Chapter 3 Multiple Integral

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- Triple Integrals in Cartesian Coordinates
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3.4 Triple Integrals

Definition

If f is a function defined over a closed, bounded solid region G, then the triple integral of f over G is defined as

$$\iiint_G f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

3.4.1 Iterated Integration

Just as for double integrals, the practical method for evaluating triple integrals is to expressed them as iterated integrals as in the following theorem:

Theorem

If $f(x, y, z)$ is continuous over a rectangle solid G: $a \le x \le b,\ c \le y \le d,\ k \le z \le l$, then the triple integral may be evaluated by the iterated integral

$$\iiint_G f(x, y, z)\, dV = \int_k^l \int_c^d \int_a^b f(x, y, z)\, dxdydz$$

The iterated integration can be performed in any order (with appropriate adjustments) to the limits of integration:

$$dx\,dy\,dz \qquad dx\,dz\,dy$$

$$dy\,dx\,dz \qquad dy\,dz\,dx$$

$$dz\,dy\,dx \qquad dz\,dx\,dy$$

Example

Evaluate $\iiint_G z^2 y e^x \, dV$, over the rectangular box G defined by

$$0 \leq x \leq 1,\ 1 \leq y \leq 2,\ -1 \leq z \leq 1$$

Solution

We shall evaluate the integral in the order $dx\,dy\,dz$.

$$\iiint_G z^2 y e^x \, dV = \int_{-1}^{1} \int_{1}^{2} \int_{0}^{1} z^2 y e^x \, dx dy dz$$

$$= \int_{-1}^{1} \int_{1}^{2} z^2 y \left[e^x \right]_0^1 dy dz = (e-1) \int_{-1}^{1} \int_{1}^{2} z^2 y \, dy dz$$

$$= (e-1) \int_{-1}^{1} z^2 [y^2/2] \Big|_1^2 \, dz$$

$$= \frac{3}{2}(e-1) \int_{-1}^{1} z^2 \, dz = e-1$$

3.4.2. Integral Over General Regions

We restrict our attention to continuous functions f and to certain simple types of regions.

3 types of region:

Type I – integrating over simple *xy*-solid

Type II – integrating over simple *xz*-solid

Type III – integrating over simple *yz*-solid

Definition

A solid region G is said to be of **Type 1** if it lies between the graphs of two continuous functions of x and y,

$$G = \{(x, y, z) : x, y \in R, k_1(x, y) \le z \le k_2(x, y)\}$$

where R is the projection of G onto the *xy*-plane, then

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)\,dz \right] dA$$

Type I Regions

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)\,dz\right] dA$$

Type II Regions

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy\right] dA$$

Type III Regions

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dx\right] dA$$

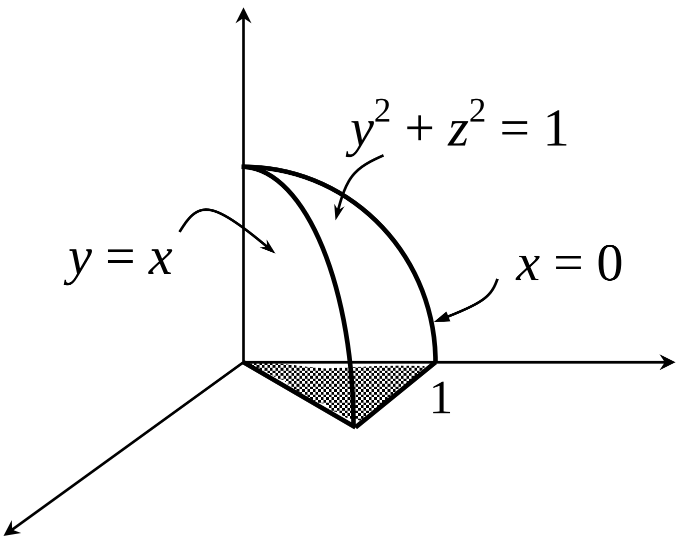
Example

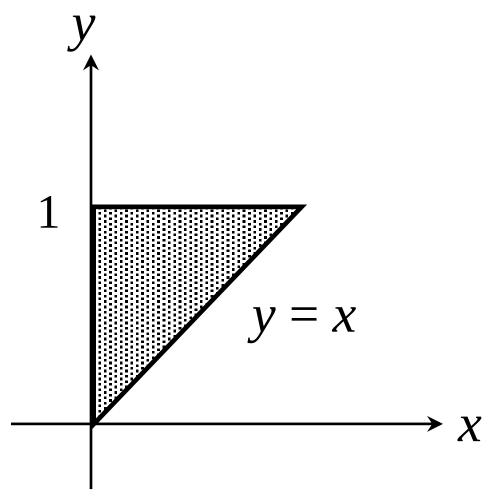
Let G be the wedge in the first octant cut from the cylindrical solid $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\,dV$$

Solution

♦ **Sketch the solid:** choose Type I





upper bounding surface: $y^2 + z^2 = 1$

lower bounding surface: *xy*-plane

♦ The *z*-limits of integration: Draw a line *L* parallel to *z*-axis passing through solid region.

As *z* increases, *L* enters *G* at $z = 0$ and leaves at $z = \sqrt{1 - y^2}$

$$\iiint_G z\, dV = \iint_R \int_0^{\sqrt{1-y^2}} [z\, dz]\, dA$$

- The x-limits of integration: Draw a line M parallel to x-axis passing through plane region R.

  As x increases, M enters R at $x = 0$ and leaves at $x = y$.

- The y-limits of integration: Choose y-limits that include all lines parallel to the x-axis.

The integral is

$$\iiint_G z\,dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy$$

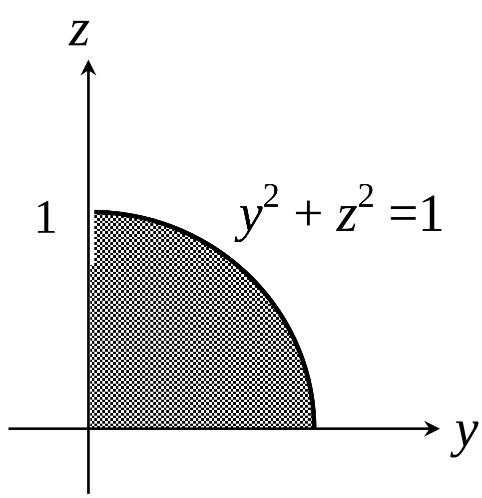
$$= \int_0^1 \int_0^y \frac{z^2}{2}\Bigg|_0^{\sqrt{1-y^2}} dx\,dy = \int_0^1 \int_0^y \frac{1}{2}(1-y^2)\,dx\,dy$$

$$= \int_0^1 (1-y^2)x\Big|_0^y dy = \frac{1}{2}\int_0^1 (y-y^3)\,dy = \frac{1}{8}$$

Alternatively, we evaluate the integral by integrating first with respect to x (Type III).

The solid is bounded in the back by the plane $x = 0$ and in the front by the plane $y = x$.

$$\iiint_G z\,dV = \iint_R \left[\int_0^y z\,dx\right] dA$$



$$\iiint_G z\,dV = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^y z\,dx\,dy\,dz$$

Question 1

In questions 1(a) - 1(b), evaluate the triple integral.

(a) $\int_{-1}^{1}\int_{0}^{2}\int_{0}^{x} x^2 \, dy \, dx \, dz$

(b) $\int_{1}^{2}\int_{0}^{z}\int_{0}^{y} e^{x} \, dx \, dy \, dz$

Question 2

Sketch the solid bounded by the graph of the given equation and express $\iiint f \;\; x, y, z \;\; dV$ as iterated integrals in six different ways.

$$x + 2y + 3z = 6, x = 0, y = 0, z = 0.$$

Question 3

In questions 3(a) - 3(b), evaluate the triple integral.

(a) $x = 0, y = 0, z = 0, 3x + 6y + z = 6.$

(b) $z = y^2, z = 0, x = 0, x = 1, y = -1, y = 1.$

Question 4

In questions 4(a) and 4(b), sketch the solid whose volume is given by the iterated integral.

(a) $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{y+6} dz\,dy\,dx$

(b) $\displaystyle\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$

3.4.3 Cylindrical Coordinates

- Generalization of polar coordinates in $\mathbb{R}^3$

- We convert a triple integral from rectangular to cylindrical coordinates by writing

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

The element of integration,

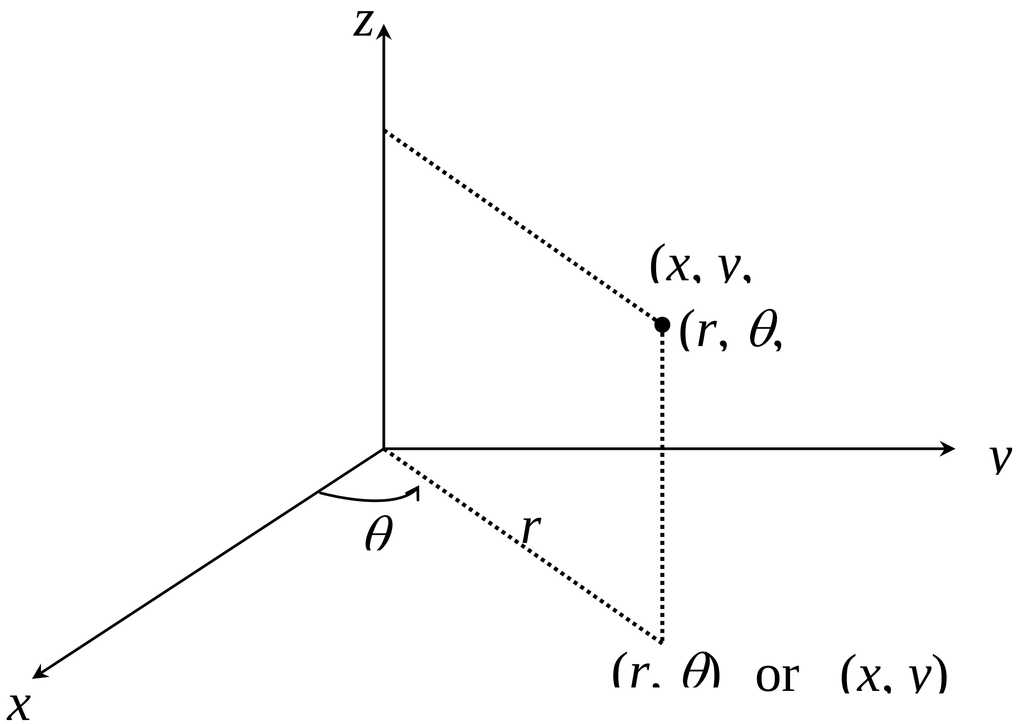
$$dV = r\,dr\,d\theta\,dz$$

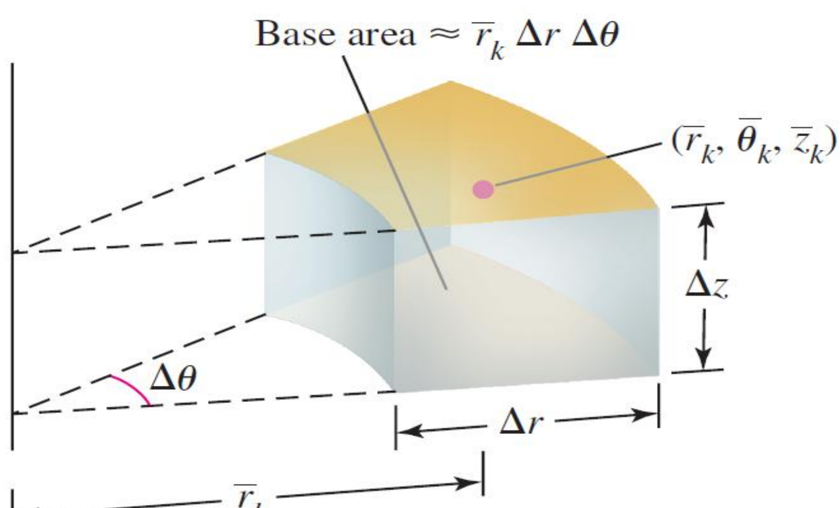
The function $f(x, y, z)$ is transform to

$$f(x, y, z) = f(r\cos\theta,\ r\sin\theta,\ z)$$

- Cylindrical coordinates are convenient for representing cylindrical surfaces and surfaces for which the *z*-axis is the axis of symmetry.

The cylindrical coordinate system





Approximate volume $\Delta V_k \approx \bar{r}_k \, \Delta r \, \Delta \theta \, \Delta z$

Theorem

Let G be a solid with upper surface $z = g_2(r, \theta)$ and lower surface $z = g_1(r, \theta)$ and let R be the projection of the solid on the xy-plane expressed in polar coordinates. Then if $f(r, \theta, z)$ is continuous on R, we have

$$\iiint_G f(r, \theta, z)\, dV = \iint_R \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, r\, dz\, dr\, d\theta$$

Example

Use cylindrical coordinates to evaluate

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2\, dz\, dy\, dx$$

Solution

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx = \iiint_G x^2\,dV$$

$$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\cos^2\theta \;\; r dz dr d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^3\cos^2\theta z\Big|_0^{9-r^2}\; dr d\theta$$

$$\vdots$$

$$= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{243}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{243}{4}\pi$$

Question 1

In questions 1(a) - 1(c), use cylindrical coordinates to find the volume of the solid bounded by the given surfaces.

(a) $z = x^2 + y^2, z = 9.$

(b) $z = x^2 + y^2, x^2 + (y-1)^2 = 1, z = 0.$

(c) $z = x^2 + y^2, x^2 + y^2 = 4, z = 0.$

Question 2

In questions 2(a) - 2(b), evaluate the integrals by changing the coordinates to cylindrical coordinates.

(a) $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{4-x^2-y^2}} z \, dz \, dx \, dy.$

(b) $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) \, dz \, dx \, dy$

3.4.4 Spherical Coordinates

Definition

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin

2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis ($0 \le \phi \le \pi$)

3. θ is the angle from cylindrical coordinates.

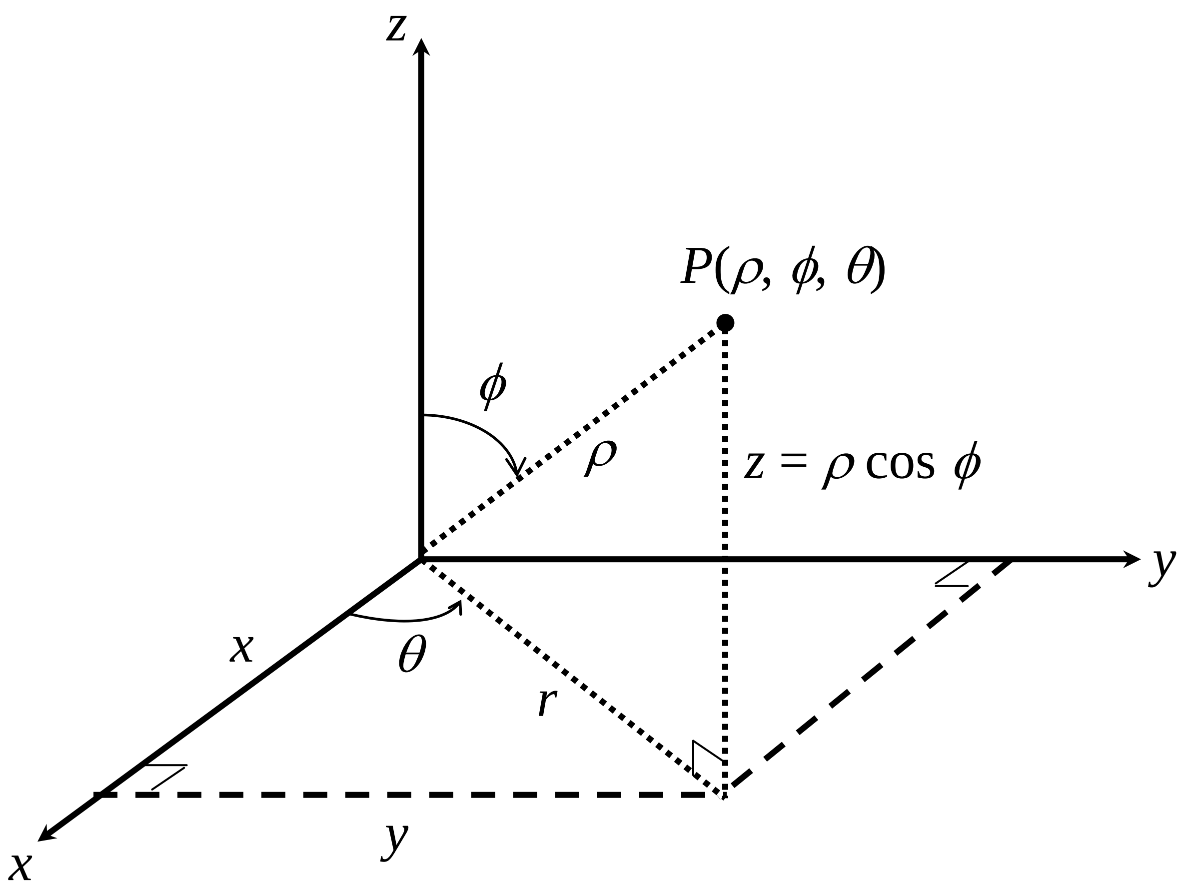
The spherical coordinate system

Since $r = \rho \sin \phi$,

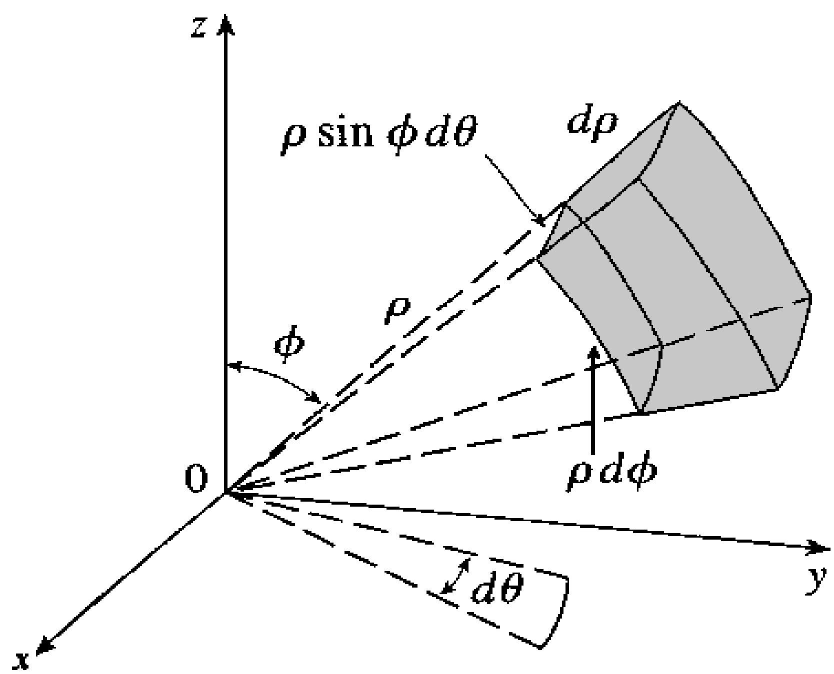
$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

and $z = \rho \cos \phi$, $x^2 + y^2 + z^2 = \rho^2$



z
P(ρ, ϕ, θ)
ϕ
ρ
z = ρ cos ϕ
y
x
θ
r
y
x



z
ρ sin ϕ dθ
dρ
ρ
ϕ
ρ dϕ
0
dθ
y
x

- The function $f(x, y, z)$ is transform to

$$f(x,y,z) = f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

- The element of integration,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

- Triple integrals in spherical coordinates are then evaluated as iterated integrals. The integral is

$$\iiint_G f(\rho, \phi, \theta)\, dV = \iiint_G f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Question 1

In questions 1(a) - 1(b), use spherical coordinates to evaluate the integrals.

(a) $\iiint_G \cos\sqrt{x^2+y^2+z^2}^{\,3}\, dV$ where G is the solid bounded by $z = \sqrt{1-x^2-y^2}$ and $z = 0$.

(b) $\iiint_G e^{\sqrt{x^2+y^2+z^2}^{\,3}}\, dV$ where G is the solid bounded by $z = \sqrt{1 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$.

Question 2

In questions 2(a) - 2(b), evaluate the integrals by changing the coordinates to spherical coordinates.

(a) $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} x^2 + y^2 + z^2 \; dz\, dy\, dx.$

(b) $\int_{0}^{\sqrt{2}} \int_{y}^{\sqrt{4-y^2}} \int_{0}^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dx\, dy.$

(c) $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx.$

3.5 Moments and Centre of Mass

3.5.1 Notation and Terminology

Lamina - a solid object that is sufficiently "flat" to be regarded as two-dimensional.

Density: mass per unit area, $\delta(x, y)$

Mass: quantity of matter in a body, m

Moment of mass: tendency of mass to produce a rotation about a point, line or plane

Positive moment – clockwise rotation

Negative moment – counterclockwise rotation

Center of Gravity/Center of Mass:

a point where a system behaves as if all its mass is concentrated there (balance point).

Centroid: center of mass of a homogeneous body

Moment of inertia: tendency to resist a change in the rotational motion about an axis.

Definition

If δ is a continuous density function on the lamina corresponding to a plane region *R*, then

- Mass, $m = \iint_R \delta(x, y)\, dA$
- Moments of mass about the *x*- and *y*-axes,

$$M_x = \iint_R y\,\delta(x, y)\, dA$$

$$M_y = \iint_R x\,\delta(x, y)\, dA$$

- Centre of mass, $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$
- If the density δ is constant, the point $(\bar{x}, \bar{y})$ is called the centroid of the region.

Example

A lamina of density $\delta(x, y) = x^2$ occupies a region R bounded by the parabola $y = 2 - x^2$ and the line $y = x$. Find

(a) mass

(b) centre of mass of the lamina.

Solution

♦ sketch the region R

(a) mass of lamina,

$$m = \iint_R \delta(x, y)\, dA = \int_{-2}^{1} \int_{x}^{2-x^2} x^2 \, dy\, dx$$

$$= \int_{-2}^{1} x^2 \, y\Big|_{x}^{2-x^2} \, dx$$

$$\therefore\ m = \int_{-2}^{1} (2x^2 - x^4 - x^3)\, dx = \frac{63}{20}$$

(b) centre of mass, $(\bar{x}, \bar{y})$

KNOW: $\bar{x} = \dfrac{M_y}{m}, \quad \bar{y} = \dfrac{M_x}{m}$

$$M_x = \iint_R y\, \delta(x, y)\, dA$$

$$= \int_{-2}^{1} \int_{x}^{2-x^2} y\, x^2\, dy\, dx = \int_{-2}^{1} x^2 \left.\frac{y^2}{2}\right|_{x}^{2-x^2} dx$$

$$\therefore\ M_x = \frac{1}{2} \int_{-2}^{1} (x^6 - 5x^4 + 4x^2)\ dx = -\frac{9}{7}$$

$$M_y = \iint_R x\, \delta(x, y)\, dA$$

$$= \int_{-2}^{1} \int_{x}^{2-x^2} x^3\ dy\ dx = \int_{-2}^{1} x^3 y\Big|_{x}^{2-x^2}\ dx$$

$$\therefore\ M_y = \int_{-2}^{1} (2x^3 - x^5 - x^4)\ dx = -\frac{18}{5}$$

From (a) we found $m = \dfrac{63}{20}$, so the centre of mass is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} = \frac{-18/5}{63/20} = -\frac{8}{7} \approx -1.14$$

$$\bar{y} = \frac{M_x}{m} = \frac{-9/7}{63/20} = -\frac{20}{49} \approx -0.41$$

In an analogous way, we can use the triple integral to find mass and the center of mass of a solid in $\mathbb{R}^3$. The density $\delta(x, y, z)$ at a point in the solid now refers to mass per unit volume.

- ♦ Mass $$m = \iiint_G \delta(x, y, z)\, dV$$

- ♦ Moments
$$M_{yz} = \iiint_G x\, \delta(x, y, z)\, dV$$

$$M_{xz} = \iiint\limits_G y\,\delta(x, y, z)\,dV$$

$$M_{xy} = \iiint\limits_G z\,\delta(x, y, z)\,dV$$

- Centre of mass

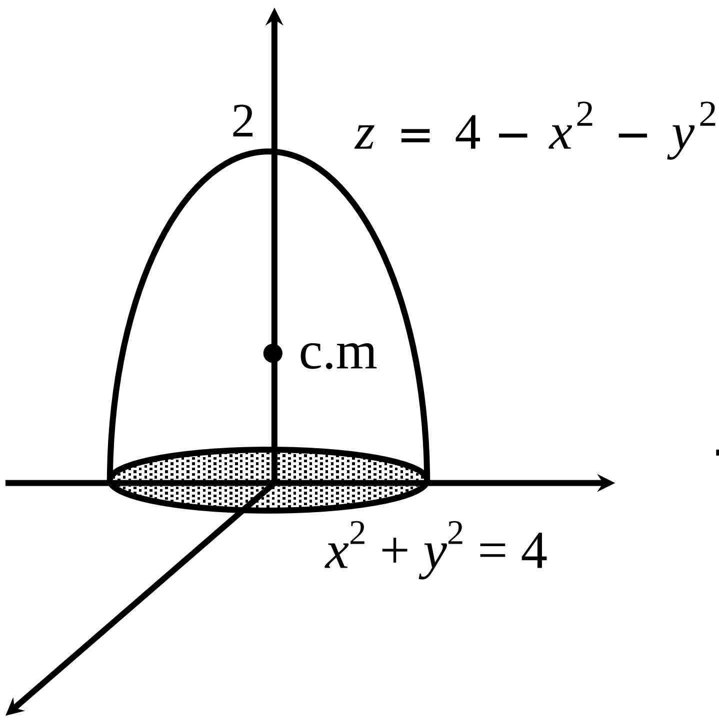
$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$$

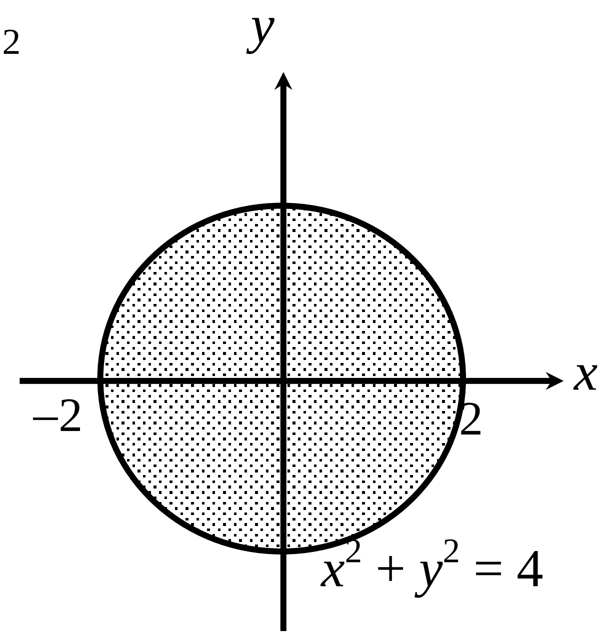
- If the density δ is constant, the point $(\bar{x}, \bar{y}, \bar{z})$ is called the centroid.

Example

Find the centroid of a solid of constant density δ bounded below by the disk $x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$.

Solution





By symmetry, $\bar{x} = \bar{y} = 0$. So we only need to find $\bar{z}$.

$$\bar{z} = \frac{M_{xy}}{m}$$

$$M_{xy} = \iiint_G z\,\delta(x, y, z)\,dV$$

$$= \iint_R \int_0^{4-x^2-y^2} z\delta\ dz\ dy\ dx$$

$$= \iint_R \delta\ \frac{z^2}{2}\Bigg|_0^{4-x^2-y^2} dy\ dx$$

$$= \frac{\delta}{2}\iint_R (4 - x^2 - y^2)^2\ dy\ dx$$

$$= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2 r\ dr\ d\theta$$

$$= \frac{\delta}{2}\int_0^{2\pi} - \frac{1}{6}(4 - r^2)^3\Bigg|_0^2\ dx$$

$$= \frac{16\delta}{3}\int_0^{2\pi} d\theta$$

$$\therefore\ M_{xy} = \frac{32\pi\delta}{3}$$

A similar calculation gives

$$m = \iiint_G \delta(x, y, z)\, dV$$

$$= \iint_R \int_0^{4-x^2-y^2} \delta\, dz\, dy\, dx = 8\pi\delta$$

Therefore $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{32\pi\delta/3}{8\pi\delta} = \dfrac{4}{3}$.

Thus the centroid is $(\bar{x}, \bar{y}, \bar{z})$ = (0, 0, 4/3).

Question

A solid is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 2$. If the density $\delta(x, y, z) = 2x$, find the centre of mass.

3.5.3 Moments of Inertia

- Also called the second moments

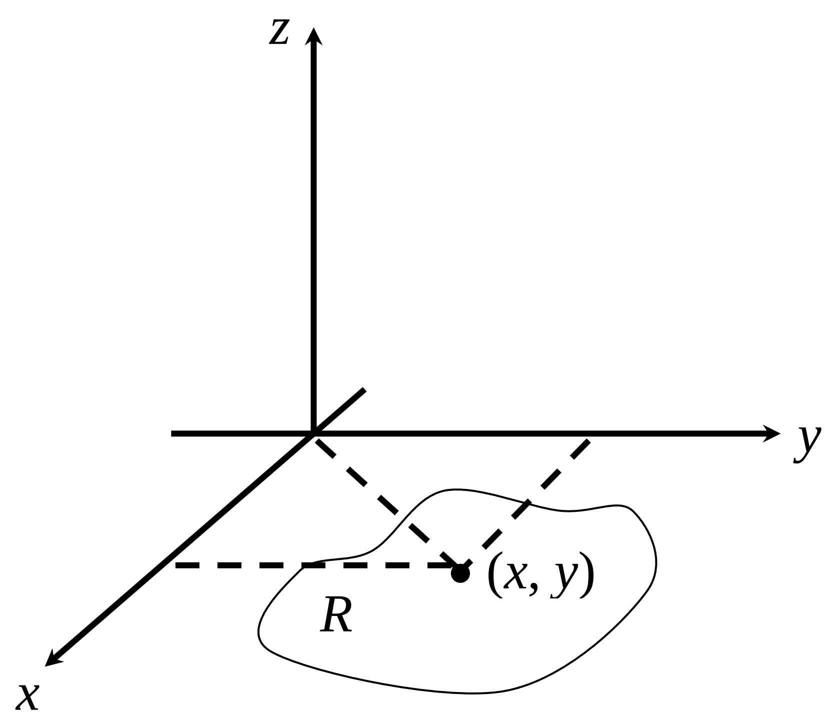
Definition

The moments of inertia of a lamina of density δ covering the planar region R about the x-, y-, and z-axis are given by

$$I_x = \iint_R y^2 \delta(x, y)\, dA$$

$$I_y = \iint_R x^2 \delta(x, y)\, dA$$

$$I_z = \iint_R (x^2 + y^2) \delta(x, y)\, dA$$



The concept of moments of inertia generalise easily to solid regions.

Suppose the solid occupies a region *R* and that the density at each point (x, y, z) in *R* is given by $\delta(x, y, z)$. The moments of inertia of the solid about the *x*-, *y*-, and *z*-axis are given by

$$I_x = \iiint_G (y^2 + z^2)\delta(x, y, z)\, dV$$

$$I_y = \iiint_G (x^2 + z^2)\delta(x, y, z)\, dV$$

$$I_z = \iiint_G (x^2 + y^2)\delta(x, y, z)\, dV$$

Question 1

A lamina of density $\delta(x, y) = x^2 y$ occupies the region *R* in the plane that is bounded by the parabola $y = x^2$ and the lines $x = 2$ and $y = 1$. Find the moments of inertia of the lamina about the *x*-axis and the *y*-axis.

Question 2

Find the moment of inertia of the "ice cream cone" *G* cut from the solid sphere $\rho \le 1$ by the cone $\phi = \frac{\pi}{3}$ about the *z*-axis. (Take $\delta = 1$)

Question 3

Find the moment of inertia of a solid hemisphere of radius 2 with respect to its axis of symmetry, if the density is proportional to the distance from the axis of symmetry.