CHAPTER 3
ROOTS OF EQUATION

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Roots of equation

Given:

\[ f(x) = ax^2 + bx + c = 0 \]  \hspace{1cm} \text{Eqn. 3.1}

To solve: use the quadratic formula eq:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]  \hspace{1cm} \text{Eqn. 3.2}

- The values calculated by equation 3.2 are called the “roots” of equation 3.1. They represent the values of \( x \) that make equation 3.1 equal to zero.

- Roots of equations can be defined as “the value of \( x \) that makes \( f(x) = 0 \)” or can be called as the zeros of the equation.
Solve $f(x) = 0$ for $x$
Illustrations of places of roots

- No roots
- Even # of roots
- Odd # of root
- One root

Graphs showing the behavior of $f(x)$ with respect to roots.
• **Example:** The Newton’s 2\textsuperscript{nd} Law for the parachutist’s velocity.

\[ v = \frac{gm}{c} \left( 1 - e^{\frac{c}{m} t} \right) \]  

Eqn. 3.3

• If the parameters are known, equation 3.3 can be used to predict the parachutist’s velocity as a function of time.

• Although the equation provides a mathematical representation of the interrelationship among the model variables & parameters, it cannot be solved for the drag coefficient, \( c \).

• **The solution** to the dilemma is provided by \( NM \) for roots of equations.

\[ f (c) = \frac{gm}{c} \left( 1 - e^{\frac{c}{m} t} \right) - v \]  

Eqn. 3.4
2 Types of Method to determine Root of Equation

1. **Bracketing methods** (two initial guesses for the root are required and these guesses must be “bracket”).
   a) Graphical method
   b) Bisection Method
   c) False-Position Method

2. **Open methods**
   a) Simple fixed-point iteration method
   b) Newton- Raphson method
   c) Secant Method
CHAPTER 3a
BRACKETING METHOD
Bracketing method – a) Graphical

- Two initial guesses for the root are required
- Guesses must be “bracket”, or be on either side of the root

“Obtaining an estimate of the root of an equation $f(x) = 0$ by making a plot of the function. The point at which it crosses the axis, represents the $x$ value for which $f(x) = 0$, provides a rough approximation of the root”.
Example 1

Use the graphical approach to determine the drag coefficient $c$ needed for a parachutist of mass $m = 68.1\text{kg}$ to have a velocity of $40\text{ m/s}$ after free falling for time $t = 10\text{s}$. Step size is 4.

*Note: the acceleration due to gravity is $9.8\text{ m/s}^2$*
Solution

• Given formula:

\[ v = \frac{gm}{c} \left( 1 - e^{\frac{c}{mt}} \right) \]

• Then, subtracting the dependent variable, \( v \) from both side of the equation to give:

\[ f(c) = \frac{gm}{c} \left( 1 - e^{-\left(\frac{c}{m}\right)} \right) - v \]

• Substitute all given values;

\[ f(c) = \frac{9.8(68.1)}{c} \left( 1 - e^{-\left(\frac{c}{68.1}\right)^{10}} \right) - 40 \]

or

\[ f(c) = \frac{667.38}{c} \left( 1 - e^{-0.146843c} \right) - 40 \]

Eqn. 5.1.1
Substitute various $c$ values into the Eq E5.1.1:

<table>
<thead>
<tr>
<th>$c$</th>
<th>$f(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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</tr>
<tr>
<td>8</td>
<td>17.653</td>
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<tr>
<td>12</td>
<td>6.067</td>
</tr>
<tr>
<td>16</td>
<td>-2.269</td>
</tr>
<tr>
<td>20</td>
<td>-8.401</td>
</tr>
</tbody>
</table>

The sign changes.

Plot a graph: resulting curve crosses the $c$ axis between 12 and 16. Rough estimate of the root $\approx 14.75$
The graphical approach for determining the roots of an equation.

$$f(c) = \frac{667.38}{14.75} \left(1 - e^{-0.146843(14.75)}\right) - 40$$

$$= 0.059 \approx 0$$

Validity of the graphical estimate check by substitute $c = 14.75$ into equation.
• It can also be checked by substituting $c = 14.75$ into equation 3.3 to give:

$$v = 9.8 \frac{(68.1)}{14.75} \left(1 - e^{-\left(\frac{14.75}{68.1}\right)10}\right)$$

$$v = 40.059 \text{ m/s}$$

which is very close to desire fall velocity of 40 m/s.
b) Bisection method

- When applying graphical method, \( f(x) \) has changed sign from +ve to –ve, where;
  \[
  f(x_l) \cdot f(x_u) < 0
  \]
- Then there is at least one real root between \( x_l \) and \( x_u \).
- In this method, we dividing halve the interval \((x_l \text{ and } x_u)\) into a number of sub-intervals. Each sub-interval is to locate the sign changes. *Alternatively, called binary chopping (interval halving)*
Step to solve bisection method

**Step 1:** Choose lower $x_l$ and upper $x_u$ guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l)f(x_u) < 0$.

**Step 2:** Estimate the root $x_r$ by:

$$x_r = \frac{x_l + x_u}{2}$$

**Step 3:** Make the following evaluations to determine in which subinterval the root lies:

- a) If $f(x_l)f(x_r) < 0$, the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and return step 2
- b) If $f(x_l)f(x_r) > 0$, the root lies in the upper subinterval. Therefore, set $x_l = x_r$ and return to step 2
- c) If $f(x_l)f(x_r) = 0$, the root equals $x_r$; terminate the computation.
Example 2

Using bisection method to solve the same problem approach as in graphical method. True value given=14.7802
Solution

- The first step in bisection method is to guess two values of the unknown ($x_l$ and $x_u$) where the function changes.
- As from the graph, the function changes sign between 12 and 16.

**Step 1a:** $x_l=12$, $x_u=16$, therefore initial estimate of the root $x_r$:

$$x_r = \frac{x_l + x_u}{2} = \frac{12 + 16}{2} = 14$$

This estimate represents a true percent relative error of $\varepsilon_t = 5.3\%$ (true value of the root is 14.7802).

$1^{st}$ iteration for $x_r=14$

$$\varepsilon_t = \frac{14.7802 - 14}{14.7802} \times 100\%$$
**Step 1b:** Next we compute the function value at the lower bound and at the midpoint

\[ f(x_l) = f(12), \quad f(x_r) = f(14) = 6.067(1.569) = 9.517 \]

\[ f(x_l) f(x_r) > 0 \text{ set } x_l = x_r = 14 \text{ and return to step } 2 \]

no sign change occurs between the lower bound and the midpoint. Consequently, the root must be located between 14 and 16. Therefore;

**Step 2: 2\textsuperscript{nd iteration}**

\[ x_r = \frac{14 + 16}{2} = 15 \]

Which represent a true percent error \( \varepsilon_t = 1.5\% \). The process can be repeated to obtain refined estimate, such as:

\[ f(14) f(15) = (1.569)(-0.425) = -0.666 \]

\[ \varepsilon_t = \frac{14.7802 - 15}{14.7802} \times 100 \% \]
Therefore, the root is between 14 and 15. The upper bound, $x_u$, is redefined as 15, and the root estimate for the 3\textsuperscript{rd} iteration is calculated as:

$$\varepsilon_t = \frac{14.7802 - 14.5}{14.7802} \times 100\%$$

Which represent a true percent error $\varepsilon_t = 1.9\%$.

The method can be repeated until the result is accurate enough to satisfy your needs.

Normally the termination of computation can be defined as:

$$|\varepsilon_a| < \varepsilon_s$$

Termination Criteria and error estimates

$$\varepsilon_a = \frac{|x_r^{new} - x_r^{old}|}{x_r^{new}} \times 100\%$$

Eqn. 5.2

$\varepsilon_a$ = approximate percent relative error
$x_r^{new}$ = root for the present iteration
$x_r^{old}$ = root from the previous iteration

Computation is terminate when $\varepsilon_a < \varepsilon_s$ as well.
Graphical depiction of the bisection method
Let say the stooping criterion, \( \varepsilon_s \), given as 0.5%. Then, continue the previous example until the approximate error, \( \varepsilon_a < \varepsilon_s \)

Recalled:

\[ \varepsilon_s = \left( 0.5 \times 10^{-2n} \right) \% \]

Solution:

Using equation 5.2, calculate \( \varepsilon_a \). For instance, \( x_r \) 1\(^{st}\) iteration and \( x_r \) 2\(^{nd}\) iteration:

\[ \varepsilon_a = \left| \frac{15 - 14}{15} \right| \times 100\% = 6.667\% \]

\( \varepsilon_a \) is greater than \( \varepsilon_s \). Use Eq 5.2 to calculate \( \varepsilon_a \) for all iterations.
<table>
<thead>
<tr>
<th>Iteration</th>
<th>$x_l$</th>
<th>$x_u$</th>
<th>$x_r$</th>
<th>$\varepsilon_a(%)$</th>
<th>$\varepsilon_t(%)$</th>
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<td>16</td>
<td>14</td>
<td>-</td>
<td>5.279</td>
</tr>
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<td>2</td>
<td>14</td>
<td>16</td>
<td>15</td>
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<td>1.695</td>
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<td>0.641</td>
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<td>14.875</td>
<td>14.8125</td>
<td>0.422</td>
<td>0.219</td>
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</table>

Stop computation because $\varepsilon_a < \varepsilon_s$
Working with your buddy
Lets do Quiz 1 and Quiz 2
Determine the real root of \( f(x) = 4x^3 - 6x^2 + 7x - 2.3 \)

a) Graphically

b) Using bisection to locate the root. Employ initial guesses of \( x_l = 0 \) and \( x_u = 1 \) and iterate until the estimated error falls below a level of \( \varepsilon_s = 10\% \).
Solution - Graphically

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<th>f(x)</th>
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<td>0.2</td>
<td>-0.96</td>
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<tr>
<td>0.4</td>
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<td>0.6</td>
<td>0.88</td>
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<tr>
<td>0.8</td>
<td>2.16</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
Solution – Bisection method

1\textsuperscript{st} iteration

\[ x_{r1} = \frac{0 + 1}{2} = 0.5 \]

\[ f(0)f(0.5) = (-2)(0.375) = -0.75 < 0 \]

\[ x_r = x_u \]

2\textsuperscript{nd} iteration

\[ x_{r2} = \frac{0 + 0.5}{2} = 0.25 \]

\[ f(0)f(0.25) = (-2)(-0.73438) = 1.46875 > 0 \]

\[ x_r = x_l \quad \epsilon_a = \frac{0.25 - 0.5}{0.25} \times 100 \% = 100 \% > \epsilon_s \]

3\textsuperscript{rd} iteration

\[ x_{r3} = \frac{0.25 + 0.5}{2} = 0.375 \]

\[ f(0.25)f(0.375) = (-0.73438)(-0.18945) = 0.13913 > 0 \]

\[ x_r = x_l \quad \epsilon_a = \frac{0.375 - 0.25}{0.375} \times 100 \% = 33 \% > \epsilon_s \]
4\textsuperscript{th} iteration
\[ x_{r,4} = \frac{0.375 + 0.5}{2} = 0.4375 \]
\[ f(0.375) f(0.4375) = (-0.18945)(0.08667) = -0.01642 < 0 \]
\[ x_r = x_u \quad \varepsilon_a = \frac{0.4375 - 0.375}{0.4375} = 14.28\% > \varepsilon_s \]

5\textsuperscript{th} iteration
\[ x_{r,5} = \frac{0.375 + 0.4375}{2} = 0.40625 \]
\[ f(0.375) f(0.40625) = (-0.18945)(-0.052459) = 0.009939 > 0 \]
\[ \varepsilon_a = \frac{0.40625 - 0.4375}{0.40625} \times 100\% = 7.692\% < \varepsilon_s \]

Stop computing
\[ x_r = \text{root} = 0.40625 \]
c) False position method

- An alternative method that exploits the graphical insight by joining $f(x_u)$ and $f(x_l)$ by a straight line, which intersection with the x-axis represents an improved estimate of the root.
- Replacement of the curve by straight line gives “False Position” or called “linear Interpolation Method”.

Figure 5.12
Using similar triangles (from fig 5.12), the intersection of the straight line with the x-axis can be estimated as:

\[ \frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u} \]  

Eqn 5.6

Rearrange

\[ x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \]  

Eqn 5.7

False Position Method

**Compare to**

\[ x_r = \frac{x_l + x_u}{2} \]

Bisection method
Example 3

Use the false-position method to determine the root of the same equation investigated in example 2.

The true value = 14.7802 until $\varepsilon_a < \varepsilon_s = 0.5\%$.

Initial guesses: $x_l = 12, x_u = 16$
Solution

Step 1: \( x_1 = 12 \) \( f(x_1) = 6.0699 \)
\( x_u = 16 \) \( f(x_u) = -2.2688 \)

1\(^{st}\) Iteration

\[
x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} = 16 - \frac{(-2.2688)(12 - 16)}{(6.0699) - (-2.2688)} = 14.9113
\]

\[
\varepsilon_r = \frac{14.7802 - 14.9113}{14.7802} \times 100\% = 0.00852\%
\]

- \( x_r = 14.9113, \ f(x_r) = -0.2543, \) check: \( f(x_l) f(x_r) = -1.5426 < 0 \)
- Therefore, \( x_u = x_r = 14.9113 \)
2\textsuperscript{nd} Iteration

Step 1: \( x_1 = 12 \) \( f(x_1) = 6.0699 \)
\[ x_u = 14.9113 \quad f(x_u) = -0.2543 \]

\[ x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \]
\[ = 14.9113 - \frac{(-0.2543)(12 - 14.9113)}{(6.0699) - (-0.2543)} = 14.7942 \]

\[ \varepsilon_i = \frac{14.7942 - 14.7942}{14.7802}100\% = 0.09\% \]

\[ \varepsilon_a = \frac{14.7942 - 14.9113}{14.7942}100\% = 0.79\% > \varepsilon_i \]

Step 3: \( f(x_l) f(x_r) = f(12) f(14.7942) = 0.1704 \)
\( f(x_l) f(x_r) < 0 \), set \( x_u = x_r = 14.7942 \) and return to step 1
3rd Iteration

Step 1: $x_1 = 12$ \quad f(x_1) = 6.0699

\[ x_u = 14.7942 \quad f(x_u) = -0.02802 \]

\[
x_r = 14.7942 - \frac{(-0.02802)(12 - 14.7943)}{(6.0699) - (-0.02802)} = 14.78146
\]

\[
\varepsilon_t = \frac{14.7802 - 14.78146}{14.7802} \times 100\% = 0.00852\%
\]

\[
\varepsilon_a = \frac{14.78146 - 14.7942}{14.78146} \times 100\% = 0.0862\% < \varepsilon_s
\]

Stop the computation

\[\therefore \text{ The real root is } 14.78146\]
**Comparison between 2 methods**

### Bisection Algorithm Results
**Example 4.4 with $E_s = 0.5\%$**

<table>
<thead>
<tr>
<th>Iter</th>
<th>$x_l$</th>
<th>$x_u$</th>
<th>$x_r$</th>
<th>$E_a$</th>
<th>$E_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.00</td>
<td>16.00</td>
<td>14.000</td>
<td>-</td>
<td>5.279</td>
</tr>
<tr>
<td>2</td>
<td>14.00</td>
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<td>1.896</td>
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<td>0.218</td>
</tr>
</tbody>
</table>

### False Position Algorithm Results
**Example 4.4 with $E_s = 0.5\%$**

<table>
<thead>
<tr>
<th>Iter</th>
<th>$x_l$</th>
<th>$x_u$</th>
<th>$x_r$</th>
<th>$E_a$</th>
<th>$E_t$</th>
</tr>
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<tbody>
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<td>3</td>
<td>12.00</td>
<td>14.794</td>
<td>14.7817</td>
<td>0.085</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Let's do Quiz 3
Quiz 5

Determine the real root of:

\[ f(x) = -26 + 85x - 91x^2 + 44x^3 - 8x^4 + x^5 \]

a) Graphically

b) Using bisection to locate the root to \( \varepsilon_s = 10\% \). Employ initial guesses of \( x_l = 0.5 \) and \( x_u = . \)

c) Perform the same computation as in b) but use the false-position method and \( \varepsilon_s = 0.2\% \).
Solution – graphical methods

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
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<tbody>
<tr>
<td>0.5</td>
<td>-1.47813</td>
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<tr>
<td>0.6</td>
<td>0.321632</td>
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<tr>
<td>0.7</td>
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<tr>
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<tr>
<td>0.9</td>
<td>3.140543</td>
</tr>
<tr>
<td>1</td>
<td>3.7</td>
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CHAPTER 3b
OPEN METHOD
Open methods – a) Simple fixed point iteration

- Re-arranging the function $f(x)=0$, so that $x$ is on the left side of the equation, i.e.
  \[ x = g(x) \] \[ \text{-------- 6.1} \]

- Transformation can be accomplished either by algebraic manipulation or by simply adding $x$ to both sides of the original equation.
  e.g. \[ x^2 - 2x + 3 = 0 \]
  can be simply manipulated to yield
  \[ x = \frac{x^2 + 3}{2} \]
  e.g. \[ \sin x = 0 \]

By adding $x$ to both sides to yield \[ x = \sin x + x \]
• Equation 6.1 provides formula to predict new value of \( x \) as a function of an old value of \( x \). Thus, given initial guess at the root \( x_i \), to compute new estimate \( x_{i+1} \) as expressed by the iterative formula;

\[
x_{i+1} \Rightarrow x_{i+1} = g(x_i)
\]

\[\text{------- 6.2}\]

• Error estimator:

\[
\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%
\]
Example 4

Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$. Given true value of the root $= 0.56714329$
when \( f(x) = 0 \), \( x = e^{-x} \)

\( x_{i+1} = e^{-x} \) starting with initial guess of \( x_0 = 0 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( x_{i+1} = e^{-x} )</th>
<th>( \varepsilon_a(%) )</th>
<th>( \varepsilon_t(%) )</th>
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</thead>
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<td></td>
<td>38.3</td>
<td>11.8</td>
</tr>
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</table>

\[
\varepsilon_a = \frac{0.367879 - 1}{0.367879} \times 100\%
\]
## Solution

<table>
<thead>
<tr>
<th>i</th>
<th>$x_i$</th>
<th>$\varepsilon_a(%)$</th>
<th>$\varepsilon_t(%)$</th>
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<td>2.20</td>
</tr>
<tr>
<td>8</td>
<td>0.560115</td>
<td>3.48</td>
<td>1.24</td>
</tr>
<tr>
<td>9</td>
<td>0.571143</td>
<td>1.93</td>
<td>0.705</td>
</tr>
<tr>
<td>10</td>
<td>0.564879</td>
<td>1.11</td>
<td>0.399</td>
</tr>
</tbody>
</table>
b) Newton-Raphson Method

- The most widely used of all root-locating formulas. In this method, if the initial guess at the root is $x_i$, a tangent can be extended from the point $[x_i, f(x_i)]$.
- The point where this tangent crosses the $x$ axis usually represents an improved estimate of the root.
- Based on Taylor series expansion;

\[
f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}
\]

- Rearrange to yield Newton-Raphson formula

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

----- 6.6
Newton-Raphson

- A convenient method for functions whose derivatives can be evaluated analytically. Figure 6.5, pg 139 shows graphical depiction of the Newton-Raphson method.

A tangent i.e. $f'(x_i)$ is extrapolated down to x axis to provide an estimate of the root at $x_{i+1}$.
Example 5

Use Newton-Raphson method to locate the root of

\[ f(x) = e^{-x} - x, \ f(x) = 0. \]

Given true value of the root = 0.56714329
Solution

The first derivative of the function can be evaluated as:

\[ f'(x) = e^{-x} - 1 \]

Substitute into NR formula:

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

\[ x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1} \]

Start the iteration with \( x_0 = 0 \)
Solution

<table>
<thead>
<tr>
<th>i</th>
<th>$x_i$</th>
<th>$x_{i+1}$</th>
<th>$\varepsilon_a$ (%)</th>
<th>$\varepsilon_t$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>-</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.566311003</td>
<td>100</td>
<td>11.8</td>
</tr>
<tr>
<td>2</td>
<td>0.566311003</td>
<td>0.567143165</td>
<td>11.7</td>
<td>0.147</td>
</tr>
<tr>
<td>3</td>
<td>0.567143165</td>
<td>0.567143290</td>
<td>0.15</td>
<td>0.000022</td>
</tr>
<tr>
<td>4</td>
<td>0.567143290</td>
<td></td>
<td>2.2 x 10^{-5}</td>
<td>&lt;10^{-8}</td>
</tr>
</tbody>
</table>

$\varepsilon_a = \frac{0.566311003 - 0.5}{0.566311003} \times 100\%$

- $\varepsilon_a$ and $\varepsilon_t$ in NR decreases much faster than simple – fixed point iteration
c) The Secant Method

- If derivatives extremely difficult or inconvenient to evaluate, “the secant method” is used.
- The derivatives can be approximated by a backward finite divided difference
  \[ f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i} \]
- The technique is similar to NR (estimate of the root is predicted by extrapolating a tangent of the function to x-axis), but the secant method uses a difference, while NR uses derivative to estimate the slope (figure 6.7, pg 145).
- Substitute \( f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i} \) in \( x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \) will yield;
  \[ x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \]

-----  6.7
• Equation 6.7 is called a secant method. This approach requires two initial estimates of $x$. However, $f(x)$ is not required to change signs between the estimates and hence it is not classified as a bracketing method.
Example 6

Use the secant method to estimate the root of

\( f(x) = e^{-x} - x, \quad f(x) = 0. \) Given true value of the root

\( = 0.56714329. \) Start with initial estimate of \( x_{i-1} = \)

0 and \( x_0 = 1.0 \)
Solution

1st iteration \( x_{i-1} = 0 \) \( f(x_{i-1}) = 1 \)
\( x_0 = 1.0 \) \( f(x_0) = -0.63212 \)
\[ x_1 = 1 - \frac{-0.63212(0 - 1)}{1 - (-0.63212)} = 0.61270 \]
\[ \varepsilon_t = \left| \frac{0.56714329 - 0.61270}{0.56714329} \right| 100\% = 8.03\% \]
Solution

2nd iteration \( x_0 = 0 \quad f(x_0) = -0.63212 \)
\[ x_1 = 0.6127 \quad f(x_1) = -0.0708 \]
\[ x_1 = 0.6127 - \frac{-0.07081(1 - 0.61270)}{-0.63212 - (-0.07081)} = 0.56384 \]
\[ \varepsilon_t = \left| \frac{0.56714329 - 0.56384}{0.56714329} \right| \times 100\% = 0.58\% \]

3rd iteration \( x_1 = 0.61270 \quad f(x_1) = -0.0708 \)
\[ x_2 = 0.56384 \quad f(x_2) = 0.00518 \]
\[ x_1 = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - (-0.00518)} = 0.56717 \]
\[ \varepsilon_t = \left| \frac{0.56714329 - 0.56717}{0.56714329} \right| \times 100\% = 0.0048\% \]
Work with your buddy and do Quiz 6 & Quiz 7
## Comparison

<table>
<thead>
<tr>
<th>Bracketing Methods</th>
<th>Open Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>- 2 initial guesses</td>
<td>- 1 initial guess <em>(Simple Fixed point iteration and Newton –Raphson method)</em> or 2 initial guesses (Secant Method)</td>
</tr>
<tr>
<td>- Root <em>is located within</em> lower and upper interval, repeating will result in closer estimates of true value of the root</td>
<td>- not necessary bracket the root</td>
</tr>
<tr>
<td>- <em>Convergent</em> due to it move closer to the true value as the computation progress</td>
<td>- Sometimes <em>diverge /move away</em> from the true root as computation progress</td>
</tr>
<tr>
<td></td>
<td>- but if converge (closer to true value) more quickly than bracketing method.</td>
</tr>
</tbody>
</table>
Question?
THE END

Thank You for the Attention