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Groups and Graphs in Probability Theory

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Abstract. In this paper, G denotes a dihedral group of order $2n$ and Ω denotes the set of all subsets of all commuting elements of size two in the form of (a,b), where a and b commute and $|a|=|b|=2$. By extending the concept of commutativity degree, the probability that an element of a group fixes a set can be acquired using the group actions on set. In this paper, the probability that an element of G fixes the set Ω under regular action is computed. The results obtained are then applied to graph theory, more precisely to generalized conjugacy class graph and orbit graph.

INTRODUCTION

Throughout this paper, Γ denotes a simple undirected graph. In the following, we state a brief history related to the commutativity degree and graph theory

The commutativity degree is a concept that used to determine the abelianness of groups. The following is the definition of the commutativity degree:

Definition 1 [1] Let G be a finite non-abelian group. Suppose that x and y are two random elements of G. The probability that x and y commute is defined as:

$$P(G)=\frac{\left|\left\{(x,y)\in G\times G: xy=yx\right\}\right|}{|G|^2}.$$

It was proven that the probability of two elements of group commute is less than or equal to 5/8 for all finite non-abelian groups [2] and [3]. The concept of commutativity degree has been generalized and extended by several researchers. One of these extensions is the probability that a group element fixes a set [4]. The following is the definition of the probability that a group element fixes a set.

Definition 2[4] Let G be a finite group and let S be a set of elements of G of size two in the form of (a,b), where a,b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω. The probability that an element of a group fixes the set is given as follows:

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$$P_G(\Omega)=\frac{\left|\{(\omega,g)\in\Omega\times G: g\omega=\omega, g\in G, \omega\in\Omega\}\right|}{|\Omega||G|}.$$

El-sanfaz*et al*. [4] provided the theorem that is used to compute the probability that a group element fixes a set, given in the following.

Theorem 1[4] Let G be a finite group and let S be a set of elements of G of size two in the form of (a,b), where a,b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω, then the probability that an element of a group fixes a set is given by $P_G(\Omega)=\frac{K(\Omega)}{|\Omega|}$, where $K(\Omega)$ is the number of orbits under the group action of Ω in G.

Since this work is a connection between algebraic theory and graph theory, the followings are some basic concepts of graph theory that are needed in this paper. These concepts can be found in one of the references ([5], [6]).

A graph Γ is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise, Γ is called undirected. Two vertices are adjacent if they are linked by an edge. A complete graph is a graph where each ordered pair of distinct vertices are adjacent, denoted by K_n.

A non-empty set S of $V(\Gamma)$ is called an independent set of Γ if there is no adjacent between two elements of S in Γ. Meanwhile, the independent number is the number of vertices in the maximum independent set and it is denoted by $\alpha(\Gamma)$. However, the maximum number c for which Γ is c-vertex colorable is known as chromatic number, denoted by $\chi(\Gamma)$. The diameter is the maximum distance between any two vertices of Γ, denoted by $d(\Gamma)$. Furthermore, a clique is a complete subgraph in Γ, while the clique number is the size of the largest clique in Γ and is denoted by $\omega(\Gamma)$. The dominating set $X\subseteq V(\Gamma)$ is a set where for each v outside X, there exists $x\in X$ such that v is adjacent to x. The minimum size of X is called the dominating number and it is denoted by $\gamma(\Gamma)$ ([5]-[7]).

In 1990, a new graph called conjugacy class graph was introduced by Bertram *et al.* [8]. The vertices of this graph are non-central conjugacy classes i.e $|V(\Gamma)|=K(G)-|Z(G)|$, where $K(G)$ is the number of conjugacy class of a group and $Z(G)$ is the center of a group G. A pair of vertices of this graph are connected by an edge if their cardinalities are not coprime. Omer *et al*. [9] extended the work on conjugacy class graph by defining the generalized conjugacy class graph whose vertices are non-central orbits in which two vertices are adjacent if their cardinalities are not coprime.

Recently, Omer *et al*. [10] introduced a new graph called the orbit graph whose vertices are non-central orbits under group action on a set. Two vertices are adjacent if they are conjugate.

This paper is divided into three sections. The first section focuses on background of some topics in group theory and graph theory, while the second section provides some earlier and recent publications that are related to the probability that a group element fixes a set and graph theory. In the third section, we present our results on which include the probability that a group element fixes a set, generalized conjugacy class graph and orbit graph.

PRELIMINARIES

In this section, some works that are related to the probability that an element of group fixes a set and graph theory are stated. We commence with brief information about the probability that a group element fixes a set, followed by some related work on graph theory more precisely to graph related to conjugacy classes.

In 2014, the probability that a group element fixes a set was introduced by El-sanfaz *et al.* [4]. The probability was determined for dihedral groups [4]. Also, the probability that a group element fixes a set is obtained for metacyclic 2-groups of negative type [11]. In addition, the probability that an element of group fixes a set is computed for metacyclic 2-groups of positive type [12]. The probability is also computed for semi-dihedral groups and quasi-dihedral groups [13]. Moreover, El-sanfaz *et al*. [14] found the probability that a group element fixes a set for metacyclic 2-groups under regular action.

In this paper, the orbits and their sizes that are obtained under group action on a set are used to apply the results to graph theory, specifically to generalized conjugacy class graph and orbit graph.

Some related works on conjugacy class graph, generalized conjugacy class graph and orbit graph are stated as follows:

Bianchi *et al*. [15] studied the regularity of the conjugacy class graph. In addition, Moreto *et al*. [16] classified the groups in which conjugacy classes sizesare not coprime for any five distinct classes. Furthermore, You *et al.* [17] classified the groups in which conjugacy classes are not set-wise relatively prime for any four distinct classes. Moreover, Moradipour*et al.* [18] used the conjugacy class graph to find some graph properties of some finite metacyclic 2-groups. Omer *et al*. [9] extended the work in [8] and defined the generalized conjugacy class graph denoted by $\Gamma_G^{\Omega_c}$. The vertices of $\Gamma_G^{\Omega_c}$ are non-central orbits in which two vertices are connected by an edge if their cardinalities are not coprime. The generalized conjugacy class graph was found for dihedral groups [9] and metacyclic 2-groups [19]. In addition, El-sanfaz *et al.* [13] determined the generalized conjugacy class graph of semi-dihedral groups and quasi-dihedral groups. The generalized conjugacy class graph is also found for metacyclic 2-groups [14].

In [10], Omer *et al.* introduced the orbit graph whose vertices are non-central orbits under group action on a set. The following is the definition of the orbit graph.

Definition 3 [10] Let G be a finite group and Ω a set of G. Let A be the set of commuting element in Ω, i.e. $\{\omega\in\Omega:\omega g=g\omega, g\in G\}$. The orbit graph Γ_G^{Ω} consists of two sets, namely vertices and edges denoted by $V\left(\Gamma_G^{\Omega}\right)$ and $E\left(\Gamma_G^{\Omega}\right)$, respectively. The vertices of Γ_G^{Ω} are non-central elements in Ω but not in A, that is $\left|V\left(\Gamma_G^{\Omega}\right)\right|=\Omega-|A|$, the number of edges are $\left|E\left(\Gamma_G^{\Omega}\right)\right|=\sum_{i=1}^{\left|V\left(\Gamma_G^{\Omega}\right)\right|}\binom{v_i}{2}$, where v the size of orbit under group action of G on Ω. Two vertices v_1 and v_2 in Γ_G^{Ω} are adjacent if one of the following conditions is satisfied.

(i) If there exists $g\in G$, such that $gv_1=v_2$

(ii) If the vertices of Γ_G^{Ω} are conjugate that is, $v_1=g^{v_2}$.

The orbit graph was found for solvable groups, dihedral groups, quaternion groups [20] and *p*-groups [21].

Since the group action used in this research is regular, the definition of regular action is given in the following:

Definition 4 [22] Suppose G is a finite group that acts on a set S. G acts regularly on S if the action is transitive and $\mathrm{Stab}_G=1$ for all $x\in S$. Also, the action is regular if for s_1 and s_2 belongs to S, there exists an element $g\in G$ such that $gs_1=s_2$.

RESULTS AND DISCUSSION

This section consists of two parts. The first part focuses on the probability that a dihedral group element fixes a set under regular action, while the second part applies the obtained results to graph theory, namely to generalized conjugacy class graph and orbit graph.

The Probability that a Dihedral Group Element Fixes a Set

In this section, the probability that an element of dihedral group fixes a set under regular action is determined.

Theorem 2: Let G be a finite non-abelian dihedral group of order $2n$. Let Ω be the set of all subsets of commuting elements of G of size two of the form of (a,b), where a and b commute and $|a|=|b|=2$. If G acts regularly on Ω, then $P_G(\Omega)=\dfrac{n+1}{|\Omega|}$.

Proof: The elements of Ω are stated as follows: There are n elements in the form of $\left(a^{\frac{n}{2}}, a^{\frac{n}{2}+i}b\right), 0 \le i \le 2n,$ and there are $\frac{n}{2}$ elements of the form $\left(a^i b, a^{\frac{n}{2}+i}b\right)$, $0 \le i \le 2n.$ Then $|\Omega| = \frac{3n}{2}.$ If G acts regularly on $\Omega,$ then there exists an element $g \in G$ such that $g\omega_1 = \omega_2$ for $\omega_1, \omega_2 \in \Omega.$ Thus, the orbits under group action on Ω are described as follows: There are one orbit in the form of $\left(a^i b, a^{\frac{n}{2}+i}b\right)$, $0 \le i \le 2n$ of size $\frac{n}{2}$ and n orbits in the form of $\left(a^{\frac{n}{2}}, a^i b\right), 0 \le i \le 2n$ of size one. Thus there are $n+1$ orbits. Using Theorem 1, then $P_G(\Omega) = \frac{n+1}{|\Omega|}.$

Example 1: Let G be a finite non-abelian dihedral group of order 8, namely D_4 . Let Ω be the set of all subsets of commuting elements of G of size two of the form of (a,b), where a and b commute and $|a| = |b| = 2.$. If G acts regularly on $\Omega,$ then $P_G(\Omega) = \frac{5}{6}.$

Solution: The elements of order two in G are a^2, b, ab, a^2b and a^3b. Thus the elements of Ω are $(a^2,b), (a^2,ab), (a^2,a^2b), (a^2,a^3b), (b,a^2b)$ and $(ab,a^3b).$ Thus $|\Omega| = 6.$ Since the action is regular, thus there exists an element $g \in G$ and $\omega_1, \omega_2 \in \Omega$ such that $g\omega_1 = \omega_2.$ Therefore, there are four orbits in the form of $\left(a^{\frac{n}{2}}, a^i b\right), 0 \le i \le 2n$ and one orbit is in the form of $\left(a^i b, a^{\frac{n}{2}+i}b\right)$, $0 \le i \le 2n.$ Hence $K(\Omega) = 5.$ Based on Theorem 1, $P_G(\Omega) = \frac{5}{6}.$

In next section, the results on the orbits and their sizes are applied to generalized conjugacy class graph and orbit graph.

Generalized Conjugacy Class Graph and Orbit Graph of Dihedral Groups

In this section, the results on orbits and their sizes that are obtained in previous section are applied to generalized conjugacy class graph and orbit graph. The results are given in the following theorem, started with the generalized conjugacy class graph.

Theorem 3: Let G be a finite non-abelian dihedral group of order $2n$. Let Ω be the set of all subsets of commuting elements of G of size two of the form of (a,b), where a and b commute and $|a| = |b| = 2.$ If G acts regularly on $\Omega,$ then the generalized conjugacy class graph $\Gamma_G^{\Omega_c} = K_e.$

Proof: Based on Theorem 2, there are $n+1$ orbits, one of them of has size $\frac{n}{2}$ and n of them have size one. Thus, the number of vertices in $\Gamma_G^{\Omega_c}$ is $n+1$ i.e. $\left|V\left(\Gamma_G^{\Omega_c}\right)\right| = n+1.$ Since two vertices are adjacent if their cardinalities are set-wise relatively prime, as gcd($\frac{n}{2}$,1)=1. Thus $\Gamma_G^{\Omega_c}$ consists of $n+1$ isolated vertices. Therefore $\Gamma_G^{\Omega_c}$ is an empty graph.

The following is the orbit graph of dihedral groups.

Theorem 4: Let G be a finite non-abelian dihedral group of order $2n$, where n is even. Let Ω be the set of all subsets of commuting elements of G of size two of the form of (a,b), where a and b commute and $|a|=|b|=2$. If G acts regularly on Ω, then the orbit graph, $\Gamma_G^{\Omega} = K_{\frac{n}{2}}$.

Proof: Based on Theorem 2, the number of elements of Ω is $|\Omega| = \frac{3n}{2}$. Using Definition 3, the number of vertices of Γ_G^{Ω} is $\frac{3n}{2}$ i.e. $\left|V\left(\Gamma_G^{\Omega}\right)\right| = \frac{3n}{2}$. Now two vertices $w_1, w_2 \in \Gamma_G^{\Omega}$ are adjacent if there exists $g \in G$ such that $gw_1 = w_2$. Again using Theorem 2, the vertices which are in the form of $\left(a^i b, a^{\frac{n}{2}+i} b\right), 0 \le i \le 2n$ are adjacent to each other, thus Γ_G^{Ω} consist of a complete component of $K_{\frac{n}{2}}$ and there are n isolated vertices in the form of $\left(a^{\frac{n}{2}}, a^i b\right)$, $0 \le i \le 2n$. The proof then follows.

Example 2: Let G be a finite non-abelian dihedral group of order 8, namely D_4. Let Ω be the set of all subsets of commuting elements of G of size two of the form of (a,b), where a and b commute and $|a|=|b|=2$. If G acts regularly on Ω, then $\Gamma_G^{\Omega} = K_2 \bigcup \left(a^2, b\right) \bigcup \left(a^2, ab\right) \bigcup \left(a^2, a^2 b\right) \bigcup \left(a^2, a^3 b\right)$.

Solution: Based on Definition 3 and Example 1, the number of vertices in Γ_G^{Ω} is six. According to Example 1, there are five orbits, where one orbit has size two and four orbits have size one. Based on vertices adjacency of the orbit graph, Γ_G^{Ω} consists of one complete components of K_2 and four isolated vertices, namely $\left(a^2, b\right), \left(a^2, ab\right), \left(a^2, a^2 b\right)$ and $\left(a^2, a^3 b\right)$, as claimed.

Based on Theorem 4, the following corollary is concluded.

Corollary 1: Let G be a finite non-abelian dihedral group of order $2n$. Let Ω be the set of all subsets of commuting elements of G of size two of the form of (a,b), where a and b commute and $|a|=|b|=2$. If G acts regularly on Ω and $\Gamma_G^{\Omega} = K_{\frac{n}{2}}$, then $\chi\left(\Gamma_G^{\Omega}\right) = \omega\left(\Gamma_G^{\Omega}\right) = \frac{n}{2}$.

Proof: The chromatic number and the clique number are equal to $\frac{n}{2}$ since the orbit graph consists of a complete component of $\frac{n}{2}$, thus the maximum number of complete sub graphs is equal to the minimum number of coloring vertices. The proof is complete.

CONCLUSION

In this paper, the probability that an element of dihedral group fixes a set under regular action is computed. The results obtained are then applied to generalized conjugacy class graph and orbit graph. Some graph properties including the chromatic number and the clique number are also obtained.

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