

Chapter 9: Complex Numbers

- 9.1 Imaginary Number
- 9.2 Complex Number
 - definition
 - argand diagram
 - equality of complex number
- 9.3 Algebraic operations on complex number
 - addition and subtraction
 - multiplication
 - complex conjugate
 - division
- 9.4 Polar form of complex number
 - modulus and argument
 - multiplication and division of complex number in polar form
- 9.5 De Moivre's Theorem
 - definition
 - n^{th} power of complex number
 - n^{th} roots of complex number
 - proving some trigonometric identities
- 9.6 Euler's formula
 - definition
 - n^{th} power of complex number
 - n^{th} roots of complex number
 - relationship between circular and hyperbolic functions

9.1 Imaginary numbers

Consider:

$$x^2 = -4$$

This equation has no real solution. To solve the equation, we will introduce an imaginary number.

Definition 9.1 (Imaginary Number)

The imaginary number i is defined as:

$$i^2 = -1$$

Therefore, using the definition, we will get,

$$x^2 = -4$$

$$x = \sqrt{-4}$$

$$= \sqrt{4(-1)}$$

$$= \sqrt{4i^2}$$

$$= \pm 2i$$

Example: Express the following as imaginary numbers

a) $\sqrt{-25}$

b) $\sqrt{-8}$

9.2 Complex Numbers

Definition 9.2 (Complex Numbers)

If z is a **complex number**, then it can be expressed in the form :

$$z = x + iy,$$

where $x, y \in R$ and $i = \sqrt{-1}$.

x : **real part**

y : **imaginary part**

Or frequently represented as :

$$Re(z) = x \text{ and } Im(z) = y$$

Example:

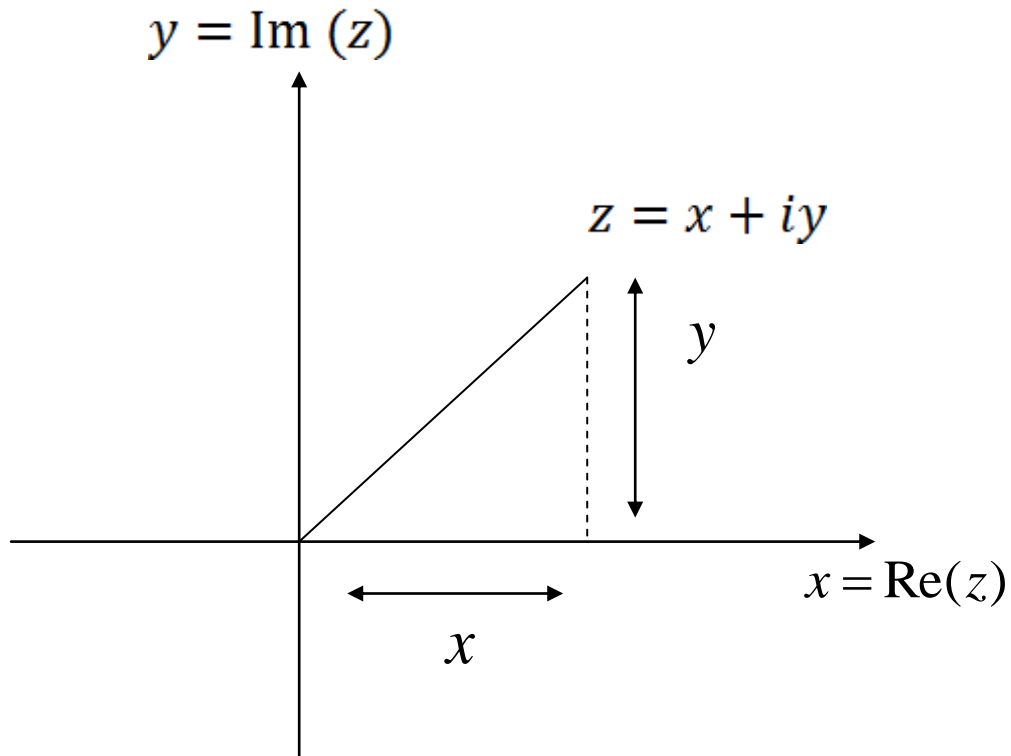
Find the real and imaginary parts of the following complex numbers

(a) $z_1 = 2 + 3i$

(b) $4i^2 + i - 2i^3$

9.2.1 Argand Diagram

We can graph complex numbers using an **Argand Diagram**.



Example:

Sketch the following complex numbers on the same axes.

(a) $z_1 = 3 + 2i$

(b) $z_2 = 3 - 2i$

(c) $z_3 = -3 - 2i$

(d) $z_4 = -3 + 2i$

9.2.2 Equality of Two Complex Numbers

Given that $z_1 = a + bi$ and $z_2 = c + di$

where $z_1, z_2 \in C$.

Two complex numbers are equal iff the real parts and the imaginary parts are respectively equal.

So, if $z_1 = z_2$, then $a = c$ and $b = d$.

Example 1:

Solve for x and y if given $3x + 4i = (2y + x) + xi$.

Example 2:

Solve $(3 + 4i)^2 - 2(x - iy) = x + iy$ for real numbers x and y .

Example 3:

Solve the following equation for x and y where

$$xy - 2i + x + 2xyi - 5 = \frac{3}{2} - 3i$$

9.3 Algebraic Operations on Complex Numbers

9.3.1 Addition and subtraction

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, then

$$z_1 \pm z_2 = (a + c) \pm (b + d)i$$

Example:

Given $Z_1 = -2 + 2i$, $Z_2 = 1 - \frac{\sqrt{3}}{2}i$ and $Z_3 = 4 - 6i$. Find

a) $Z_1 - Z_2$ b) $Z_1 + Z_3$

9.3.2 Multiplication

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, and k is a constant, then

$$\begin{aligned} \text{(i)} \quad z_1 \cdot z_2 &= (a + bi) \cdot (c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

$$\text{(ii)} \quad kz_1 = ka + kbi$$

Example:

Given $Z_1 = -2 + 2i$, and $Z_2 = 4 - 6i$. Find Z_1Z_2 .

9.3.3 *Complex Conjugate*

If $z = a + bi$ then the **conjugate** of z is denoted as $\bar{z} = a - bi$.

Note that $z \cdot \bar{z} = a^2 + b^2$

9.3.4 *Division*

If we are dividing with a complex number, the denominator must be converted to a real number. In order to do that, multiply both the denominator and numerator by complex conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

Example 1:

Given that $z_1 = 1 - 2i$, $z_2 = -3 + 4i$. Find $\frac{z_1}{z_2}$, and express it in $a + bi$ form.

Example 2:

Given $z_1 = 2 + i$ and $z_2 = 3 - 4i$, find $\frac{1}{z_1} + \frac{1}{z_2}$ in the form of $a + ib$.

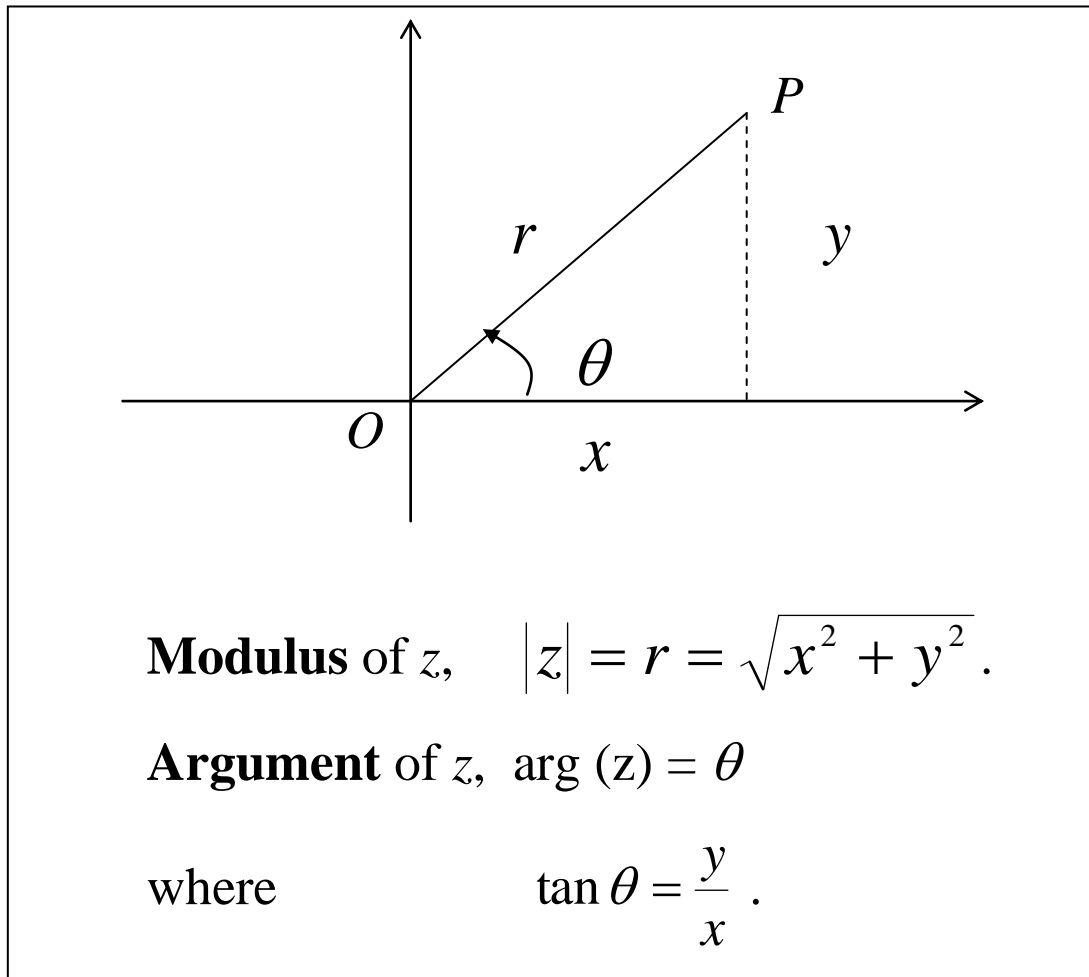
Example 3:

Given $Z = \frac{-2+3i}{3-2}$. Find the complex conjugate, \bar{Z} . Write your answer in $a + ib$ form.

Example 4:

Given $Z_1 = -2 + 2i$, and $Z_2 = 4 - 6i$. Find $\frac{2}{\bar{Z}_1 + \bar{Z}_2}$.

9.4 Polar Form of Complex Numbers



From the diagram above, we can see that

$$x = r \cos \theta \quad y = r \sin \theta$$

Then, z can be written as

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \\ &= r \operatorname{cis} \theta \quad (z \text{ in polar form}) \end{aligned}$$

Example 1:

Express $z = -2 - \sqrt{3}i$ in polar form.

Example 2:

Express $\frac{2+3i}{1-i}$ in polar form.

Example 3

Given that $z_1 = 2 + i$ and $z_2 = -2 + 4i$, find z such that

$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. Give your answer in the form of $a + ib$. Hence,

find the modulus and argument of z .

9.5 De Moivre's Theorem

9.5.1 The n -th Power Of A Complex Number

Definition 9.5 (De Moivre's Theorem)

If $z = r(\cos\theta + i\sin\theta)$ and $n \in R$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example 1:

- a) Write $z = 1 - i$ in the polar form.

Then, using De Moivre's theorem, find z^4 .

- b) Use D'Moivre's formula to write $(-1 - i)^{12}$ in the form of $a + ib$.

9.5.2 The n -th Roots of a Complex Number

A complex number w is a n -th root of the complex number z if

$w^n = z$ or $w = z^{\frac{1}{n}}$. Hence

$$\begin{aligned}w &= z^{\frac{1}{n}} \\&= [r(\cos \theta + i \sin \theta)]^{\frac{1}{n}}, \\&= r^{1/n} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right) \text{ for } k = 0, 1, 2, \dots, n-1\end{aligned}$$

Substituting $k = 0, 1, 2, \dots, n-1$ yields the n th roots of the given complex number.

Example 1:

Find all the roots for the following equations:

(a) $z^3 = 27$ (b) $z^4 = (\sqrt{3} + i)$.

Example 2:

Solve $z^4 + (-1 + i) = 0$ and express them in $a + ib$ form.

Example 3:

Find all cube roots of $-26 - 8i$.

Example 4:

Solve $z^3 + 8 = 0$. Sketch the roots on the argand diagram.

9.5.3 De Moivre's Theorem to Prove Trigonometric Identities

De-Moivre's theorem can be used to prove some trigonometric identities. (with the help of Binomial theorem or Pascal triangle.)

Example:

Prove that

$$\cos 5\theta = 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta \text{ and}$$

$$\sin 5\theta = 16\sin^5 \theta - 20\sin^3 \theta + 5\sin \theta.$$

Solution:

The idea is to write $(\cos \theta + i \sin \theta)^5$ in two different ways.

We use both the Pascal triangle and De Moivre's theorem, and compare the results.

From Pascal triangle,

$$\begin{aligned}(\cos \theta + i \sin \theta)^5 &= \cos^5 \theta + i 5 \cos^4 \theta \sin \theta - 10 \cos \theta \sin^2 \theta \\ &\quad - i 10 \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta. \\ &= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + \\ &\quad i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta).\end{aligned}$$

Also, by De Moivre's Theorem, we have

$$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta.$$

and so

$$\begin{aligned} \cos 5\theta + i \sin 5\theta &= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + \\ &\quad i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta). \end{aligned}$$

Equating the real parts gives

$$\begin{aligned} \cos 5\theta &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta. \\ &= \cos^5 \theta - 10 \cos^3 \theta(1 - \cos^2 \theta) + 5 \cos \theta(1 - \cos^2 \theta)^2 \\ &= \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta - 10 \cos^3 \theta + \\ &\quad 5 \cos^5 \theta. \\ &= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta. \text{ (proved)} \end{aligned}$$

Equating the imaginary parts gives

$$\begin{aligned} \sin 5\theta &= 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta \\ &= 5(1 - \sin^2 \theta)^2 \sin \theta - 10(1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta \\ &= 5(1 - 2 \sin^2 \theta + \sin^4 \theta) \sin \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \\ &\quad \sin^5 \theta \\ &= 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \\ &\quad \sin^5 \theta \\ &= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \text{ (proved)}. \end{aligned}$$

9.6 Euler's Formula

Definition 9.6

Euler's formula states that

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It follows that

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

From the definition, if z is a complex number with modulus r and $\text{Arg}(z)$, θ ; then

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= re^{i\theta} \end{aligned} \quad (z \text{ in euler form})$$

Example:

Express the following complex numbers in the form of $re^{i\theta}$

(a) $3 + i$

(b) $2 - 4i$

9.6.1 The n -th Power Of A Complex Number

We know that a complex number can be express as $z = re^{i\theta}$, then

$$z^2 = r^2 e^{i2\theta}$$

$$z^3 = r^3 e^{i3\theta}$$

$$z^4 = r^4 e^{i4\theta}$$

\vdots

$$z^n = r^n e^{in\theta}$$

Example 1:

Given $z = 2 + 2\sqrt{3}i$. Find the modulus and argument of z^5 .

Example 2:

Find $(\sqrt{3} - i)^{40}$ in the form of $a + ib$.

Example 3:

Express the complex number $z = -1 + \sqrt{3}i$ in the form of $re^{i\theta}$.

Then find

(a) z^2

(b) z^3

(c) z^7

9.6.2 The n -th Roots Of A Complex Number

The n -th roots of a complex number can be found using the Euler's formula. Note that:

$$z = re^{i(\theta+2k\pi)}$$

Then,

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\left(\frac{\theta+2k\pi}{2}\right)i}, \quad k = 0,1$$

$$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{\left(\frac{\theta+2k\pi}{3}\right)i}, \quad k = 0,1,2$$

⋮

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\left(\frac{\theta+2k\pi}{n}\right)i}, \quad k = 0,1,2,\dots,n-1$$

Example 1:

Find the cube roots of $z = 1 + i$.

Example 2:

Given $z = -1 + i$. Find all roots of $z^{\frac{1}{3}}$ in Euler form.

Example 3:

Solve $z^3 + 8i = 0$ and sketch the roots on an Argand diagram.

9.6.3 Relationship between circular and hyperbolic functions.

Euler's formula provides the theoretical link between circular and hyperbolic functions. Since

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

we deduce that

$$\boxed{\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}} \quad \text{and} \quad \boxed{\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}. \quad (1)$$

In Chapter 8, we defined the hyperbolic function by

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}} \quad \text{and} \quad \boxed{\sinh x = \frac{e^x - e^{-x}}{2}} \quad (2)$$

Comparing (1) and (2), we have

$$\begin{aligned} \cosh ix &= \frac{e^{ix} + e^{-ix}}{2} = \cos x \\ \sinh ix &= \frac{e^{ix} - e^{-ix}}{2i} = i \sin x \end{aligned}$$

so that

$$\tanh ix = i \tan x.$$

Also,

$$\begin{aligned}\cos ix &= \frac{e^{i^2x} + e^{-i^2x}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x \\ \sin ix &= \frac{e^{i^2x} - e^{-i^2x}}{2i} = \frac{e^{-x} - e^x}{2} = \sinh x\end{aligned}\tag{3}$$

so that

$$\tan ix = i \tanh x.$$

Using these results, we can evaluate functions such as $\sin z$, $\cos z$, $\tan z$, $\sinh z$, $\cosh z$ and $\tanh z$.

For example, to evaluate

$$\cos z = \cos(x + iy)$$

we use the identity

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

and obtain

$$\cos z = \cos x \cos iy - \sin x \sin iy$$

Using results in (3), this gives

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

Example: Find the values of

a) $\sinh(3 + 4i)$

b) $\tan\left(\frac{\pi}{4} - 3i\right)$

c) $\sin\left(\frac{\pi}{4}(1 + i)\right)$

(Ans: a) $-6.548 - 7.619i$

b) $0.005 - 1.0i$

c) $0.9366 + 0.6142i$)

Pascal's Triangle

				1						
				1	1					
			1	2	1					
		1	3	3	1					
	1	4	6	4	1					
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	200	252	200	120	45	10	1

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

$$(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$$

In general:

$$(x + y)^n = c_1x^n y^0 + c_2x^{n-1}y^1 + \dots + c_{n+1}x^0 y^n$$