CHAPTER 4:

IMPROPER INTEGRALS

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4.1 L’Hopital Rule

If you are doing any limit and you get something in the form $0/0$ or $\infty/\infty$, then you should probably try to use L’Hopital rule. The basic idea of L’Hospital rule is simple. Consider the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If both the numerator and the denominator are finite at $a$ and $g(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

**Example 1:**

$$\lim_{x \to 3} \frac{x^2 + 1}{x + 2} = \frac{10}{5} = 2.$$
But what happens if both the numerator and the denominator tend to zero???

It is not clear what the limit is. In fact, depending on what functions $f(x)$ and $g(x)$ are, the limit can be anything at all!!!

**4.1.1 L’Hopital Rule for 0/0**

Suppose $\lim f(x) = \lim g(x) = 0$. Then

1. If $\lim \frac{f'(x)}{g'(x)} = L$,

   then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L$.

2. If $\lim \frac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, then so does $\lim \frac{f(x)}{g(x)}$. 
Example 2:

Find \( \lim_{x \to 0} \frac{\sin x}{x} \) by L’Hopital rule.

Example 3:

Find \( \lim_{x \to 1} \frac{2\ln x}{x - 1} \).

Example 4:

Find \( \lim_{x \to 1} \frac{e^x - 1}{x^2} \).
Example 5:

Find \( \lim_{x \to 0} \frac{1 - \cos x}{x} \).

Note: If the numerator and the denominator both tend to \(+\infty\) or \(-\infty\), L’Hopital rule still applies.

4.1.2 L’Hopital Rule for \(\infty/\infty\)

Suppose \( \lim f(x) \) and \( \lim g(x) \) are both infinite. Then

1. If \( \lim \frac{f'(x)}{g'(x)} = L \),

then \( \lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L \).
2. If \( \lim \frac{f'(x)}{g'(x)} \) tends to \(+\infty\) or \(-\infty\) in the limit, then so does \( \lim \frac{f(x)}{g(x)} \).

**Example 6:**

Find \( \lim_{x \to \infty} \frac{x}{e^x} \).

**Example 7:**

Find \( \lim_{x \to \infty} \frac{\ln(\ln x^{1000})}{\ln x} \).
4.2 Improper Integrals

The definite integral
\[ \int_{a}^{b} f(x) \, dx \]

is known as *improper integral* if either

1) one or both limits are infinite, or

2) \( f(x) \) is undefined at certain points on/in the interval.

Note: We called case: 1) as Type I
2) as Type II

4.2.1 Improper Integral Type 1

1) If \( f(x) \) is continuous in the interval \([a, \infty)\),

\[ \int_{a}^{\infty} f(x) \, dx = \lim_{T \to \infty} \int_{a}^{T} f(x) \, dx. \]
2) If $f(x)$ is continuous in the interval $(-\infty, b]$, then
\[ \int_{-\infty}^{b} f(x) \, dx = \lim_{T \to -\infty} \int_{T}^{b} f(x) \, dx. \]

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

3) If $f(x)$ is continuous in the interval $(-\infty, \infty)$, then
\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx \]
with any real number $c$.

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.
Example 8:
Determine whether the following integral are convergent or divergent:

1) \[ \int_{1}^{\infty} \frac{1}{x} \, dx \]

2) \[ \int_{0}^{\infty} xe^{-x} \, dx \]

3) \[ \int_{-\infty}^{\infty} \frac{x}{1 + x^2} \, dx \]

Example 9:
For what values of \( p \) is the integral \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) convergent?
4.2.2 Improper Integral Type 2

1) If $f(x)$ is continuous on $[a,b)$, and discontinuous at $b$, then

$$\int_a^b f(x) \, dx = \lim_{T \to b^-} \int_a^T f(x) \, dx.$$ 

2) If $f(x)$ is continuous on $(a,b]$, and discontinuous at $a$, then

$$\int_a^b f(x) \, dx = \lim_{T \to a^+} \int_T^b f(x) \, dx.$$ 

Note: the improper integrals in 1) and 2) is said to converge if the limit exists and diverge if the limit does not exist.

3) If $f(x)$ has discontinuity at $c$, where $a < c < b$, then

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$
Note: the improper integrals in 3) is said to converge if both terms converge and diverge if either term diverges.

Example 10:

Determine whether \( \int_{1}^{9} \frac{1}{\sqrt[3]{x-9}} \, dx \) converge or diverge.

Example 11:

Find \( \int_{0}^{3} \frac{1}{x-1} \, dx \) if possible.