CHAPTER 6 : VECTORS

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6.2 Planes in Space
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Review:

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Basic Concepts

What is scalar?

✓ a quantity that has only magnitude

What is vector?

✓ a quantity that has magnitude and direction

A vector can be represented by a directed line segment where

✓ length of the line segment
  - the magnitude of the vector

✓ direction of the line segment
  - the direction of the vector
✓ A vector can be written as $\overrightarrow{PQ}$, or $\mathbf{a}$. The order of the letters is important. $\overrightarrow{PQ}$ means the vector is from $P$ to $Q$ or the position vector $Q$ relative to $P$, $\overrightarrow{QP}$ means vector is from $Q$ to $P$ or the position vector $P$ relative to $Q$.

✓ If $P\left(x_1, y_1\right)$ is the initial point and $Q\left(x_2, y_2\right)$ is the terminal point of a directed line segment, $\overrightarrow{PQ}$ then **component form** of vector $\mathbf{v}$ that represents $\overrightarrow{PQ}$ is
\[ \langle v_1, v_2 \rangle = \langle x_2 - x_1, y_2 - y_1 \rangle = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \]

and the **magnitude** or the **length** of \( v \) is

\[ |v| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ = \sqrt{c_1^2 + c_2^2} \]

\[ \overrightarrow{OP} = \langle x_1 - 0, y_1 - 0 \rangle = \langle x_1, y_1 \rangle \]

\[ \overrightarrow{OQ} = \langle x_2 - 0, y_2 - 0 \rangle = \langle x_2, y_2 \rangle \]

- **Note**-

Any vector that has magnitude of 1 unit = **unit vector**.
**Example:**

Find the component form and length of the vector \( \mathbf{v} \) that has initial point \((3,-7)\) and terminal point \((-2,5)\).

**Solution:**

\[
\mathbf{v} = \langle -2 - 3, 5 + 7 \rangle = \langle -5, 12 \rangle
\]

\[
|\mathbf{v}| = \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144} = 13
\]

**Example:**

Given \( \mathbf{v} = \langle -2, 5 \rangle \) and \( \mathbf{w} = \langle 3, 4 \rangle \), find each of the following vectors:

a) \( \frac{1}{2} \mathbf{v} \)  
b) \( \mathbf{w} - \mathbf{v} \)  
c) \( \mathbf{v} + 2\mathbf{w} \)

**Answer:**

a) \( \langle -1, 5/2 \rangle \)  
b) \( \langle 5, -1 \rangle \)  
c) \( \langle 4, 13 \rangle \)
Theorem:

If \( \mathbf{a} \) is a non-null vector and if \( \hat{\mathbf{a}} \) is a unit vector having the same direction as \( \mathbf{a} \), then

\[
\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}
\]

-Note-

To verify that magnitude is 1, \( |\hat{\mathbf{a}}| = 1 \)

Example:

Find a unit vector in the direction of \( \mathbf{v} = \langle -2, 5 \rangle \) and verify that it has length 1.

Solution:

\[
\mathbf{v} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + 5^2}} = \frac{1}{\sqrt{29}} \langle -2, 5 \rangle
\]

\[
|\mathbf{v}| = \sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{1} = 1
\]
Standard Unit Vectors

✓ Three standard unit vectors are: \( \mathbf{i}, \mathbf{j} \) dan \( \mathbf{k} \)

✓ Vectors \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) can be written in components form:

\[
\mathbf{i} = \langle 1, 0, 0 \rangle,
\]

\[
\mathbf{j} = \langle 0, 1, 0 \rangle \quad \text{and}
\]

\[
\mathbf{k} = \langle 0, 0, 1 \rangle
\]

and can interpreted as

\[
\mathbf{a} = \langle x, y, z \rangle
\]

\[
= xi + yj + zk
\]
The vector \( \overrightarrow{PQ} \) with initial point \( P(x_1, y_1, z_1) \) and terminal point \( Q(x_2, y_2, z_2) \) has the standard representation

\[
\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k
\]

or

\[
PQ = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle
\]

**Example:**

Let \( \mathbf{u} \) be the vector with initial point \((2, -5)\) and terminal point \((-1, 3)\), and let \( \mathbf{v} = 2i - j \).

Write each of the following vectors as a linear combination of \( \mathbf{i} \) and \( \mathbf{j} \).

a) \( \mathbf{u} \)

b) \( \mathbf{w} = 2\mathbf{u} - 3\mathbf{v} \)
-Note-

If $\theta$ is the angle between $\vec{v}$ and the positive $x$–axis then we can write

$$x = |\vec{v}| \cos \theta \text{ and } y = |\vec{v}| \sin \theta \; ; \; |\vec{v}| = \sqrt{x_1^2 + y_2^2}.$$  

**Example:**

The vector $\mathbf{v}$ has a length of 3 and makes an angle of $30^\circ = \frac{\pi}{6}$ with the positive $x$-axis.

Write $\mathbf{v}$ as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.  

Vectors in Space

Properties of Vectors in Space

Let \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) be vectors in 3 dimensional space and \( k \) is a constant.

1. \( \mathbf{v} = \mathbf{w} \) if and only if
   \[ v_1 = w_1, v_2 = w_2, v_3 = w_3. \]

2. The magnitude of \( \mathbf{v} \) is \( |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \)

3. The unit vector in the direction of \( \mathbf{v} \) is
   \[ \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle v_1, v_2, v_3 \rangle}{|\mathbf{v}|} \]

4. \( \mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \)

5. \( k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle \)

6. Zero vector is denoted as \( \mathbf{0} = \langle 0, 0, 0 \rangle \).

7. \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \)
8. Let \( \mathbf{u} = (u_1, u_2, u_3) \),

then \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \).

9. \( \mathbf{u} + 0 = \mathbf{u} \)

10. \( \mathbf{u} + (-\mathbf{u}) = 0 \)

11. \( (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} \)

12. \( c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \)

13. \( c(d\mathbf{u}) = (cd)\mathbf{u} \)

14. \( 1(\mathbf{u}) = \mathbf{u} \) and \( 0(\mathbf{u}) = 0 \)

15. \( |c\mathbf{u}| = c|\mathbf{u}| \)

Example:

Express the vector \( \overrightarrow{PQ} \) if it starts at point \( P = (6,5,8) \) and stops at point \( Q = (7,3,9) \) in components form.
Solution:

\[ \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \]

\[ \overrightarrow{PQ} = \langle 7 - 6, 3 - 5, 9 - 8 \rangle \]

\[ \overrightarrow{PQ} = \langle 1, -2, 1 \rangle \]

Example:

Given that \( \mathbf{a} = \langle 3, 1, -2 \rangle \), \( \mathbf{b} = \langle -1, 6, 4 \rangle \). Find

(a) \( \mathbf{a} + 3\mathbf{b} \)  
(b) \( |\mathbf{b}| \)

(c) a unit vector which is in the direction of \( \mathbf{b} \).

(d) find the unit vector which has the same direction as \( \mathbf{a} + 3\mathbf{b} \).

Answer:

(a) \( \langle 0, 19, 10 \rangle \)  
(b) \( \sqrt{53} \)  
(c) \( \langle -2, \frac{41}{6}, \frac{23}{3} \rangle \)

(d) 1  
(e) \( \frac{1}{\sqrt{53}} \langle -1, 6, 4 \rangle \)  
(f) \( \frac{1}{\sqrt{461}} \langle 0, 19, 10 \rangle \)
Parallel Vector

- have same slopes
- \( \nu_1 = \lambda \nu_2 \); \( \lambda \) constants

So, there are multiples of each other.

**Example:**

Vector \( \mathbf{w} \) has initial point \((2, -1, 3)\) and terminal point \((-4, 7, 5)\). Which of the following vectors is parallel to \( \mathbf{w} \)?

**Solution:**

\[
\bar{w} = \langle 4 - 2, 7 + 1, 5 - 3 \rangle
\]

\[
\bar{w} = \langle -6, 8, 2 \rangle
\]

One example: \( 2\langle -6, 8, 2 \rangle = \langle -12, 16, 4 \rangle \)

Another is \( \frac{1}{2} \langle -6, 8, 2 \rangle = \langle -3, 4, 1 \rangle \).
Example: (Collinear Points)

Determine whether the point \( P(1,-2,3), Q(2,1,0) \)
and \( R(4,7,-6) \) lie on the same line.

Solution:

\[
\overrightarrow{PQ} = \langle 2-1, 1+2, 0-3 \rangle = \langle 1, 3, -3 \rangle
\]

\[
|\overrightarrow{PQ}| = \sqrt{1^2 + 3^2 + (-3)^2} = \sqrt{19}
\]

\[
\overrightarrow{QR} = \langle 4-2, 7-1, -6-0 \rangle = \langle 2, 6, -6 \rangle
\]

\[
|\overrightarrow{QR}| = \sqrt{2^2 + 6^2 + (-6)^2} = 2\sqrt{19}
\]

\[
\overrightarrow{PR} = \langle 4-1, 7+2, -6-3 \rangle = \langle 3, 9, -9 \rangle
\]

\[
|\overrightarrow{PR}| = \sqrt{3^2 + 9^2 + (-9)^2} = 3\sqrt{19}
\]

Thus, \( \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} \). Since one vector is a multiple of the other, the two vectors are
parallel and since they share a common point \( Q \), they must be the same line.

**The Dot Product**

*Theorem*

If \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \), then the scalar product \( \mathbf{v} \cdot \mathbf{w} \) is

\[
\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3
\]

*Note*

The dot product is also called
- the scalar product
- the inner product

The dot product of two vectors is a scalar.
The Angle between Vectors

Refer to the figure below, let

\[ \vec{u} = \vec{u}(OP) = \langle u_1, u_2, u_3 \rangle, \]

\[ \vec{v} = \vec{v}(OQ) = \langle v_1, v_2, v_3 \rangle \]

be two vectors and let \( \theta \) be the angle between them, with \( 0 \leq \theta \leq \pi \).

Compute the distance, \( c \) between points \( P \) and \( Q \) in two ways.
1) Using the Distance formula

\[ c^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \]

\[ = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \]

\[ = |\vec{u}|^2 + |\vec{v}|^2 - 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \quad ---(1) \]

2) Using the Law of Cosines

\[ c^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos \theta \quad ---(2) \]

Equating equation (1) and (2), we get

\[ \cos \theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\vec{u}||\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \]

Example:

If \( \vec{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} \), \( \vec{w} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \) and the angle between \( \vec{v} \) and \( \vec{w} \) is 60°, find \( \vec{v} \cdot \vec{w} \).
Solution:

\[
\vec{v} \cdot \vec{w} = \sqrt{2^2 + (-1)^2 + 1^2} \cdot \sqrt{1^2 + 1^2 + 2^2} \cos(\pi/3)
\]

\[
= \sqrt{6} \cdot \sqrt{6} \cos(\pi/3) = 6(1/2) = 3
\]

Example:

Given that \( \mathbf{u} = \langle 2, -2, 3 \rangle \), \( \mathbf{v} = \langle 5, 8, 1 \rangle \) and \( \mathbf{w} = \langle -4, 3, -2 \rangle \), find

(a) \( \mathbf{u} \cdot \mathbf{v} \)

(b) \( (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \)

(c) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \)

(d) the angle between \( \mathbf{u} \) and \( \mathbf{v} \)

(e) the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

Answer:

(a) -3

(b) \( \langle 12, -9, 6 \rangle \)

(c) -23
(d) 94°24'
(e) 87°46'

Example:

Let $A=(4,1,2)$, $B=(3,4,5)$ and $C=(5,3,1)$ are the vertices of a triangle. Find the angle at vertex $A$.

Answer:

79°12'

-Theorem-

The nature of an angle $\theta$, between two vectors $u$ and $v$.

✓ $\theta$ is an acute angle if and only if $u \cdot v > 0$

✓ $\theta$ is an obtuse angle if and only if $u \cdot v < 0$

✓ $\theta = 90^\circ$ if and only if $u \cdot v = 0$. The Vectors $u$ and $v$ are orthogonal / perpendicular.
Example:

Show that the given vectors are perpendicular to each other.

(a) i and j

(b) 3i-7j+2k and 10i+4j-k

-Theorem-

(Properties of Dot Product)

If u, v and w are nonzero vectors and k is a scalar,

1. \( u \cdot v = v \cdot u \)

2. \( u \cdot (v + w) = u \cdot v + u \cdot w \)

3. \( ku \cdot v = u \cdot kv \)

4. \( v \cdot v = |v|^2 \)

5. \( u \cdot 0 = 0 \cdot u = 0 \)
The Cross Products

- The cross product (vector product) $\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to $\mathbf{u}$ and $\mathbf{v}$. (illustrated in figure below)
- The direction is determined by the right hand rule.

\[ \mathbf{u} \times \mathbf{v} \]

✓ If the first two fingers of the right hand point in the directions of $\mathbf{u}$ and $\mathbf{v}$ respectively, then the thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.

Ex: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
- The length is determined by the lengths of $u$ and $v$ and the angle between them.

- If we change the order informing the cross product, then we change the direction.

Ex:

$$\vec{v} \times \vec{u} = - (\vec{u} \times \vec{v})$$

-**Theorem**-

If $u = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ and $v = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$, then,

$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$
Properties of Cross Product

(a) \( \mathbf{u} \times \mathbf{u} = 0 \)

(b) \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \)

(c) \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \)

(d) \( (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v}) \)

(e) \( \mathbf{u} \parallel \mathbf{v} \) if and only if \( \mathbf{u} \times \mathbf{v} = 0 \)

(f) \( \mathbf{u} \times 0 = 0 \times \mathbf{u} = 0 \)

Example:

1) Given that \( \mathbf{u} = \langle 3,0,4 \rangle \) and \( \mathbf{v} = \langle 1,5,-2 \rangle \),

   find

   (a) \( \mathbf{u} \times \mathbf{v} \)

   (b) \( \mathbf{v} \times \mathbf{u} \)

2) Find two unit vectors that are perpendicular to the vectors \( \mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \)

   and \( \mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} \).
Answer:

1) (a) $-20i + 10j + 15k$  (b) $20i - 10j - 15k$

2) $\pm \frac{1}{\sqrt{162}} \langle 11, -5, 4 \rangle$ (The unit vector in the opposite direction is also a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$)

Further geometry interpretation of the cross product comes from computing its magnitude.

$$|\vec{u} \times \vec{v}|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

$$|\vec{u} \times \vec{v}|^2 = \left( u_1^2 + u_2^2 + u_3^2 \right) \left( v_1^2 + v_2^2 + v_3^2 \right)$$

$$- (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$

$$= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta$$

$$= |\vec{u}|^2 |\vec{v}|^2 \left( 1 - \cos^2 \theta \right)$$

$$= |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta$$
with $\theta$ is the angle between $\vec{u}$ and $\vec{v}$.

Therefore, $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin \theta$.

From the figure above, we can see that the magnitude of the cross product is the area of the parallelogram of which arrows representing the two vectors are adjacent sides.

\[
\begin{align*}
\text{Area of a parallelogram} &= |\vec{u}||\vec{v}|\sin \theta = |\vec{u} \times \vec{v}|
\end{align*}
\]

\[
\begin{align*}
\text{Area of triangle} &= \frac{1}{2}|\vec{u} \times \vec{v}|
\end{align*}
\]
Example:

(a) Find an area of a parallelogram that is formed from vectors \( u = i + j - 3k \) and \( v = -6j + 5k \).

(b) Find an area of a triangle that is formed from vectors \( u = i + j - 3k \) and \( v = -6j + 5k \).

Answer:

(a) \( \sqrt{230} \)  (b) \( \frac{\sqrt{230}}{2} \)

Scalar Triple Product

- Theorem -

If \( \mathbf{a} = \langle x_1, y_1, z_1 \rangle \), \( \mathbf{b} = \langle x_2, y_2, z_2 \rangle \) and \( \mathbf{c} = \langle x_3, y_3, z_3 \rangle \),

then
Properties of The Scalar Triple Product

1) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \)

2) \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} \)

3) \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)

4) \( \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0 \)

5) \( (\mathbf{a} + \mathbf{d}) \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c}) \)

Example:

If \( \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k} \), \( \mathbf{b} = -6\mathbf{j} + 5\mathbf{k} \) and \( \mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k} \), evaluate

(a) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)

(b) \( (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \)

(c) \( (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \)

(d) \( \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) \)
6.1 Lines in Space

In this section we use vectors to study lines in three-dimensional space.

**How Lines Can Be Defined Using Vectors?**

The most convenient way to describe a line in space is to give a point on it and a nonzero vector parallel to it.

Suppose $L$ is a straight line that passes through $P(x_0, y_0, z_0)$ and is parallel to the vector $v = ai + bj + ck$. 
Thus, a point $Q(x, y, z)$ also lies on the line if 
\[
\overrightarrow{PQ} = tv .
\]
Let,
\[
\overrightarrow{r_0} = \overrightarrow{OP} \quad \text{and} \quad \overrightarrow{r} = \overrightarrow{OQ} ,
\]
Then
\[
\therefore \overrightarrow{PQ} = \overrightarrow{r} - \overrightarrow{r_0} .
\]
\[
\overrightarrow{r} - \overrightarrow{r_0} = tv \\
\overrightarrow{r} = \overrightarrow{r_0} + tv
\]
\[
\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t < a, b, c >
\]
**-Theorem-

*(Parametric Equations for a Line)*

The line through the point $P(x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{A} = \langle a, b, c \rangle$ has the **parametric equations**, 

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct. \]

If we let \( \mathbf{R}_0 = \langle x_0, y_0, z_0 \rangle \) denote the position vector of \( P(x_0, y_0, z_0) \) and \( \mathbf{R} = \langle x, y, z \rangle \) the position vector of the arbitrary point \( Q(x, y, z) \) on the line, then we write equation (1) in the vector form,

\[ \mathbf{R} = \mathbf{R}_0 + t\mathbf{A}. \]

**Example:**

Give the parametric equations for the line through the point \((6,4,3)\) and parallel to the vector \( \langle 2,0,-7 \rangle \).

**-Theorem-**

(Symmetric Equations for a line)
The line through the point \( P(x_0, y_0, z_0) \) and parallel to the nonzero vector \( \mathbf{A} = \langle a, b, c \rangle \) has the symmetrical equations,

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\]

**Example:**

Given that the symmetrical equations of a line in space is

\[
\frac{2x + 1}{3} = \frac{3 - y}{4} = \frac{z + 4}{2}.
\]

Find,

(a) a point on the line.

(b) a vector that is parallel to the line.
6.1.1 Angle between Two Lines

Consider two straight lines

\[ l_1 : \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \]

and

\[ l_2 : \frac{x-x_2}{d} = \frac{y-y_2}{e} = \frac{z-z_2}{f} \]

The line \( l_1 \) parallel to the vector \( \mathbf{u} = ai + bj + ck \)
and the line \( l_2 \) parallel to the vector \( \mathbf{v} = di + ej + fk \). Since the lines \( l_1 \) and \( l_2 \) are parallel to the vectors \( \mathbf{u} \) and \( \mathbf{v} \) respectively, then the angle, \( \theta \) between the two lines is given by

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|| \mathbf{u} || \, || \mathbf{v} ||}
\]
Example:

Find an acute angle between line

\[ l_1 = i + 2j + t (2i - j + 2k) \]

and line

\[ l_2 = 2i - j + k + s (3i - 6j + 2k). \]
6.1.2 Intersection of Two lines

In three dimensional coordinates (space), two line can be in one of the three cases as shown below,
Let $l_1$ and $l_2$ are given by:

\[ l_1 : \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \text{and} \quad (1) \]

\[ l_2 : \frac{x-x_2}{d} = \frac{y-y_2}{e} = \frac{z-z_2}{f} \quad \text{(2)} \]

From (1), we have $\mathbf{v}_1 = <a, b, c>$

From (2), we have $\mathbf{v}_2 = <d, e, f>$

Two lines are parallel if we can write

\[ \mathbf{v}_1 = \lambda \mathbf{v}_2 \]

The parametric equations of $l_1$ and $l_2$ are:

\[ l_1 : \begin{align*}
x &= x_1 + at \\
y &= y_1 + bt \\
z &= z_1 + ct
\end{align*} \]

\[ l_2 : \begin{align*}
x &= x_2 + ds \\
y &= y_2 + es \\
z &= z_2 + fs
\end{align*} \quad (3) \]
Two lines are intersect if there exist unique values of $t$ and $s$ such that:

\[ x_1 + a t = x_2 + ds \]
\[ y_1 + b t = y_2 + es \]
\[ z_1 + c t = z_2 + fs \]

Substitute the value of $t$ and $s$ in (3) to get $x$, $y$ and $z$. The point of intersection $= (x, y, z)$

Two lines are skewed if they are neither parallel nor intersect.

**Example:**

Determine whether $l_1$ and $l_2$ are parallel, intersect or skewed.

a) $l_1 : x = 3 + 3t, \ y = 1 - 4t, \ z = -4 - 7t$

$l_2 : x = 2 + 3s, \ y = 5 - 4s, \ z = 3 - 7s$
b) \[ l_1 : \frac{x-1}{1} = \frac{2-y}{4} = z \]
\[ l_2 : \frac{x-4}{-1} = y-3 = \frac{z+2}{3} \]

**Solutions:**

a) for \( l_1 \):

point on the line, \( P = (3, 1, -4) \)

vector that parallel to line, \( \mathbf{v}_1 = \langle 3, -4, -7 \rangle \)

for \( l_2 \):

point on the line, \( Q = (2, 5, 3) \)

vector that parallel to line, \( \mathbf{v}_2 = \langle 3, -4, -7 \rangle \)

\[ \mathbf{v}_1 = \lambda \mathbf{v}_2 \]

where \( \lambda = 1 \)

Therefore, lines \( l_1 \) and \( l_2 \) are parallel.

b) Symmetrical equations of \( l_1 \) and \( l_2 \) can be rewrite as:
\[
\begin{align*}
\ell_1 : & \quad \frac{x-1}{1} = \frac{y-2}{-4} = \frac{z-0}{1} \\
\ell_2 : & \quad \frac{x-4}{-1} = \frac{y-3}{1} = \frac{z-(-2)}{3}
\end{align*}
\]

Therefore:

for \( \ell_1 \): \( P = (1, 2, 0) \) , \( \mathbf{v}_1 = <1, -4, 1> \)

for \( \ell_2 \): \( Q = (4, 3, -2) \) , \( \mathbf{v}_2 = <-1, 1, 3> \)

\( \mathbf{v}_1 = \lambda \mathbf{v}_2 \)  ?

\( \mathbf{v}_1 \neq \lambda \mathbf{v}_2 \) \( \rightarrow \) not parallel.

In parametric eq’s:

\( \ell_1 : x = 1 + t \) , \( y = 2 - 4t \) , \( z = t \)

\( \ell_2 : x = 4 - s \) , \( y = 3 + s \) , \( z = -2 + 3s \)

\[
\begin{align*}
1 + t &= 4 - s \quad \text{(1)} \\
2 - 4t &= 3 + s \quad \text{(2)} \\
t &= -2 + 3s \quad \text{(3)}
\end{align*}
\]
Solve the simultaneous equations (1), (2), and (3) to get $t$ and $s$.

$$s = \frac{5}{4} \quad \text{and} \quad t = \frac{7}{4}$$

The value of $t$ and $s$ must satisfy (1), (2) and (3). Clearly they are not satisfying (2) i.e

$$2 - \frac{7}{4} = 3 + \frac{5}{4} \quad ? \quad \Rightarrow \quad \frac{1}{4} \neq \frac{17}{4}$$

Therefore, lines $l_1$ and $l_2$ are not intersect.

This implies the lines are skewed!

**Example:**

Let $L_1$ and $L_2$ be the lines

$L_1 : x = 1 + 4t, y = 5 - 4t, z = -1 + 5t$

$L_2 : x = 2 + 8t, y = 4 - 3t, z = 5 + t$

(a) Are the lines parallel?

(b) Do the lines intersect?
Distance from a point Q to a line that passes through point P parallel to vector v is equal to the length of the component of PQ perpendicular to the line.

\[ d = |P\vec{Q}| \sin \theta \]

where \( \theta \) is the angle between v and vector \( P\vec{Q} \).
Since

$$|v \times \vec{PQ}| = |v| |\vec{PQ}| \sin \theta,$$

so we have the shortest distance of $Q$ from $L$ as

$$d = \frac{|v \times \vec{PQ}|}{|v|}.$$

**Example:**
Find the distance from the point $(0, 0, 0)$ to the line,

$$\frac{x - 5}{3} = \frac{y - 5}{4} = \frac{z + 3}{-5}.$$

**Example:**
Find the distance from the point $(2, 1, 3)$ to the line,

$$x = 2 + 2t, \ y = 1 + 6t, \ z = 3.$$
Example:

Find the shortest path from the point $Q(2, 0, -2)$ to the line

$$l : \frac{x-2}{3} = \frac{y+1}{1} = \frac{z-1}{2}$$

6.2 Planes in Space

Suppose that $\alpha$ is a plane. Point $P(x_0, y_0, z_0)$ and $Q(x, y, z)$ lie on it. If $N = ai + bj + ck$ is a non-null vector perpendicular (orthogonal) to $\alpha$, then $N$ is perpendicular to $PQ$.

![Diagram of a plane with vectors and points](image-url)
Thus,

\[ PQ \cdot N = 0 \]

\[ \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0 \]

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

**Conclusion**

The equation of a plane can be determined if a point on the plane and a vector orthogonal to the plane are known.

**Theorem**

(Equation of a Plane)

The plane through the point \( P(x_0, y_0, z_0) \) and with the nonzero normal vector \( \mathbf{N} = \langle a, b, c \rangle \) has the equation
Point-normal form:
\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

Standard form:
\[ ax + by + cz = d \quad \text{with} \quad d = ax_0 + by_0 + cz_0 \]

Example:

1. Give an equation for the plane through the point \((2, 3, 4)\) and perpendicular to the vector \((-6, 5, -4)\).

2. Give an equation for the plane through the point \((4, 5, 1)\) and parallel to the vectors \(A = (-2, 0, 1)\) and \(B = (0, 1, -4)\).
3. Give parametric equations for the line through the point (5, -3, 2) and perpendicular to the plane $6x + 2y - 7z = 5$.

### 6.2.1 Intersection of Two Planes

Intersection of two planes is a line. ($L$)

To obtain the equation of the intersecting line, we need

- a point on the line $L$ which is given by solving the equations of the two planes.
- a vector parallel to the line $L$ which is
\( N_1 \times N_2 \)

If \( N_1 \times N_2 = \langle a, b, c \rangle \), then the equation of the line \( L \) in symmetrical form is

\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}
\]

**Example:**

Find the equation of the line passing through \( P(2, 3, 1) \) and parallel to the line of intersection of the planes \( x + 2y - 3z = 4 \) and \( x - 2y + z = 0 \).

**Answer:**

\[
\frac{x-2}{-4} = \frac{y-3}{-4} = \frac{z-1}{-4}
\]
6.2.2 Angle between Two Planes

-Properties of Two Planes-

✓ An angle between the crossing planes is an angle between their normal vectors.

\[
\cos \theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|}
\]

✓ Two planes are parallel if and only if their normal vectors are parallel,

\[
\vec{N}_1 = \lambda \vec{N}_2.
\]

✓ Two planes are orthogonal if and only if

\[
\vec{N}_1 \cdot \vec{N}_2 = 0.
\]

Example:

Find the angle between plane \(3x + 4y = 0\) and plane \(2x + y - 2z = 5\).
6.2.3 *Angle between a Line and a Plane*

Let $\alpha$ be the angle between the normal vector $\mathbf{N}$ to a plane $\pi$ and the line $L$. Then we have

$$\cos \alpha = \frac{\mathbf{v} \cdot \mathbf{N}}{||\mathbf{v}|| \cdot ||\mathbf{N}||}$$

where $\mathbf{v}$ is vector parallel to $L$. Furthermore, if the angle between the line $L$ and the plane $\pi$, then

$$\alpha + \theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{2} - \alpha$$
\[
\sin \theta = \sin \left( \frac{\pi}{2} - \alpha \right) = \cos \alpha
\]

\[
\sin \theta = \frac{\mathbf{v} \cdot \mathbf{N}}{|\mathbf{v}||\mathbf{N}|}
\]

**Example:**

Find the angle between the plane 
\[3x - 2y + z = 5\] and the line 
\[
\frac{x - 3}{2} = \frac{y + 2}{-1} = \frac{z - 3}{3} \cdot
\]
6.2.4 **Shortest Distance from a Point to a Plane**

(a) From a Point to a Plane

-**Theorem**-

The distance $D$ between a point $Q(x_1, y_1, z_1)$ and the plane $ax + by + cz = d$ is

$$D = \left| \frac{N \cdot PQ}{|N|} \right| = \left| \frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Where $P(x_0, y_0, z_0)$ is any point on the plane.
**Example:**

Find the distance $D$ between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

**Example:**

1) Show that the line

$$
\frac{x - 1}{3} = \frac{y}{-2} = \frac{z + 1}{1}
$$

is parallel to the plane $3x - 2y + z = 1$.

2) Find the distance from the line to the plane in part (a).

**(b) Between two parallel planes**

The distance between two parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is given by

$$
D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}
$$
Example:

Find the distance between two parallel planes

\[ x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7. \]

-Note-

Both formulas can also be used to compute the distance between 2 skewed lines.

(c) Between two skewed lines
Assume \( L1 \) and \( L2 \) are skew lines in space containing the points \( P \) and \( Q \) and are parallel to vectors \( u \) and \( v \) respectively.

Then the shortest distance between \( L1 \) and \( L2 \) is the perpendicular distance between the two lines and its direction is given by a vector normal to both lines.

So, the distance between the two lines is absolute value of the scalar projection of \( PQ \) on the normal vector.

\[
d = |PQ \cos \theta|
= \left| \frac{N \cdot PQ}{|N|} \right| = \left| \frac{(u \times v) \cdot PQ}{|u \times v|} \right|
\]
Example:
Find the shortest distance from P(1, -1, 2) to the plane \( 3x - 7y + z = 5 \).

Example:
Find the shortest distance between the skewed lines.

\[
l_1 : x = 1 + 2t, \ y = -1 + t, \ z = 2 + 4t \\
l_2 : x = -2 + 4s, \ y = -3s, \ z = -1 + s
\]

Example:
Find the distance between the lines

\[
L_1 : i + 2j + 3k + t(i - k) \\
L_2 : x = 0, \ y = 1 + 2t, \ z = 3 + t
\]
Example:

Find the distance between the lines $L_1$ through the points $A(1, 0, -1)$ and $B(-1, 1, 0)$ and the line $L_2$ through the points $C(3, 1, -1)$ and $D(4, 5, -2)$.