UNIVERSITI TEKNOLOGI MALAYSIA
SSE 1893 ENGINEERING MATHEMATICS
TUTORIAL 4

1. Evaluate the following line integral

(i) $\int_{\mathcal{C}} x^2 y \, ds$, $\quad \mathcal{C}$: curve $x = 3\cos t,\ y = 3\sin t,\ 0 \le t \le \frac{1}{2}\pi$.

(ii) $\int_{\mathcal{C}} (\sin x + \cos x) \, ds$, $\quad \mathcal{C}$: line segment from $(0,0)$ to $(\pi, 2\pi)$.

(iii) $\int_{\mathcal{C}} (2x + 9z) \, ds$, $\quad \mathcal{C}$: curve $x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$.

(iv) $\int_{\mathcal{C}} (y - x) \, dx + x^2 \, dy$, $\quad \mathcal{C}$: curve $y = x^2$ from point $(0,0)$ to $(4,16)$.

(v) $\int_{\mathcal{C}} (x^2 + y^2) \, dx + 2xy \, dy$, $\quad \mathcal{C}$: curve $x = t^2,\ y = t^3,\ 0 \le t \le \frac{3}{2}$.

(vi) $\int_{\mathcal{C}} (x + y + z) \, dx + x \, dy - yz \, dz$,
$\mathcal{C}$: line segment from $(1,2,1)$ to $(2,1,0)$.

2. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for the given function $\mathbf{F}$ and curve $\mathcal{C}$.

(i) $\mathbf{F}(x,y) = x^3 y \, \mathbf{i} + (x - y) \, \mathbf{j}$, $\quad \mathcal{C}$: parabola $y = x^2$ from point $(-2,4)$ to $(1,1)$.

(ii) $\mathbf{F}(x,y) = (x^2 - y^2) \, \mathbf{i} + x \, \mathbf{j}$, $\quad \mathcal{C}$: curve $\mathbf{r}(t) = t^{2/3} \, \mathbf{i} + t \, \mathbf{j},\ -1 \le t \le 1$.

(iii) $\mathbf{F}(x,y) = (x^3 - y^3) \, \mathbf{i} + xy^2 \, \mathbf{j}$, $\quad \mathcal{C}$: curve $x = t^2,\ y = t^3,\ -1 \le t \le 0$.

(iv) $\mathbf{F}(x,y,z) = 2x \, \mathbf{i} + xz \, \mathbf{j} + yz \, \mathbf{k}$, $\quad \mathcal{C}$: line segment from $(2,2,1)$ to $(0,2,1)$ to $(0,0,3)$.

(v) $\mathbf{F}(x,y,z) = yz \, \mathbf{i} - xz \, \mathbf{j} + xy \, \mathbf{k}$, $\quad \mathcal{C}$: curve $x = e^t,\ y = e^{3t},\ z = e^{-t}$, $0 \le t \le 1$.

(vi) $\mathbf{F}(x,y,z) = z \, \mathbf{i} + x \, \mathbf{j} + y \, \mathbf{k}$, $\quad \mathcal{C}$: curve $x = \sin t,\ y = 3\sin t,\ z = \sin^2 t$, $0 \le t \le \frac{1}{2}\pi$.

3. Show that the line integral is independent of path and evaluate the line integral by choosing a convenient path or by using a potential function.

(i) $\int_{\mathcal{C}} y^2 \, dx + 2xy \, dy$, $\quad \mathcal{C}$: line segment from $(1,2)$ to $(0,0)$ to$(1,3)$.

(ii) $\int_{\mathcal{C}} (y^2 + 2xy) \, dx + (x^2 + 2xy) \, dy$, $\quad \mathcal{C}$: line segment from $(-1,2)$ ke $(0,3)$ to $(3,1)$.

(iii) $\int_{\mathcal{C}} (6xy^3 + 2z^2) \, dx + 9x^2 y^2 \, dy + (4xz + 1) \, dz$, $\quad \mathcal{C}$: line segment from $(0,0,0)$ ke $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

(iv) $\int_{\mathcal{C}} (y + z) \, dx + (x + z) \, dy + (x + y) \, dz$, $\quad \mathcal{C}$: line segment from $(0,0,0)$ ke $(0,1,0)$ to $(-1,0,\pi)$.

4. Show that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Hence find a potential function and evaluate the line integral along $\mathcal{C}$.

(i) $\mathbf{F}(x, y) = e^y\,\mathbf{i} + xe^y\,\mathbf{j}$, $\mathcal{C}$: hemisphere $y = \sqrt{1 - x^2}$ from $(1, 0)$ to $(-1, 0)$.

(ii) $\mathbf{F}(x, y) = (10x - 7y)\,\mathbf{i} - (7x - 2y)\,\mathbf{j}$, $\mathcal{C}$: curve $y = 1 - x^2$ from point $(0, 1)$ to $(1, 0)$.

(iii) $\mathbf{F}(x, y) = y \sin x\,\mathbf{i} - \cos x\,\mathbf{j}$, $\mathcal{C}$: curve $\mathbf{r}(t) = (t + \sin t)\,\mathbf{i} - t \cos t\,\mathbf{j}$, $0 \le t \le \pi$.

(iv) $\mathbf{F}(x, y, z) = 3x^2\,\mathbf{i} + 6y^2\,\mathbf{j} + 9z^2\,\mathbf{k}$, $\mathcal{C}$: line segment from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$.

(v) $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$, $\mathcal{C}$: line segment from $(0, 0, 0)$ to $(1, 1, 1)$ to $(0, 0, 1)$.

(vi) $\mathbf{F}(x, y, z) = 8xz\,\mathbf{i} + (1 - 6yz^3)\,\mathbf{j} + (4x^2 - 9y^2z^2)\,\mathbf{k}$, $\mathcal{C}$: line segment from $(1, 2, 0)$ to $(1, 2, 3)$ to $(0, 0, 2)$.

5. Evaluate the following integral using Green's theorem.

(i) $\oint_{\mathcal{C}} x^2y\,dx + x\,dy$, $\mathcal{C}$: line segment from $(0, 0)$ to $(1, 2)$ to $(2, 0)$ to $(0, 0)$.

(ii) $\oint_{\mathcal{C}} 3y\,dx - x^2\,dy$, $\mathcal{C}$: line segment from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$.

(iii) $\oint_{\mathcal{C}} (2x + y^2)\,dx + (x^2 + 2y)\,dy$, $\mathcal{C}$: boundary of the region bounded by $y = 0$, $x = 2$ dan $y = \frac{1}{4}x^3$.

(iv) $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, $\mathbf{F}(x, y) = (e^x - y^3)\,\mathbf{i} + (\cos y + x^3)\,\mathbf{j}$, $\mathcal{C}$: curve $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.

(v) $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, $\mathbf{F}(x, y) = y^3\,\mathbf{i} + x^3\,\mathbf{j}$ $\mathcal{C}$: circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

(vi) $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, $\mathbf{F}(x, y) = (-3y + 2xy^2)\,\mathbf{i} + 3xy\,\mathbf{j}$ $\mathcal{C}$: curve $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$.

6. (i) Show that for curve $\mathcal{C}$ with the parametric equations $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, then $\oint_{\mathcal{C}} \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy = 2\pi$.

(ii) Show that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ for $f(x, y) = \frac{-y}{x^2 + y^2}$ and $g(x, y) = \frac{x}{x^2 + y^2}$.

Using Green's theorem show that $\oint_{\mathcal{C}} \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy = 0$.

Why is this result different from (i)?

7. The area of region R bounded by the curve $\mathcal{C}$ is given by the line integral

$$\frac{1}{2}\left[\oint_{\mathcal{C}} -y\,dx + x\,dy\right].$$

By using this result show that the area of an ellipse with the parametric equations $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ is πab.

8. Compute the work done by the force field $\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} - yx^2\,\mathbf{j} + 3xz\,\mathbf{k}$ acting on an object moving along the straight line from point $(1, 1, 0)$ to $(1, 1, 1)$ to $(0, 0, 0)$.

9. Given the force field, $\mathbf{F}(x, y) = (e^y + ye^x)\,\mathbf{i} + (xe^y + e^x)\,\mathbf{j}$. Compute the work done if the motion starts from point $P(2, 0)$ to point $Q(-2, 0)$ along the hemisphere with the parametric equations $x = 2\cos\theta,\ y = 2\sin\theta,\ 0 \le \theta \le \pi$.

10. Compute the work done by the force $\mathbf{F}(x, y) = (x^2 - y)\,\mathbf{i} + x\,\mathbf{j}$ acting on an insect as it moves along a circle with radius 2.

11. Session 2001/02 Sem I

a) Evaluate the line integral

$$\int_{\mathcal{C}} (x^2 + y^2 - x)dx + y\sqrt{x^2 + y^2}dy$$

with $\mathcal{C}$ is the path consist of a line segment from $(0, 0)$ to $(1, 0)$ along the $x-$axis, and from $(1, 0)$ to $(-1, 0)$ along a semicircle $y = \sqrt{1 - x^2}$.

(10 Marks)

b) Given $\mathbf{F}(x, y, z) = (x^2+y^2)\,\mathbf{i} + (\lambda xy)\,\mathbf{j} + z\,\mathbf{k}$. Show that the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, is independent of the path if $\lambda = 2$. By using this value of λ, evaluate the above line integral if $\mathcal{C}$ is the path that connects the points $A(1, 0, 1)$ to $B(2, 1, 1)$.

(10 Marks)

12. Session 2001/02 Sem II

a) If $\mathbf{F}(x, y, z) = (3x^2+6y)\,\mathbf{i} - (14yz)\,\mathbf{j} + (20xz^2)\,\mathbf{k}$, evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, from $(0, 0, 0)$ to the point $(1, 1, 1)$ with the path $\mathcal{C}$ as the line segment that connects the point $(0, 0, 0)$ to $(1, 0, 0)$, $(1, 0, 0)$ to $(1, 1, 0)$, and from $(1, 1, 0)$ to $(1, 1, 1)$

(6 Marks)

b) Show that the vector function $\mathbf{F}(x, y, z) = (2xy + z^3)\,\mathbf{i} + (x^2)\,\mathbf{j} + (3xz^2)\,\mathbf{k}$ is a conservative field and find the potential function for $\mathbf{F}$. Hence find the work done in this field to move an object from a point $(1, -2, 1)$ to a point $(3, 1, 4)$.

(7 Marks)

c) By using the Green's theorem, evaluate

$$\oint_{\mathcal{C}} (xy + y^2)dx + (x^2)dy$$

if $\mathcal{C}$ is the boundary of a region bounded by $y = x^2$ and $y = x$ in the anticlockwise manner.

(7 Marks)

13. Session 2002/03 Sem I

a) Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ with

$$\mathbf{F}(x, y, z) = (x^2 + y)\,\mathbf{i} - (3xz)\,\mathbf{j} + (z - 4y)\,\mathbf{k},$$

with the path $\mathcal{C}$ as the line segment that connects the point $(1, 3, 2)$ to $(2, 0, 0)$ and from $(2, 0, 0)$ to $(-2, 0, 0)$ along the semicircle $y = \sqrt{4 - x^2}$ on the $xy-$plane.

(7 Marks)

b) Show that the vector function

$$\mathbf{F}(x,y,z) = (ye^{-x})\,\mathbf{i} + (ze^{y} - e^{-x})\,\mathbf{j} + (e^{y})\,\mathbf{k}$$

is a conservative field and find the potential function for $\mathbf{F}$. Hence find the work done in this field to move an object from a point $(1,1,1)$ to $(1,2,3)$ to $(3,3,4)$.

(7 Marks)

c) By using the Green's theorem, evaluate

$$\oint_{\mathcal{C}} (2xy^3)dx + (4x^2y^2)dy$$

if $\mathcal{C}$ is the boundary of a region bounded by $y = x + 2$, $x-$axis and $x = \sqrt{y}$ in the anticlockwise manner.

(6 Marks)

14. Session 2002/03 Sem II

a) Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ with

$$\mathbf{F}(x,y,z) = (y^2 + z)\,\mathbf{i} + (x^2y)\,\mathbf{j} + (xz)\,\mathbf{k},$$

with the path $\mathcal{C}$ as the line segment that connects the point $A(1,1,1)$ to $B(7,3,4)$.

(5 Marks)

b) Determine whether the vector function

$$\mathbf{F}(x,y,z) = e^{y+2z}\,[\mathbf{i} + x\,\mathbf{j} + 2x\,\mathbf{k}]$$

is a conservative field. If it is a conservative field, obtain the potential function for $\mathbf{F}$. Hence find the work done in this field to move an object from a point $(1,0,-1)$ to $(0,0,0)$ to $(0,1,1)$.

(7 Marks)

c) By using the Green's theorem in the $xy-$plane, show that if A is the area of the region bounded by a simple closed curved $\mathcal{C}$, then

$$\frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx = \text{ Area } \quad \text{A}$$

Hence by using this formula, show that the area of a circle with radius a is πa^2.

(8 Marks)

15. Session 2005/06 Sem I

a) Evaluate $\int_{\mathcal{C}} xy^2dx + xy^2dy$ along the path $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ as shown in the Figure A. Also evaluate this integral along the straight line path $\mathcal{C}_3$ from $(0,2)$ to $(3,5)$.

(8 Marks)

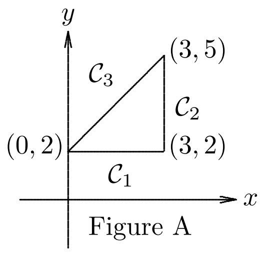

Figure A

b) Determine whether the vector function $\mathbf{F}(x, y) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$ is a conservative field. If it is, find the corresponding potential function $\phi(x, y)$.

(7 Marks)

c) Use Green's theorem to evaluate $\oint_{\mathcal{C}} 4x^2y\,dx + 2y\,dy$ where $\mathcal{C}$ is the anticlockwise boundary of a triangular region with vertices $A(0, 0)$, $B(1, 2)$ and $C(0, 2)$.

(5 Marks)

16. Session 2006/07 Sem I

a) Evaluate the line integral

$$\int_{\mathcal{C}} x\,dx - xy\,dy + yz\,dz,$$

where $\mathcal{C}$ is the line segment from $(0, 0, 0)$ to $(2, 1, 2)$.

(6 Marks)

b) A force $\mathbf{F}(x, y) = (e^x \sin y - y)\,\mathbf{i} + (e^x \cos y - x - 2)\,\mathbf{j}$ acts on an object moving in a plane. Show that $\mathbf{F}$ is conservative and find a scalar potential for $\mathbf{F}$. Use this potential function to find the work done by $\mathbf{F}$ to move the object from the point $(0, 0)$ to $(1, \frac{\pi}{2})$.

(7 Marks)

c) Use Green's theorem to evaluate

$$\oint_{\mathcal{C}} [x^2 + 2y + \sin(x^2)]\,dx + [x + y + \cos(y^2)]\,dy$$

where $\mathcal{C}$ is the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant oriented counterclockwise.

(7 Marks)

17. Session 2007/08 Sem II

a) Evaluate

$$\int_{\mathcal{C}} \frac{e^{-z}}{x^2 + y^2}\,ds$$

where $\mathcal{C}$ is the path given by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\,\mathbf{k}$ for $0 \le t \le 2\pi$.

(7 Marks)

b) Given the vector field

$$\mathbf{F}(x, y, z) = (e^x \sin y - z \sin x)\,\mathbf{i} + (e^x \cos y)\,\mathbf{j} + \cos x\,\mathbf{k}.$$

Show that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path $\mathcal{C}$. Hence find a potential function evaluate the line integral along the curve $\mathcal{C}$, from $(0,0,0)$ to $(1,1,1)$ to (π,π,π).

(7 Marks)

c) Use Green's theorem to evaluate the integral

$$\oint_{\mathcal{C}} [e^x + y^2]\, dx + [e^y + x^2]\, dy$$

where $\mathcal{C}$ is the boundary of the region enclosed by $y = x^2$ and $y = x$, and curve $\mathcal{C}$ is oriented counterclockwise.

(6 Marks)

ANSWERS FOR TUTORIAL 4

1. (i) 27 **(ii)** $2\sqrt{5}$ **(iii)** $\frac{1}{6}\left(14\sqrt{14}-1\right)$

(iv) $141\frac{1}{3}$ **(v)** 29.426 **(vi)** $2\frac{5}{6}$

2. (i) 3 **(ii)** $\frac{6}{5}$ **(iii)** $\frac{-7}{44}$

(iv) $\frac{-2}{3}$ **(v)** $1-e^3$ **(vi)** $3\frac{5}{6}$

3. (i) 5 **(ii)** 14 **(iii)** 5

(iv) $-\pi$

4. (i) -2 **(ii)** 4 **(iii)** π

(iv) 6 **(v)** 0 **(vi)** -2

5. (i) $-\frac{1}{3}$ **(ii)** -4 **(iii)** $2\frac{2}{35}$

(iv) 24π **(v)** 0 **(vi)** 48π

8. $\frac{3}{2}$ **9.** -4 **10.** 8π

11. (a) $-2\frac{1}{6}$ **(b)** $\phi = \frac{1}{3}x^3 + xy^2 + \frac{1}{2}z^2 + c$ Workdone$=4\frac{1}{3}$

12. (a) $7\frac{2}{3}$ **(b)** $\phi = x^2y + xz^3 + c$ Workdone$=202$ **(c)** $-\frac{1}{20}$

13. (a) $\frac{1}{2}(41-4\pi)$ **(b)** $\phi = -ye^{-x} + ze^y + c$ Workdone$=77.8424$ **(c)** $29\frac{13}{15}$

14. (a) $167\frac{1}{2}$ **(b)** $\phi = xe^{y+2z} + c$ Workdone$=-e^{-2}$

15. (a) $\mathcal{C}_1 \cup \mathcal{C}_2 : 135$ $\mathcal{C}_3 : 148\frac{1}{2}$ **(b)** $\phi = e^x \sin y + c$ **(c)** $-\frac{2}{3}$

16. (a) $2\frac{2}{3}$ **(b)** $\phi = e^x \sin y - xy - 2y + c$ Workdone$=e - \frac{3\pi}{2}$ **(c)** $-\frac{1}{6}$

17. (a) $\frac{\sqrt{5}}{4}(1-e^{-2\pi})$ **(b)** $\phi = e^x \sin y + z\cos x + c$ Workdone$=-\pi$ **(c)** $\frac{1}{30}$