**Question 1 (20 marks)**

a) Show that the differential equation

$$\left(2 + \sin^{-1} y\right) dx + \frac{x - 3}{\sqrt{1 - y^2}}\, dy = 0, \qquad -1 < y < 1,$$

is an exact equation. Hence, find its solution for $y(0) = 1$.

**(7 marks)**

b) Find the solution to the Bernoulli's equation

$$x\frac{dy}{dx} + y = 2y^{\frac{1}{2}}, \qquad y > 0.$$

**(7 marks)**

c) The growth rate of a bacteria in a culture can be represented by the following differential equation

$$\frac{dP}{dt} = kP,$$

where $P(t)$ is the population of bacteria at time $t$ (hour), and $k$ is a constant. If at time $t = 0$ the population of bacteria are 500 and it increases to 1000 after one hour, find the population of the bacteria after two hours.

**(6 marks)**

**Question 2 (15 marks)**

a) Solve the boundary value problem,

$$y^{'''} + 3y^{''} + 4y^{'} + 12y = 0$$

subject to $y(0) = 1$, $y(\pi) = 1$ and $y^{'}(0) = 1$.

**(7 marks)**

b) The equation of motion for an undamped system at resonance is governed by

$$m\frac{d^2y}{dt^2} + ky = F_0 \cos\gamma t,$$

where $m$ is the mass, $k$ is the spring constant and $F_0 \cos\gamma t$ is the external force. Find $y(t)$ if $m = 5$kg, $k = 500$ N/m, $F_0 = 500$ N,$\gamma = 20$, $y(0) = 0$ and $y'(0) = 4$.

**(8 marks)**

**Question 3 (15 marks)**

a) Given

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ t-1, & t \ge 1. \end{cases}$$

Write $f(t)$ in terms of unit step function. Hence, find $\mathcal{L}\{f(t)\}$.

**(3 marks)**

b) Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2(s^2+s-2)}\right\}$.

**(6 marks)**

c) Use the results in part (a) and (b), or otherwise to find the solution to the initial value problem

$$y'' + y' - 2y = f(t), \qquad y(0) = y'(0) = 0,$$

where the function $f(t)$ given in part (a).

**(6 marks )**

**Question 4 (15 marks)**

a) Find $\mathcal{L}\left\{t\cosh 2t\right\}$.

**(3 marks)**

b) Use the convolution theorem to find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$.

**(5 marks)**

c) Given that $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\begin{aligned}\frac{dx}{dt} &= 2x-3y,\\ \frac{dy}{dt} &= -x+2y,\end{aligned}$$

subject to the initial conditions $x=0$, $y=1$ when $t=0$. Solve for $x(t)$ by using Laplace transform.

**(7 marks)**

**Question 5 (15 marks)**

Given a function

$$\begin{aligned} f(x) &= \begin{cases} x+\pi, & -\pi \le x < 0 \\ 0, & 0 < x \le \pi \end{cases} \\ f(x) &= f(x+2\pi). \end{aligned}$$

a) Sketch the graph of $y=f(x)$ for $-3\pi \le x \le 3\pi$. Hence, determine if the function is even, odd or neither.

**(4 marks)**

b) Show that the Fourier series of $f(x)$ is given by

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{2\cos(2n-1)x}{\pi(2n-1)^2} - \frac{\sin nx}{n}\right\}.$$

**(8 marks)**

c) By choosing an appropriate value of $x$, find the exact value for the following series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

**(3 marks)**

**Question 6 (20 marks)**

a) Use the D'Alembert solution method to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \qquad -\infty < x < \infty,$$

subject to initial conditions

$$u(x,0) = e^x, \qquad \frac{\partial u}{\partial t}(x,0) = e^{-x}.$$

**(5 marks)**

b) Find the solution of the following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \qquad t > 0,$$

that satisfies the boundary conditions

$$u_x(0,t) = 0, \qquad u_x(2,t) = 0, \qquad t > 0,$$

and initial conditions

$$u(x,0) = 2x(1-x), \qquad 0 < x < 2.$$

**(15 marks)**