Part A (55%)

QUESTION 1 (6 MARKS)

Express $5\cosh x + 13\sinh x$ in the form of $R\sinh(x+\alpha)$. Find R and show that $\alpha = \ln\left(\frac{3}{2}\right)$.

QUESTION 2 (6 MARKS)

Given $y = 2x\cos^{-1}(\cos x) + \tan^{-1}(\sinh x)$, find $\frac{dy}{dx}$.

QUESTION 3 (6 MARKS)

Evaluate

$$\int_3^5 \frac{1}{\sqrt{x^2-6x+13}}\,dx.$$

QUESTION 4 (6 MARKS)

Determine whether the improper integral converges or diverges

$$\int_1^4 \frac{1}{(x-2)^{\frac{2}{3}}}\,dx.$$

QUESTION 5 (7 MARKS)

Find the Maclaurin's expansion for $\ln(x+1)$ up to x^4. Use the series to:

(i) approximate the following integral

$$\int_0^{0.5} \ln(x+1)\,dx,$$

(ii) compute the following limit

$$\lim_{x \to 0} \frac{\ln(x+1)}{x}.$$

QUESTION 6 (6 MARKS)

Find the equation of the plane π that contains the line $L_1 : x+2 = 3-y = \frac{z}{3}$ and the point (3, 1, -5).

QUESTION 7 (6 MARKS)

Given the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$. Find all the eigenvalues for A. Then, find an eigenvector corresponding to the smallest eigenvalue of the matrix.

QUESTION 8 (6 MARKS)

Consider the polar equation $r = 2 + 4\cos\theta$. Show that the graph of the equation is symmetrical about the x-axis. Hence sketch the graph. **(Use the polar grid provided)**

QUESTION 9 (6 MARKS)

Simplify the following complex number and express your answer in the form $a + ib$, where a and b are real numbers.

$$\left(\frac{7+i}{3+4i}\right)^{43}.$$

Part B (45%)

QUESTION 10 (15 MARKS)

Given two planes, π_1 and π_2.

$$\pi_1 : x - 5y + 2z = 8, \quad \pi_2 : 3x + 5y + z = 10$$

(i) Find two unit vectors parallel to the line of intersection of the two planes. **(5 marks)**

(ii) Find an equation for the plane that passes through the point (1, 5, -2) and is perpendicular to the planes π_1 and π_2. **(4 marks)**

(iii) Find the acute angle between the planes π_1 and π_2. **(3 marks)**

(iv) Obtain the distance between the point (4, -3, 8) and the plane π_2. **(3 marks)**

QUESTION 11 (15 MARKS)

Consider the following system of linear equations:

$$x + y + z = 2$$
$$x + 2y + 3z = 5$$
$$2x + 3y + (m-3)z = m$$

(i) Write the linear system as a matrix equation. **(2 marks)**

(ii) Obtain the reduced echelon matrix of the system. **(4 marks)**

(iii) Find the values of m such that (a) the system has a unique solution, (b) the system has infinitely many solutions, (c) the system is inconsistent. **(4 marks)**

(iv) Taking $m = 1$, solve the system of equations using Cramer's Rule. **(5 marks)**

QUESTION 12 (15 MARKS)

Given the Cartesian equation of a curve C_1 is $y^2 = (x^2 + y^2)^{1/2} - x^2 + y$.

(i) Convert the Cartesian equation of C_1 to polar form. **(3 marks)**

(ii) Let C_2 be the polar equation $r = \cos 2\theta$. Determine the symmetries of the curves C_1 and C_2. Then sketch the graphs on the same diagram. **(Use the polar grid provided)**
(7 marks)

(iii) Find the points of intersection of the two curves C_1 and C_2.
(5 marks)

QUESTION 13 (15 MARKS)

(a) Solve $z^4 = -8iz$. Sketch all the roots on a single Argand diagram.
(9 marks)

(b) Use de Moivre's theorem to show that

$$\cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta.$$

Hence, or ortherwise, find the smallest positive solution of

$$4 \cos^3 \theta - 3 \cos \theta = 1.$$

(6 marks)

Trigonometric
$\cos^2 x + \sin^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$\cot^2 x + 1 = \text{cosec}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$
$2 \sin x \cos y = \sin(x+y) + \sin(x-y)$
$2 \sin x \sin y = -\cos(x+y) + \cos(x-y)$
$2 \cos x \cos y = \cos(x+y) + \cos(x-y)$

Logarithm
$a^x = e^{x \ln a}$
$\log_a x = \dfrac{\log_b x}{\log_b a}$

Hyperbolic
$\sinh x = \dfrac{e^x - e^{-x}}{2}$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$
$\cosh^2 x - \sinh^2 x = 1$
$1 - \tanh^2 x = \text{sech}^2 x$
$\coth^2 x - 1 = \text{cosech}^2 x$
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sinh 2x = 2 \sinh x \cosh x$
$\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$
$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Inverse Hyperbolic
$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}),\ -\infty < x < \infty$
$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}),\ x \geq 1$
$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right),\ -1 < x < 1$

Differentiations	Integrations	Differentiations	Integrations
$\frac{d}{dx}[k] = 0$, k constant.	$\int k\,dx = kx + C$, k constant.	$\frac{d}{dx}[\sec x] = \sec x \tan x$.	$\int \sec x \tan x\,dx = \sec x + C$.
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.	$\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x$.	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$.
$\frac{d}{dx}[e^x] = e^x$.	$\int e^x dx = e^x + C$.	$\frac{d}{dx}[\cosh x] = \sinh x$.	$\int \sinh x\,dx = \cosh x + C$.
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{x}$.	$\int \frac{dx}{x} = \ln\|x\| + C$.	$\frac{d}{dx}[\sinh x] = \cosh x$.	$\int \cosh x\,dx = \sinh x + C$.
$\frac{d}{dx}[\cos x] = -\sin x$.	$\int \sin x\,dx = -\cos x + C$.	$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$.	$\int \operatorname{sech}^2 x\,dx = \tanh x + C$.
$\frac{d}{dx}[\sin x] = \cos x$.	$\int \cos x\,dx = \sin x + C$.	$\frac{d}{dx}[\coth x] = -\operatorname{cosech}^2 x$.	$\int \operatorname{cosech}^2 x\,dx = -\coth x + C$.
$\frac{d}{dx}[\tan x] = \sec^2 x$.	$\int \sec^2 x\,dx = \tan x + C$.	$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$.	$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$.
$\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$.	$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$.	$\frac{d}{dx}[\operatorname{cosech} x] = -\operatorname{cosech} x \coth x$.	$\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x + C$.

Differentiations of Inverse Functions
$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}},\ \lvert x\rvert < 1.$
$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}},\ \lvert x\rvert < 1.$
$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}.$
$\frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}.$
$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1.$
$\frac{d}{dx}[\mathrm{cosec}^{-1} x] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1.$
$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2+1}}.$
$\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}},\ \lvert x\rvert > 1.$
$\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2},\ \lvert x\rvert < 1.$
$\frac{d}{dx}[\coth^{-1} x] = \frac{1}{1-x^2},\ \lvert x\rvert > 1.$
$\frac{d}{dx}[\mathrm{sech}^{-1} x] = \frac{-1}{x\sqrt{1-x^2}},\ 0 < x < 1.$
$\frac{d}{dx}[\mathrm{cosech}^{-1} x] = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}},\ x \neq 0.$

Integrations Resulting in Inverse Functions
$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C.$
$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C.$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \sec^{-1}(x) + C.$
$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}(x) + C.$
$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + C,\ x > 0.$
$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C,\ \lvert x\rvert < 1.$
$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C,\ \lvert x\rvert > 1.$
$\int \frac{dx}{x\sqrt{1-x^2}} = -\mathrm{sech}^{-1}(x) + C,\ \lvert x\rvert < 1.$
$\int \frac{dx}{\lvert x\rvert\sqrt{1+x^2}} = -\mathrm{cosech}^{-1} x + C,\ x \neq 0.$