CHAPTER 1
FURTHER TRANSCENDENTIAL FUNCTIONS

1.1 Hyperbolic Functions

1.1.1 Definition of Hyperbolic Functions
1.1.2 Graphs of Hyperbolic Functions
1.1.3 Hyperbolic Identities

1.2 Inverse Functions

1.2.1 Inverse Trigonometric Functions
1.2.2 Inverse Trigonometric Identities
1.2.3 Inverse Hyperbolic Functions
1.2.4 Log Form of the Inverse Hyperbolic Functions
### 1.1.1 Definition of Hyperbolic Functions

**Hyperbolic Sine**, pronounced “shin”.
\[
\sinh x = \frac{e^x - e^{-x}}{2}
\]

**Hyperbolic Cosine**, pronounced “cosh”.
\[
\cosh x = \frac{e^x + e^{-x}}{2}
\]

**Hyperbolic Tangent**, pronounced “tanh”.
\[
\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}
\]

**Hyperbolic Secant**, pronounced “shek”.
\[
\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}
\]

**Hyperbolic Cosecant**, pronounced “coshek”.
\[
\text{cosech } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}
\]

**Hyperbolic Cotangent**, pronounced “coth”.
\[
\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}
\]
1.1.2 Graphs of Hyperbolic Functions

Since the hyperbolic functions depend on the values of $e^x$ and $e^{-x}$, its graphs is a combination of the exponential graphs.

(i) **Graph of sinh $x$**

From the graph, we see

(i) $\sinh 0 = 0$.

(ii) The domain is all real numbers

(iii) The curve is symmetrical about the origin, i.e.

$$\sinh (-x) = -\sinh x$$

(iv) It is an increasing one-to-one function.
We see from the graph of \( y = \cosh x \) that:

(i) \( \cosh 0 = 1 \)
(ii) The domain is all real numbers.
(iii) The value of \( \cosh x \) is never less than 1.
(iv) The curve is symmetrical about the \( y \)-axis, i.e.
\[ \cosh (\text{-}x) = \cosh x \]
(v) For any given value of \( \cosh x \), there are two values of \( x \).
We see

(i) \( \tanh 0 = 0 \)

(ii) \( \tanh x \) always lies between \( y = -1 \) and \( y = 1 \).

(iii) \( \tanh (-x) = -\tanh x \)

(iv) It has horizontal asymptotes \( y = \pm 1 \).
1.1.3 Hyperbolic Identities

For every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions.

1. \( \cosh^2 x - \sinh^2 x = 1 \)

2. \( 1 - \tanh^2 x = \text{sech}^2 x \)

3. \( \coth^2 x - 1 = \text{cosech}^2 x \)

4. \( \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \)

5. \( \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \)

6. \( \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \)

7. \( \sinh 2x = 2 \sinh x \cosh x \)

8. \( \cosh 2x = \cosh^2 x + \sinh^2 x \)
   \[ = 2 \cosh^2 x - 1 \]
   \[ = 2 \sinh^2 x + 1 \]

9. \( \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \)
Some of the hyperbolic identities follow exactly the trig. identities; others have a difference in sign.

<table>
<thead>
<tr>
<th>Trig. Identities</th>
<th>Hyperbolic Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sec \theta = \frac{1}{\cos \theta} )</td>
<td>( \text{sech} \theta = \frac{1}{\cosh \theta} )</td>
</tr>
<tr>
<td>( \cosec \theta = \frac{1}{\sin \theta} )</td>
<td>( \text{cosech} \theta = \frac{1}{\sinh \theta} )</td>
</tr>
<tr>
<td>( \cot \theta = \frac{1}{\tan \theta} )</td>
<td>( \coth \theta = \frac{1}{\tanh \theta} )</td>
</tr>
<tr>
<td>( \cos^2 \theta + \sin^2 \theta \equiv 1 )</td>
<td>( \cosh^2 \theta - \sinh^2 \theta \equiv 1 )</td>
</tr>
<tr>
<td>( 1 + \tan^2 \theta \equiv \sec^2 \theta )</td>
<td>( 1 - \tanh^2 \theta \equiv \text{sech}^2 \theta )</td>
</tr>
<tr>
<td>( 1 + \cot^2 \theta \equiv \cosec^2 \theta )</td>
<td>( \coth^2 \theta - 1 \equiv \text{cosech}^2 \theta )</td>
</tr>
<tr>
<td>( \sin 2A \equiv 2 \sin A \cos A )</td>
<td>( \sinh 2A \equiv 2 \sinh A \cosh A )</td>
</tr>
<tr>
<td>( \cos 2A \equiv \cos^2 A - \sin^2 A )</td>
<td>( \cosh 2A \equiv \cosh^2 A + \sinh^2 A )</td>
</tr>
<tr>
<td>( \equiv 1 - 2 \sin^2 A )</td>
<td>( \equiv 1 + 2 \sinh^2 A )</td>
</tr>
<tr>
<td>( \equiv 2 \cos^2 A - 1 )</td>
<td>( \equiv 2 \cosh^2 A - 1 )</td>
</tr>
</tbody>
</table>
**Examples 1.1**

1. By using definition of hyperbolic functions,
   a) Evaluate sinh(-4) to four decimal places.
   
   b) Show that $2 \cosh^2 x - 1 = \cosh 2x$

2. a) By using identities of hyperbolic functions, show that
   
   $\frac{1 - \tanh^2 x}{1 + \tanh^2 x} = \sech 2x$

   b) Solve the following for $x$, giving your answer in 4dcp.
   
   $\cosh 2x = \sinh x + 1$

3. Solve for $x$ if given $2\cosh x - \sinh x = 2$.

4. a) By using definition of hyperbolic functions, proof that
   
   $\cosh^2 x - \sinh^2 x = 1$

   b) Solve $\cosh x = 4 - \sinh x$. Use 4 dcp.
**1.2 Inverse Functions**

**Definition 1.2 (Inverse Functions)**

If \( f : X \to Y \) is a one-to-one function with the domain \( X \) and the range \( Y \), then there exists an inverse function,

\[
f^{-1} : Y \to X
\]

where the domain is \( Y \) and the range is \( X \) such that

\[
y = f(x) \iff x = f^{-1}(y)
\]

Thus, \( f^{-1}(f(x)) = x \) for all values of \( x \) in the domain \( f \).

**Note:**

The graph of inverse function is reflections about the line \( y = x \).
1.2.1 Inverse Trigonometric Functions

Trigonometric functions are periodic hence they are not one-to-one. However, if we restrict the domain to a chosen interval, then the restricted function is one-to-one and invertible.

(i) Inverse Sine Function

Look at the graph of \( y = \sin x \) shown below

The function \( f(x) = \sin x \) is not one to one. But if the domain is restricted to \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), then \( f(x) \) is one to one.
**Definition:**

The inverse sine function is defined as

\[ y = \sin^{-1} x \iff x = \sin y \]

where \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \) and \( -1 \leq x \leq 1 \).

The function \( \sin^{-1} x \) is sometimes written as arcsin \( x \).

The graph of \( y = \sin^{-1} x \) is shown below

\[ f(x) = \sin^{-1} x \]

\[ f(x) = \arcsin x \]
(ii) Inverse Cosine Function

Look at the graph of \( y = \cos x \) shown below.

The function \( f(x) = \cos x \) is not one to one. But if the domain is restricted to \([0, \pi]\), then \( f(x) \) is one to one.

**Definition:**

The inverse cosine function is defined as

\[
y = \cos^{-1} x \iff x = \cos y
\]

where \(0 \leq y \leq \pi\) and \(-1 \leq x \leq 1\).
The graph of \( y = \cos^{-1} x \) is shown below

\[
\begin{align*}
\arccos x &= f(x) \\
\arccos x &= \cos^{-1} x
\end{align*}
\]

(iii) Inverse Tangent Function

Look at the graph of \( y = \tan x \) shown below

The function \( f(x) = \tan x \) is not one to one. But if the domain is restricted to \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), then \( f(x) \) is one to one.
**Definition:**

The inverse tangent function is defined as

\[ y = \tan^{-1} x \iff x = \tan y \]

where \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\) and \(-\infty \leq x \leq \infty\).

The graph of \( y = \tan^{-1} x \) is shown below

\[ f(x) = \tan^{-1} x \]
\[ f(x) = \arctan x \]
(iv) Inverse Cotangent Function

Domain:
Range:

(v) Inverse Secant Function

Domain:
Range:
(vi) Inverse Cosecant Function

Domain:
Range:
### Table of Inverse Trigonometric Functions

<table>
<thead>
<tr>
<th>Functions</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sin^{-1} x$</td>
<td>$[-1, 1]$</td>
<td>$[-\frac{\pi}{2}, \frac{\pi}{2}]$</td>
</tr>
<tr>
<td>$y = \cos^{-1} x$</td>
<td>$[-1, 1]$</td>
<td>$[0, \pi]$</td>
</tr>
<tr>
<td>$y = \tan^{-1} x$</td>
<td>$(-\infty, \infty)$</td>
<td>$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$</td>
</tr>
<tr>
<td>$y = \csc^{-1} x$</td>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$y = \sec^{-1} x$</td>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$y = \cot^{-1} x$</td>
<td>$(-\infty, \infty)$</td>
<td>$(0, \pi)$</td>
</tr>
</tbody>
</table>

- It is easier to remember the restrictions on the domain and range if you do so in terms of quadrants.
- $\sin^{-1} x \neq \frac{1}{\sin x}$ whereas $(\sin x)^{-1} = \frac{1}{\sin x}$. 
1.2.2 Inverse Trigonometric Identities

The definition of the inverse functions yields several formulas.

**Inversion formulas**

<table>
<thead>
<tr>
<th>Formula</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1}(\sin x) = x )</td>
<td>for (-1 \leq x \leq 1)</td>
</tr>
<tr>
<td>( \sin^{-1}(\sin y) = y )</td>
<td>for (-\frac{\pi}{2} \leq y \leq \frac{\pi}{2})</td>
</tr>
<tr>
<td>( \tan^{-1}(\tan x) = x )</td>
<td>for all (x)</td>
</tr>
<tr>
<td>( \tan^{-1}(\tan y) = y )</td>
<td>(-\frac{\pi}{2} &lt; y &lt; \frac{\pi}{2})</td>
</tr>
</tbody>
</table>

➢ These formulas are valid only on the specified domain

**Basic Relation**

<table>
<thead>
<tr>
<th>Formula</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} )</td>
<td>for (0 \leq x \leq 1)</td>
</tr>
<tr>
<td>( \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} )</td>
<td>for (0 \leq x \leq 1)</td>
</tr>
<tr>
<td>( \sec^{-1} x + \csc^{-1} x = \frac{\pi}{2} )</td>
<td>for (0 \leq x \leq 1)</td>
</tr>
</tbody>
</table>
### Negative Argument Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1}(-x) = -\sin^{-1}x )</td>
<td>( \sec^{-1}(-x) = \pi - \sec^{-1}x )</td>
</tr>
<tr>
<td>( \tan^{-1}(-x) = -\tan^{-1}x )</td>
<td>( \csc^{-1}(-x) = -\csc^{-1}x )</td>
</tr>
<tr>
<td>( \cos^{-1}(-x) = \pi - \cos^{-1}x )</td>
<td>( \cot^{-1}(-x) = \pi - \cot^{-1}x )</td>
</tr>
</tbody>
</table>

### Reciprocal Identities

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) )</td>
<td>for (</td>
</tr>
<tr>
<td>( \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) )</td>
<td>for (</td>
</tr>
<tr>
<td>( \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) )</td>
<td>for (</td>
</tr>
</tbody>
</table>
Examples 1.2:

1. Evaluate the given functions.

(i) $\sin (\sin^{-1} 0.5)$  \hspace{1cm} (ii) $\sin (\sin^{-1} 2)$

(iii) $\sin^{-1}(\sin 0.5)$  \hspace{1cm} (iv) $\sin^{-1}(\sin 2)$

2. Evaluate the given functions.

(i) $\arccsc(-2)$  \hspace{1cm} (ii) $\csc^{-1} (\sqrt{2})$

(iii) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

3. For $-1 \leq x \leq 1$, show that

(i) $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) $\cos (\sin^{-1} x) = \sqrt{1-x^2}$
1.2.3 Inverse Hyperbolic Functions

The three basic inverse hyperbolic functions are \( \sinh^{-1} x \), \( \cosh^{-1} x \), and \( \tanh^{-1} x \).

**Definition (Inverse Hyperbolic Function)**

\[
\begin{align*}
y &= \sinh^{-1} x \iff x = \sinh y \quad &\text{for all } x \text{ and } y \in \mathbb{R} \\
y &= \cosh^{-1} x \iff x = \cosh y \quad &\text{for } x \geq 1 \text{ and } y \geq 0 \\
y &= \tanh^{-1} x \iff x = \tanh y \quad &\text{for } -1 \leq x \leq 1, y \in \mathbb{R}
\end{align*}
\]

**Graphs of Inverse Hyperbolic Functions**

(i) \( y = \sinh^{-1} x \)

Domain: \( \quad \) Range:
(ii) \( y = \cosh^{-1} x \)

Domain: \( \quad \) Range: 

(iii) \( y = \tanh^{-1} x \)

Domain: \( \quad \) Range: 
1.2.4 Log Form of the Inverse Hyperbolic Functions

It may be shown that

(a) \( \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \)

(b) \( \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \)

(c) \( \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \)

(d) \( \coth^{-1} x = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right) \)

(e) \( \sech^{-1} x = \ln \left( \frac{1+\sqrt{1-x^2}}{x} \right) \)

(f) \( \cosech^{-1} x = \ln \left( \frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|} \right) \)
Inverse Hyperbolic Cosine (Proof)

If we let \( y = \cosh^{-1} x \), then

\[
x = \cosh y = \frac{e^y + e^{-y}}{2}
\]

Hence,

\[
2x = e^y + e^{-y}
\]

On rearrangement,

\[
(e^y)^2 - 2xe^y + 1 = 0
\]

Hence, (using formula \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \))

\[
e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}
\]

Since \( e^y > 0 \),

\[
\therefore e^y = x + \sqrt{x^2 - 1}
\]

Taking natural logarithms,

\[
y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})
\]
Proof for $\sinh^{-1} x$

\[ y = \sinh^{-1} x \]

\[ x = \sinh y = \frac{e^y - e^{-y}}{2} \]

\[ \therefore 2x = e^y - e^{-y} \text{ (multiply with } e^y) \]

\[ 2xe^y = e^{2y} - 1 \]

\[ e^{2y} - 2xe^y - 1 = 0 \]

\[ e^y = x \pm \sqrt{x^2 + 1} \]

Since $e^y > 0$,

\[ \therefore e^y = x + \sqrt{x^2 + 1} \]

Taking natural logarithms,

\[ y = \sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right) \]

In the same way, we can find the expression for $\tanh^{-1} x$ in logarithmic form.
Examples 1.3: Evaluate

1) \( \sinh^{-1}(0.5) \)

2) \( \cosh^{-1}(0.5) \)

3) \( \tanh^{-1}(-0.6) \)