & 0 & -40 & 80 & -40 \\ 0 & 0 & 0 & 0 & 5 & -40 & 800{,}035 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} (0+20{,}000) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 355.556\times 10^6 \end{bmatrix}$$

Since no heat generation Q occurs in this problem,

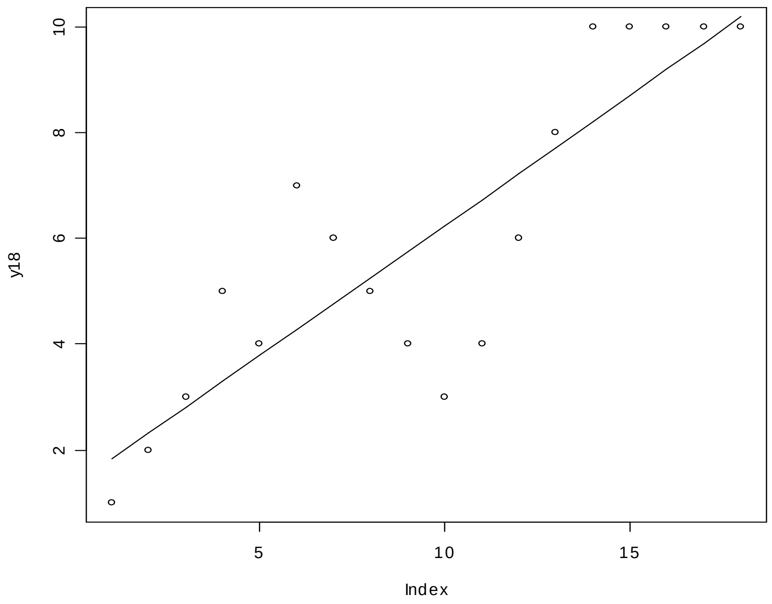
we get $\mathbf{r}_Q$=[0 0 0]^T, $\mathbf{R}$=[0 0 0 0 0 0 0]^T.

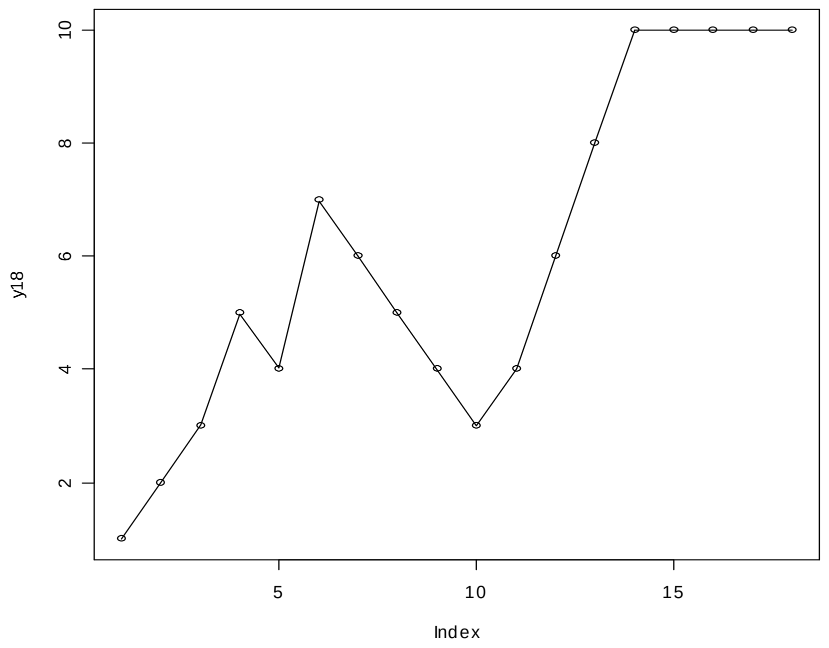
Given y_0=20ºC, y_∞=800ºC and h=25 W/m$^2\cdot$ºC,

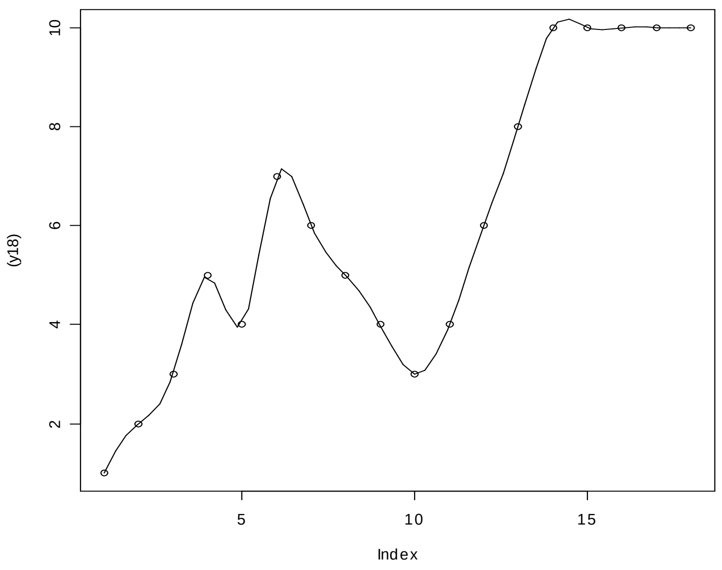
this linear system can be solved and we get

[y_1, y_2, y_3, y_4, y_5, y_6, y_7]=**[304.76, 211.91, 119.05, 88.10, 57.14, 38.57, 20.00] ºC**

Introduction to Cubic Splines







Spline

Cubic Spline Interpolation

f, function defined on interval $[a,b]$ $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

Cubic Spline Interpolant : f on $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

(1) On each subinterval $[x_j,x_{j+1}]$, $j=0,1,\ldots,n-1$, coincides with cubic polynomial

$$s(x)=s_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3$$

(2) s interpolates f at $x_0, x_1, \ldots, x_n$.

(3) s is continuous on $[a,b]$;

(4) s' is continuous on $[a,b]$;

(5) s'' is continuous on $[a,b]$;

Function s composed of n different cubic polynomial (n intervals)
Total = $4n$ unknowns.
Interpolation provided n+1 equations.
Continuity of spline and first two derivatives contributes
$3(n-1)=3n-3$ equations (continuity apply at interior points $x_1,x_2,\ldots,x_{n-1}$ only).
Definition of cubic spline provide $n+1+3(n-1)=4n-2$ equations.
– two more equations will have to be specified.

Two different types of additional constraints – boundary conditions :
Not-a-knot boundary conditions
Clamped (or complete) boundary conditions.

Cubic Spline Interpolation

Interpolation:

$s_j(x_j)=a_j=f(x_j)$, $j=0,1,\ldots,n\text{-}1$.

Continuity of spline:

$a_{j+1}=a_j+b_jh_j+c_jh^2{}_j+d_jh^3{}_j$, $j=0,1,\ldots,n-2$.

Continuity of spline derivative:

$b_{j+1}=b_j+2c_jh_j+3d_jh^2{}_j$, $j=0,1,\ldots,n-2$.

Continuity of spline second derivative:

$c_{j+1}=c_j+3d_jh_j$, $j=0,1,\ldots,n-2$.

For simplicity, let $h_j=x_{j+1}-x_j$.

$a_n=f(x_n)=f(b)$.

Interpolation conditions directly provide values of a_j.

Solve for continuity of spline second derivative for d_j:

$$d_j = \frac{c_{j+1}-c_j}{3h_j} \qquad \textbf{(1)}$$

Combine with continuity of spline and first derivative gives:

$$a_{j+1} = a_j + b_jh_j + c_jh_j^2 + \frac{c_{j+1}-c_j}{3}h_j^2 = a_j + b_jh_j + \frac{c_{j+1}+2c_j}{3}h_j^2 \qquad \textbf{(2)}$$

$$b_{j+1} = b_j + 2c_jh_j + \left(c_{j+1}-c_j\right)h_j = b_j + \left(c_{j+1}+c_j\right)h_j \qquad \textbf{(3)}$$

Solve eq (2) for b_j:

$$b_j = \frac{a_{j+1}-a_j}{h_j} - \frac{2c_j+c_{j+1}}{3}h_j \qquad \textbf{(4)}$$

Cubic Spline Interpolation

Substitute eq (4) into eq (3):

$$h_{j-1}c_{j-1}+2\left(h_{j-1}+h_j\right)c_j+h_jc_{j+1}=\frac{3}{h_j}\left(a_{j+1}-a_j\right)-\frac{3}{h_{j-1}}\left(a_j-a_{j-1}\right) \quad , j=1,2,\ldots,n-1. \qquad \textbf{(5)}$$

Eq (5) produce tridiagonal system of equations

Equations for $j=0$ and $j=n$ depend on type of boundary conditions.

Solve tridiagonal system in eq(5) to compute c_j.

Use eq(1) to compute d_j,

Eq(4) is used to compute b_j.

Use $a_j=f(x_j)$.

$s(x)=s_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3$

Not-a-Knot boundary conditions

When no information other than the value of f at each interpolating point is available (boundary Point), not-a-knot BC is applied.

on $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

On subinterval $[x_j,x_{j+1}]$, $j=0,1,\ldots,n-1$,

$s(x)=s_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3$

$\rightarrow$ s''' be continuous at $x=x_1$ and $x=x_{n-1}$.

$$\frac{d^3}{dx^3}s_0(x_1)=6d_0, \quad \frac{d^3}{dx^3}s_1(x_1)=6d_1$$

$$\frac{d^3}{dx^3}s_{n-2}(x_{n-1})=6d_{n-2}, \quad \frac{d^3}{dx^3}s_{n-1}(x_{n-1})=6d_{n-1}$$

$\rightarrow d_0=d_1, \; d_{n-2}=d_{n-1}.$

Cubic Spline Interpolation

Not-a-Knot boundary conditions

$\rightarrow d_0 = d_1,\ d_{n\text{-}2} = d_{n\text{-}1}$. Using eq (1),

$$d_j = \frac{c_{j+1} - c_j}{3h_j} \quad \textbf{(1)}$$

$$d_0 = \frac{c_1 - c_0}{3h_0} = d_1 = \frac{c_2 - c_1}{3h_1} \qquad d_{n-2} = \frac{c_{n-1} - c_{n-2}}{3h_{n-2}} = d_{n-1} = \frac{c_n - c_{n-1}}{3h_{n-1}}$$

Become:

$$h_1 c_0 - (h_0 + h_1) c_1 + h_0 c_2 = 0, \quad \textbf{(6)}$$

$$h_{n-1} c_{n-2} - (h_{n-2} + h_{n-1}) c_{n-1} + h_{n-2} c_n = 0. \quad \textbf{(7)}$$

Eq (6) & (7) do not preserve the tridiagonal structure.

Solve eq (6) & (7) for c_0 & c_{n}. $\rightarrow$

$$c_0 = \left(1 + \frac{h_0}{h_1}\right) c_1 - \frac{h_0}{h_1} c_2, \quad \textbf{(8)}$$

$$c_n = -\frac{h_{n-1}}{h_{n-2}} c_{n-2} + \left(1 + \frac{h_{n-1}}{h_{n-2}}\right) c_{n-1}. \quad \textbf{(9)}$$

Substitute c_0 from eq (8) into (5), for j=1, we get

$$\left(3h_0 + 2h_1 + \frac{h_0^2}{h_1}\right) c_1 + \left(h_1 - \frac{h_0^2}{h_1}\right) c_2 = \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \quad \textbf{(10)}$$

Substitute c_n from eq (9) into (5), for j=n–1, we get

$$\left(h_{n-2} - \frac{h_{n-1}^2}{h_{n-2}}\right) c_{n-2} + \left(3h_{n-1} + 2h_{n-2} + \frac{h_{n-1}^2}{h_{n-2}}\right) c_{n-1} = \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \quad \textbf{(11)}$$

Cubic Spline Interpolation

Not-a-Knot boundary conditions

$$h_{j-1}c_{j-1}+2(h_{j-1}+h_j)c_j+h_jc_{j+1}=\frac{3}{h_j}(a_{j+1}-a_j)-\frac{3}{h_{j-1}}(a_j-a_{j-1}) \quad , j=2,3,\ldots,n-2. \quad \textbf{(5.a)}$$

$$\left(3h_0+2h_1+\frac{h_0^2}{h_1}\right)c_1+\left(h_1-\frac{h_0^2}{h_1}\right)c_2=\frac{3}{h_1}(a_2-a_1)-\frac{3}{h_0}(a_1-a_0) \quad \textbf{(10)}$$

$$\left(h_{n-2}-\frac{h_{n-1}^2}{h_{n-2}}\right)c_{n-2}+\left(3h_{n-1}+2h_{n-2}+\frac{h_{n-1}^2}{h_{n-2}}\right)c_{n-1}=\frac{3}{h_{n-1}}(a_n-a_{n-1})-\frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \quad \textbf{(11)}$$

(5.a), (10), (11) produce complete tridiagonal system for $c_1, c_2, \ldots, c_{n-1}$.
$\rightarrow$ strictly diagonally dominant $\rightarrow$ nonsingular matrix $\rightarrow$ unique solution for c_j.

Example: $h_j=100$

$$\begin{bmatrix} 600 & 0 & & & & & \\ 100 & 400 & 100 & & & & \\ & 100 & 400 & 100 & & & \\ & & 100 & 400 & 100 & & \\ & & & 100 & 400 & 100 & \\ & & & & 100 & 400 & 100 \\ & & & & & 0 & 600 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix}=0.03\begin{bmatrix} 0 \\ 0.001 \\ -0.003 \\ 0.007 \\ -0.002 \\ 0 \\ 0 \end{bmatrix}$$

Temperature, K	Emittance, E	Temperature, K	Emittance, E
300	0.024	800	0.083
400	0.035	900	0.097
500	0.046	1000	0.111
600	0.058	1100	0.125
700	0.067		

Apply (8) & (9) to compute c_0 & c_8. $a=300=x_0<x_1<x_2<\ldots<x_7<x_8=1100=b$

On each subinterval $[x_j,x_{j+1}]$, $j=0,1,\ldots,8-1$, coincides with cubic polynomial

$s(x)=s_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3$ $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

Cubic Spline Interpolation

Not-a-Knot boundary conditions

$a=x_0<x_1<x_2<\ldots<x_{n\text{-}1}<x_n=b$

$a=300=x_0<x_1<x_2<\ldots<x_7<x_8=1100=b$

On each subinterval $[x_j,x_{j+1}]$, $j=0,1,\ldots,8–1$, coincides with cubic polynomial

$s(x)=s_j(x)=a_j+b_j(x–x_j)+c_j(x–x_j)^2+d_j(x–x_j)^3$

j	a_j	b_j	c_j	d_j
0	0.024	0.00012256410	-0.00000018846	0.0000000063
1	0.035	0.00010371795	0	0.0000000063
2	0.046	0.00012256410	0.00000018846	-0.0000000214
3	0.058	0.00009602564	-0.00000045385	0.0000000391
4	0.067	0.00012333333	0.00000072692	-0.0000000360
5	0.083	0.00016064103	-0.00000035385	0.0000000147
6	0.097	0.00013410256	0.00000008846	-0.0000000029
7	0.111	0.00014294872	0	-0.0000000029

Cubic Spline Interpolation

Clamped boundary conditions

If the values $f'(a)$ and $f'(b)$ are known, better apply clamped (or complete) BC:
$s'(a) = f'(a)$ and $s'(b) = f'(b)$.

$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$

On each subinterval $[x_j, x_{j+1}]$, $j = 0, 1, \ldots, n-1$, coincides with cubic polynomial

$s(x) = s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$

$$b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{2c_j + c_{j+1}}{3} h_j \qquad \textbf{(4)}$$

Start with $x = a$, $f'(a) = s'(a) = s_0'(a) = b_0$.

Eq (4) with $j = 0$, we get

$$f'(a) = \frac{a_1 - a_0}{h_0} - \frac{2c_0 + c_1}{3} h_0 \quad \Rightarrow \quad 2h_0 c_0 + h_0 c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \qquad \textbf{(12)}$$

At $x = b$, $f'(b) = s'(b) = s_n'(b) = b_n$.

$$b_{j+1} = b_j + 2c_j h_j + (c_{j+1} - c_j) h_j = b_j + (c_{j+1} + c_j) h_j \qquad \textbf{(3)}$$

Use eq (3) and eq(4), we get

$$h_{n-1} c_{n-1} + 2h_{n-1} c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \qquad j = 1, 2, \ldots, n-1, \qquad \textbf{(13)}$$

$$h_{j-1} c_{j-1} + 2(h_{j-1} + h_j) c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) \quad , j = 2, 3, \ldots, n-2. \qquad \textbf{(5.a)}$$

(5.a), (12), (13) produce complete tridiagonal system for $c_1, c_2, \ldots, c_{n-1}$.
→ strictly diagonally dominant → nonsingular matrix → unique solution for c_j.

Cubic Spline Interpolation

Clamped boundary conditions

Example: $h_j=0.5$

$$h_{j-1}c_{j-1}+2(h_{j-1}+h_j)c_j+h_jc_{j+1}=\frac{3}{h_j}(a_{j+1}-a_j)-\frac{3}{h_{j-1}}(a_j-a_{j-1})$$

, $j=2,3,\ldots,n-2$.

Exact: $f(x)=(x+1)e^{-x}$

$$\begin{bmatrix} 1 & 0.5 & & & \\ 0.5 & 2 & 0.5 & & \\ & 0.5 & 2 & 0.5 & \\ & & 0.5 & 2 & 0.5 \\ & & & 0.5 & 1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}=\begin{bmatrix} -3.20868 \\ -3.89232 \\ -1.59504 \\ -0.50304 \\ -0.05940 \end{bmatrix}$$

x	***f*(*x*)**	***f′*(*x*)**
-1.0	0.00000	2.71828
-0.5	0.82436	
0.0	1.00000	
0.5	0.90980	
1.0	0.73576	-0.36788

Apply eq (1) &(4), we get

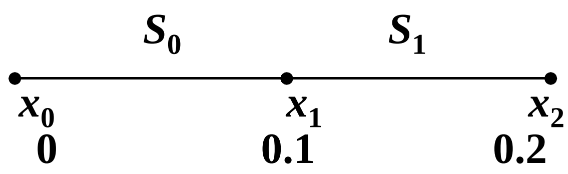
j	a_j	b_j	c_j	d_j
0	0.00000	2.71828000000	-2.62214571429	0.96605142857
1	0.82436	0.82067285714	-1.17306857143	0.46856571429
2	1.00000	-0.00097142857	-0.47022000000	0.22272571429
3	0.90980	-0.30414714286	-0.13613142857	0.09653142857

$$d_j=\frac{c_{j+1}-c_j}{3h_j} \quad \textbf{(1)}$$

$$b_j=\frac{a_{j+1}-a_j}{h_j}-\frac{2c_j+c_{j+1}}{3}h_j \quad \textbf{(4)}$$

Cubic Spline for Boundary value problem

E.g. $\boldsymbol{y''-2y'=e^{x}}$, $0\le x\le 0.2$, step size (h=0.1), initial condition, $y(0)=1$, $y'(0.2)=1.7622$.
Analytical solution: $\boldsymbol{y=1+e^{2x}-e^{x}}$.



$s_0(x_1)=s_1(x_1)$

$$a_0+b_0(h_0)+c_0\,(h_0)^2+d_0\,(h_0)^3=a_1 \qquad \textbf{(1)}$$

$s'_0(x_1)=s'_1(x_1)$

$$b_0+2c_0\,(h_0)+3d_0\,(h_0)^2=b_1 \qquad \textbf{(2)}$$

$s''_0(x_1)=s''_1(x_1)$

$$2c_0+6d_0\,(h_0)=2c_1 \qquad \textbf{(3)}$$